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ON THE METHOD FOR STUDY OF THE CRITICAL BEHAVIOUR OF SPIN MODELS IN ARBITRARY SPACE DIMENSION

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Abstract

A method for study of critical behaviour in non-integer space dimension is discussed. Critical exponents of the Ising-like and $O(m)$-symmetrical systems are calculated in the case, when the dimension of space is non-integer. Calculations are performed in the frames of the fixed-dimension field theoretical approach. Renormalization group functions in the Callan-Symanzik scheme are considered directly in non-integer dimensions. Perturbation theory expansions are resummed with the use of Padé-Borel transformation.

1 Introduction

During last years much attention has been payed to the investigation of critical behaviour of model spin lattice systems in the case when the dimension of the lattice is non-integer. Besides the pure academic interest such a problem statement has several other reasons. In different models under consideration variations of dimension of the order parameter (m) and space (d) are used to link the results to exact ones or to results from other calculational methods (e.g. ϵ-expansions near two and four dimensions). Besides, there exist models, where some new phenomena appear starting from some (non-integer) space dimension.

Moreover, the problem of non-integer space dimension has wide application in the theory of fractals. Detailed analysis of fractal lattices has lead to the controversial conjecture that some fractal lattices could interpolate standard regular lattices in non-integer dimensions. From the other side, as a result of investigation of the second-order phase transition on different fractal lattices it appeared that lattices of the same fractal dimension are characterized by different sets of critical exponents (see [1-4]). Now the resulting general belief is that, in the case of fractals, neither fractal, no any other dimension can lead to the universal dependence of the critical exponents. The point is that the fractal description involves several factors (beside the fractal dimension) that can vary independently of one another, such as raminification, connectivity, lacunarity. Being scale invariant, but not translationally invariant fractal lattices may interpolate the results for hypercubic lattices only in the limit of zero lacunarity, where translational invariance is recovered [1,3,5]. The universality hypothesis if it exists in the case of fractals should be essentially reviewed [6].

One should note, that the study of critical phenomena has benefited greatly from investigation of the dimensional dependence of critical exponents and other characteristics of critical behaviour. Consideration of the space dimension (or its deviation from some definite fixed value) as the continuous variable and its choice as a perturbation theory series expansion parameter gives the possibility to obtain results for integer d as well. Here one should mention not only the famous $\epsilon = 4 - d$ expansion [7] whose application in the theory of critical phenomena allowed to obtain the reliable values of critical exponents for a whole range of $3d$ models (see [8-11]) but $\epsilon' = d - 1$ expansion, introduced for the near-planar interface [12-14] and droplet [15] models, $\epsilon^{1/2}$-expansion for the weakly diluted Ising model [16, 17], etc. Expansion in the terms of inverse dimension for study of the second-order phase transition began, perhaps, with the analysis of the spin-1/2 Ising model on a simple hypercubic d-dimensional lattices [18].

Often the aim of investigations is the calculation of the critical exponents of some model directly at non-integer d. In this context let us mention here some papers devoted to such a problem statement.In the case of the Ising model the Kadanoff lower-bound renormalization transformation was used to obtain the values of the thermal and magnetic critical exponents for all integral values of $z = 2^d$ between 3 and 16, this leads, naturally, to critical exponents values for some non-integer d [19]. Later on critical exponents for the Ising model in non-integer dimension were calculated [5] by resummation of the five terms of the ϵ-expansion avaible [20]. Real-space renormalization group approach [1,3] as well as series expansions [4] were applied to this problem as well. Free energy of the d-dimensional Ising model was studied in [21] on the base of variational method derived from high-temperature series expansion. Accurate values of the critical exponents between one and two dimensions were obtained in [22-24] by applying finite-size-scaling methods to numerical transfer- matrix data; the transfer-matrix being written in a fashion which enables the interpolation to non-integer dimensions.

Whereas the values of critical exponents for the Ising model in general dimensions were obtained already by means of different methods and even at the presence of quenched randomness [25,26,27] it is not in the case of general value of field component number m for the case of $O(m)$-symmetric model. This model was studied by $2 + \epsilon$ expansion technique close to the special case of the planar model,which corresponds to $m = d = 2$ in [28]. In the case of arbitrary m the method based on the study of physical branch of the renormalization group equation solution was applyed to this model at general dimension in [29,30].

In this paper we present an alternative method for investigation of the dimensional dependence of the critical exponents. Our work involves the fixed dimension renormalization group approach at arbitrary dimension originated in [27,28]. Following the idea of Parisi [31] to perform the calculations directly in 2 and 3 dimensions we proposed to consider the renormalization group functions directly at the arbitrary non- integer d. Such scheme of calculations, avoiding the applica tion of ϵ-expansion will be used in this article. The results will be presented in the following order. In Sec.2 we shall briefly describe the method applied; Sec.3 will be devoted to the computation of the integrals arising within the calculati onal scheme under consideration; in Sec.4 we give the estima tes of critical exponents for the Ising model and for the model with continuous symmetry; Sec.5 contains the conclusions.

2 Problem statement and the renormalization procedure

In this paper we concentrate on the two models commonly used in the phase transition theory: the Ising model and $O(m)$-symmetric model [1]. The Ising model has a Hamiltonian which in the absence of external magnetic field is given by:

$$H = J \sum_{\langle i,j \rangle} S_i S_j, \tag{2.1}$$

where spins $S_i = \pm 1$, and summation is over nearest-neighbour sites on a lattice. As it is well known now [8-11] one can describe the long-distance properties of such a model in the neighbourhood of a second-order phase transition in the terms of continuous Euclidian field theory with the Lagrangian:

$$\mathcal{L}(\phi) = \int \mathrm{d}^d R \left\{ \frac{1}{2} \left[|\nabla \phi|^2 + m_0^2 \phi^2 \right] + \frac{\lambda_0}{4!} \phi^4 \right\}, \tag{2.2}$$

where m_0^2 is a linear function of the temperature, λ_0 is the bare coupling, $\phi = \phi(R)$ is an one-component field.

One of the ways to generalize this model is to introduce into (2.2) a multiplet of m fields forming a representation of $O(m)$-group. In this case the Lagrangian reads:

$$\mathcal{L}(\phi) = \int \mathrm{d}^d R \left\{ \frac{1}{2} \left[|\nabla \vec{\phi}|^2 + m_0^2 |\vec{\phi}|^2 \right] + \frac{\lambda_0}{4!} |\vec{\phi}|^4 \right\}, \tag{2.3}$$

where $\vec{\phi} = \vec{\phi}(R)$ is the vector field $\vec{\phi} = (\phi^1, \phi^2, \ldots; \phi^m)$. And in corresponding spin Hamiltonian stends the scalar product of m-component vectors $\vec{S} = (S^1, S^2, \ldots, S^m)$:

$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_j. \tag{2.4}$$

In order to study the critical properties of the field theories (2.2). (2.3) in general space dimension d, we use the standard procedure of renormalization of one-particle irreducible vertex function

$$\Gamma^{(L,N)}(p_1, .., p_L; k_1, .., k_N; m_0^2, \lambda_0, d)$$

at zero external momenta and nonzero mass (see [9,11] for example). Asymptotically close to the critical point, the renormalized vertex functions

$$\Gamma_R^{(N)}(\{k_j\}; m^2, \lambda; d)$$

satisfy the homogeneous Callan-Symanzik equation [9,11]:

$$\left[m \frac{\partial}{\partial m} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \frac{N}{2} \gamma_\phi(\lambda) \right] \Gamma_R^{(N)}(\{k_j\}; m^2, \lambda; d) = 0, \tag{2.5}$$

[1]Extension of the method under consideration for the models with quenched disorder at noninteger d see in [26,27].

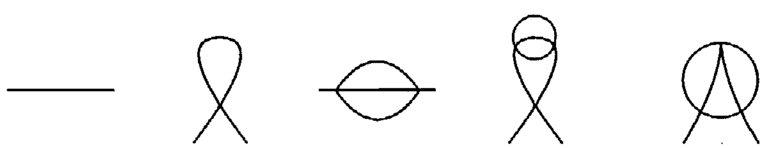

Figure 1:

here λ and m are the renormalized coupling constant and mass. The equation may be treated, in principle, for arbitrary non-integer fixed space dimension d. Functions $\beta(\lambda)$ and $\gamma_\phi(\lambda)$ are defined as follows:

$$\beta(\lambda) = -(4-d)\left[\frac{\partial \ln \lambda_0}{\partial \lambda}\right]^{-1}, \tag{2.6}$$

$$\gamma_\phi(\lambda) = \beta(\lambda)\frac{\partial \ln Z_\phi(\lambda)}{\partial \lambda}, \tag{2.7}$$

here the renormalization constant Z is given by:

$$Z_\phi^{-1} = \frac{\partial}{\partial k^2}\Gamma^{(2)}(k; m_0^2, \lambda_0)|_{k^2=0}.$$

It is implied that the bare parameters m_0^2, λ_0 are expressed here in terms of the renormalized ones. In the stable fixed point, coordinate of which are determined by zero of β-function, γ_ϕ gives the value of the pair correlation function critical exponents η. The correlation length critical exponent ν can be calculated from the consideration of the two-point vertex function with ϕ^2 insertion, $\Gamma^{(1,2)}(\{0\}; m_0^2, \lambda_0; d)$. The massive field theory normalization condition for this vertex function implies the following definition of the renormalization constant Z_{ϕ^2}:

$$Z_{\phi^2} = \Gamma^{(1,2)}(\{0\}; m_0^2, \lambda_0; d).$$

Using this relation, one can calculate the γ-function

$$\gamma_{\phi^2}(\lambda) = \beta(\lambda)\frac{\partial \ln Z_{\phi^2}(\lambda)}{\partial \lambda}, \tag{2.8}$$

which at the fixed point gives the value of the combination 2-$\nu^{-1} - \eta$ (ν being correlation length critical exponent). The other critical exponents now can be obtained on the base of ν and η using the familiar scaling relations.

Imposing the zero momentum renormalization conditions for conventionally defined 2-pt. and 4-pt. single-particle irreducible vertex functions $\Gamma_R^{(2)}(k, -k; m_0^2, \lambda_0; d)$, $\Gamma_R^{(4)}(\{k_i\}; m_0^2, \lambda_0; d)$ one obtains the expressions, shown by graphs on figs.1 and 2 in three loop approximation (the labeling of [32] is preserved). To every internal line i corresponds a propagator $(1+k_i^2)$, integration over internal momenta is imposed and momentum conservation law is carried out in every point. Finally we obtain the following expressions

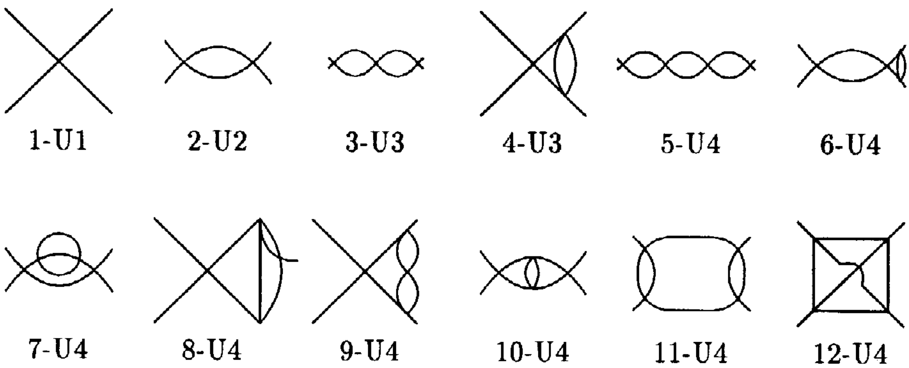

Figure 2:

for the β- and γ-functions of the $O(m)$-symmetric model (2.3) (the case of the Ising model corresponds to $m = 1$ of course) [33]:

$$\beta(u) = -(4-d)\left[u - u^2 + \beta_2 u^3 + \beta_3 u^4\right], \tag{2.9}$$

$$\gamma_\phi = -(4-d)\frac{2(m+2)}{(m+8)^2}u^2\left[2i_2 + (4i_2 - 3i_8)u\right], \tag{2.10}$$

$$\gamma_{\phi^2} = (4-d)\frac{m+2}{m+8}u\left[1 + \gamma_2 u + \gamma_3 u^2\right]. \tag{2.11}$$

We have used the change of the variables $u = (m+8)D\lambda/6$ and $\beta(u) = 6\beta(\lambda)/[(m+8)D]$, D being one-loop integral:

$$D = \frac{1}{(2\pi)^d}\int \frac{d\vec{k}}{(k^2+1)^2},$$

to define a convenient numerical scale in which the first two coefficients of $\beta(u)$ are -1 and 1. The indices 2 and 3 refer to the two- and three-loop parts of the corresponding functions:

$$\beta_2 = \frac{8}{(m+8)^2}\left[(m+2)i_2 + (5m+22)(i_1 - 1/2)\right], \tag{2.12}$$

$$\begin{aligned}\beta_3 = {} & \frac{1}{(m+8)^3}[(-32m^2 - 488m - 1424) + \\ & +4(31m^2 + 430m + 1240)i_1 + (m+2)(m+8)(8+3d)i_2 - \\ & -12(m+2)(m+8)i_3 - 48(m^2+20m+60)i_4 - \\ & -24(2m^2+21m+58)i_5 - 6(3m^2+22m+56)i_6 - \\ & -24(5m+22)i_7 - 12(m+2)(m+8)i_8],\end{aligned} \tag{2.13}$$

$$\gamma_2 = \frac{1}{(m+8)}(1 - 2i_1), \tag{2.14}$$

$$\gamma_3 = \frac{1}{(m+8)^2}[10(m+8) - (44m+280)i_1 + (8-3d)(m+2)i_2 +$$

Graph No.	Integral value
2-U2	1
3-U3	1
4-U4	i_1
5-U4	1
6-U4	i_1
7-U4	i_3
8-U4	i_4
9-U4	i_5
10-U4	i_6
11-U4	i_5
12-U4	i_7

Table 1.

Graph No	Integral value $\frac{\partial}{\partial k^2}\|_{k^2=0}$	$\frac{\partial}{\partial m_0^2}$
2-M1	0	-1
3-S2	i_2	$-3i_1$
4-M3	0	$-3i_6 - 2i_3$
5-S3	i_8	$-4i_4 - i_5$

Table 2.

$$+12(m+2)i_3 + 24(m+8)i_4 + 6(m+8)i_5 + 18(m+2)i_6]. \qquad (2.15)$$

Beside the explicit dependence on the space dimension d, considered here functions depend on d via the loop integrals $i_1 - i_8$. Correspondence between graphs of 2- and 4-point one-particle irreducible vertex function $\Gamma^{(2)}, \Gamma^{(4)}$ and their numerical integral values $i_1 - i_8$ follows from the Tables 1 and 2. Substituting integrals $i_1 - i_8$ into (2.10), (2.12) - (2.15) by their ϵ-expansion one can develop corresponding ϵ-expansion technique in order to extract the critical behaviour generated by the Hamiltonian (2.4). In this case well-known ϵ^3-results [34] are re-reproduced. From the other hand, being interested in the critical behaviour at non-integer d it is possible to calculate the corresponding values of the loop integrals $i_1 - i_8$ directly for arbitrary d. We shall briefly describe these calculations in the following section.

3 Compilation of 2-pt. and 4-pt. graphs in field theory in non-integer dimension

The simplest way to obtain the expressions for setting up the numerical procedure is to make use of the Feynman parameters method. Note, that initial multiplicity of two-loop integrals is $2 \times d$ and that of three-loop ones is $3 \times d$(i.e. the first one changes from 0 to 8 and the second one from 0 to 12 when d changes from 0 to 4).

Making use of the formula for folding many denominators into one (see [11] for example):

$$\frac{1}{a_1^{\alpha_1} a_2^{\alpha_2} \dots a_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 + \alpha_2 \dots \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\dots\Gamma(\alpha_n)} \int dx_1 dx_2 \dots dx_{n-1} \times \qquad (3.1)$$
$$\times \frac{x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \dots x_n^{\alpha_n - 1}(1 - x_1 - x_2 \dots - x_{n-1})^{\alpha_n - 1}}{[x_1 a_1 + x_2 a_2 \dots + x_{n-1} a_{n-1} + (1 - x_1 - x_2 \dots - x_{n-1}) a_n]^{\alpha_1 + \alpha_2 \dots + \alpha_n}}$$

here the integration over the Feynman parameters, x_i, extends over the domain:

$$0 \le x_i \le 1; \; x_1 + x_2 + \dots + x_{n-1} \le 1,$$

one can obtain the expressions for the integrals $i_1 - i_8$ which are given in the Appendix. These expressions, depending on the dimension of space as on the parameter are to be evaluated numerically for general value of d.

Numerical calculation of the multidimensional integrals was set up by means of successive integration over each variable. Integrals were represented in the following form:

$$\int_a^b dx F(x) = \int_{-1}^1 dx(1-x)^\alpha(1+x)^\beta f(x) \tag{3.2}$$

and then calculated by means of Gauss method [35]. In this case quadratures were constructed on the base of Jacobi polynomials orthogonalized on the interval [-1,1]. Jacobi polynomials correspond to the weight function $p(x) = (1-x)^\alpha(1+x)^\beta$ which is used in (3.2). Abscissas and weighting coefficients were defined using the recurrence relations for orthogonal polynomials with the help of the method applied in [36]. Because of the fact, that choice of the exponents of the weight functions α, β depends on the value of external integrating variables, it was necessary to apply formulas of high orders and this essentially increased the running time.

The computer running time was the major factor in limitting accuracy. Estimates of numerical accuracy were made by studying the apparent convergence of the integrals with mesh size in conjunction with extensive testing of known functions and, whenever possible, by running the same integral in different ways.

Let us note, that, due to the form of the functions to be integrated, running time for calculation of integrals at low dimension d was essentially lower, if compared with the time, necessary for calculation of the same integral with the same accuracy at high d. Nevertheless if the value of some integral is to be defined for some fixed d with higher accuracy it can be done, using the appropriate integral representations. The same concerns the case, when the value of the integrals is needed for some concrete value of d.

The dependence of the loop integrals on the space dimension d for continuous change of d is shown on the figs.3,4 and the corresponding numerical values are listed in [49].

4 Estimates of critical exponents

4.1 Resummation procedure

As it is well known, series (2.9)-(2.11) are asymptotic and in order to obtain the reliable information on the base of these expressions one should make use of some resummation procedure. In order to start let us choose here the Padé-Borel method as the simplest one. As we'll see it works well when the dimension of space d is not very low. It is interesting to note that considered here three-loop approximation allows one to proceed without any resummation as well. The point is that as it was mentioned in [31], considered directly at fixed dimension β-function does not possess stable fixed point in two-loop approximation for dimensions of space less then $d = 3.5$ (if $m = 1$). So in two-loop approximation the resummation is needed not only to improve the results but to restore the presence of a zero of the β-function as well.

The scheme of resummation is as follows. Starting from the function

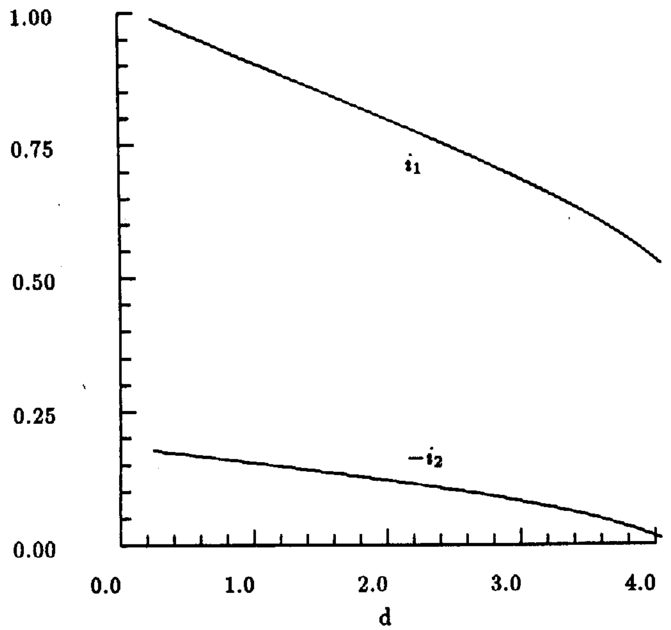

Figure 3: Two-loop integrals i_1, i_2 as the functions of space dimension d.

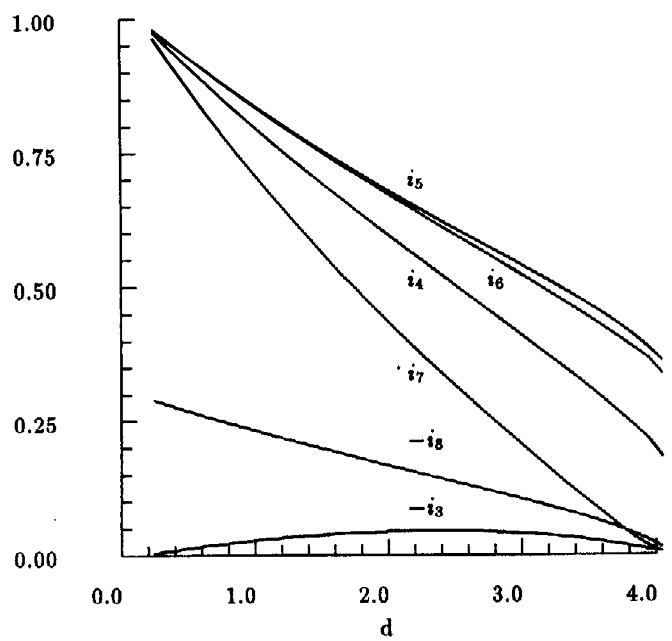

Figure 4: Three-loop integrals i_3 - i_8 as the functions of space dimension d.

$f(u)$ which is given by its Taylor expansion:

$$f(u) = \sum_{j\geq 0} c_j u^j, \tag{4.1}$$

one constructs the Borel transform:

$$F(ut) = \sum_{j\geq 0} \frac{c_j}{j!}(ut)^j. \tag{4.2}$$

Then one represents (4.2) in the form of Padé-approximant $F^P(ut)$ (in our case we have used [2/1] Padé-approximant) and the resummed function is given by

$$f^R(u) = \int_0^\infty dt e^{-t} F^P(ut). \tag{4.3}$$

As it was mentioned in [37], one can construct the Borel transform for the whole β-function as well as one can represent β-function in the form $\beta(u) = -u^{-1}f(u)$ and then resum the function $f(u)$. In the last case for the resummed function in three-loop approximation one obtains:

$$f^R(u) = -(4-d)u\left\{E(x)\left[1 - xz_1 + x^2 z_2\right] + z_1 x - xz_2(x-1)\right\}, \tag{4.4}$$

where $x = z^{-1}$, $z = -c_3 u/c_2$, $z_1 = (c_1 - c_3/c_2)u$, $z_2 = (c_2 - c_1 c_3/c_2)u^2$, $c_1 = f^{(1)}$, $c_2 = f^{(2)}/2!$, $c_3 = f^{(3)}/3!$. $f^{(1)}$, $f^{(2)}$ and $f^{(3)}$ are the one- two-and three-loop parts of the function $f(u)$. For function $E(x)$ we have:

$$E(x) = xe^x E_1(x),$$

the function

$$E_1(x) = e^{-x}\int_0^\infty dt e^{-t}(x+t)^{-1} \tag{4.5}$$

is connected with the exponential integral by the relation $E_1(x \pm i0) = -Ei(-x) \mp i\pi$ [38].

In the case of constructing of Borel-transform for the whole β-function, given by the expression (2.9) one obtains:

$$\beta^R(u) = -(4-d)uy\left\{\left[1 - y\xi_1 + y^2\xi_2\right]\left[1 - E(y)\right] + \xi_1 + (2-y)\xi_2\right\}, \tag{4.6}$$

where $y = \xi^{-1}$, $\xi = -b_3 u/b_2$, $\xi_1 = (b_1 - b_3/b_2)u$, $\xi_2 = (b_2 - b_1 b_3/b_2)u^2$, $b_1 = -1/2!$, $b_2 = \beta_2/3!$, $b_3 = \beta_3/4!$. β_2 and β_3 are two-and three-loop parts of the β-function given by (2.12) and (2.13), correspondingly.

In the table 3 we compare some of our results in the case of integer $d = 1, 2, 3$ obtained in [33] within two$-(2LA)$ and three-loop $(3LA)$ approximations by resummation of the function $-u^{-1}\beta(u)$ (referred as $2LA(1)$ and $3LA(1)$), and by the resummation of the function $\beta(u)$ (referred as $2LA(2)$ and $3LA(2)$), with the values of ν obtained in different renormalization group technique. "$3d$, fixed" means the resummation of six-loop expansion at fixed dimension $d = 3$[39], "$2d$, fixed" means the resummation of four-loop expansion at fixed dimension $d = 2$[39], "ϵ^4", "ϵ^5" means the resummation of ϵ-expansion containing terms up to ϵ^4 and ϵ^5 correspondingly [37,20].

	$m=0$	$m=1$	$m=2$	$m=3$
$d=3$				
$3d$, fixed	0.588±0.0015	0.630±0.0015	0.669±0.0020	0.705±0.0030
$\sim \varepsilon^4$	0.589	0.632	0.676	0.713
$\sim \varepsilon^4$		0.628±0.001	0.665±0.001	0.698±0.001
2LA(1)	0.592	0.632	0.668	0.699
(2)	0.590	0.628	0.663	0.693
3LA(1)	0.587	0.627	0.662	0.694
(2)	0.588	0.628	0.664	0.696
$d=2$				
$2d$, fixed		0.92+0.3		
$\sim \varepsilon^4$	0.77	1.03		
2LA(1)	0.785	0.981		
(2)	0.756	0.926		
3LA(1)	0.742	0.914		
(2)	0.754	0.941		
$d=1$				
$\sim \varepsilon^5$	1.3			
2LA(1)	1.386			
(2)	1.131			
3LA(1)	1.052			
(2)	1.147			

Table 3. Comparison of our calculations with those obtained for some integer d in different RG frameworks: "$d=3$, fixed" - resummed six-loop expansion in fixed dimension $d=3$ [39]; "$d=2$, fixed" - resummed four-loop expansion in fixed dimension $d=2$ [39]; "ϵ^4", "ϵ^5" - resummed ϵ-expansion in ϵ^4 – [37] and ϵ^5 – [20] approximations; $2LA(1)$, $2LA(2)$ - our results in two-loop approximation obtained by resumming the function $\beta(u)$ (1) and $-u^{-1}\beta\ (u)$ (2); $3LA(1)$, $3LA(2)$ - the same in three-loop approximation [33].

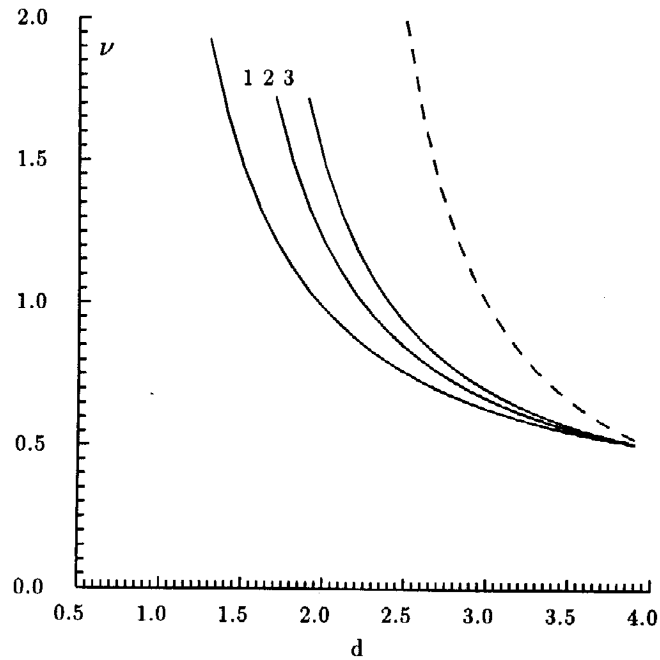

Figure 5: Correlation length critical exponent ν obtained in two-loop approximation as a function of d. Curve 1 - $m = 1$, curve 2 - $m = 2$, curve 3 - $m = 3$, dashed curve - $m = \infty$.

On fig. 5 we show the behaviour of the critical exponent ν for $m = 1$ (curve 1), $m = 2$ (curve 2), $m = 3$ (curve 3) and $m = \infty$ (dashed curve). These values were obtained in the two-loop approximation; fixed points were calculated on the base of the formula (4.6). Let us note that in the case $m = \infty$ we obtain the exact result for the spherical model [41]$\nu = (d-2)^{-1}$. For the case of finite m we obtain that ν as a function of d has singularity at some $d = d_c$ and is negative for $d < d_c$. A negative value of ν implies that the system has no transition at finite temperature. Such a critical dimension for $m = 2, m = 3$ is $d_c = 2$ [42,43], for Ising model the singularity lies at $d_c = 1$. Values of d_c corresponding to behaviour of ν shown on Fig.5 differ from those obtained from rigorous considerations [42,43].

Again, examining values of the critical exponents for $m = 1-3$ obtained within the Padé-Borel method in two- and three-loop approximations and comparing them for integer d with some data from Table 3, one can see that they are underestimated passing from two-loop approximation to three-loop one with increase of m. In the region near $d = 3$ these changes reflect in the third digit after point and of course do not qualitatively change the picture of the phase transition. The value of ν is essentially underestimated near $d = 2$ (for $m = 2, m = 3$) and $d = 1$ (for $m = 1$), where it leads to the qualitatively incorrect answer about evidence of phase transition (positive and finite ν). One can see that within simple Padé-Borel resummation procedure in 3-loop approximation the values of critical exponents are obtained with sufficient reliability starting from $d = 1.0$ for $m = 0$, $d = 1.5$ for $m = 1$ and from $d = 2.5$ for $m = 2, 3, ..$

One of the ways to obtain more accurate results in three-loop approximation and to ensure their correct asymptotic in the region of small d is to improve the resummation procedure This problem will be considered in Sec.4.3.

4.2 Influence of two- and three-loop integrals

Before improving the resummation procedure under consideration let us discuss here one problem more, which appears in the calculations under consideration. Namely let us find the answer for the question: what accuracy in determination of the loop integrals is needed in order to ensure the required accuracy in determination of the critical exponents? The point is that determination of the values of loop integrals for concrete value of space dimension d is based on their representation as multidimensional integrals. Their numerical evaluation is to be set up by means of successive integration over each *variable*. In this case the computer running time is the major factor in limiting accuracy. Let us note as well, that due to the form of the functions to be integrated, running time for calculation of integrals at low dimensions is essentially shorter if compared with the time necessary for calculation of the same integral with the same accuracy at high d. So the problem of sufficient accuracy in determination of the value of loop integrals especially in the region close to $d = 4$ appears.

So let us study the influence of the "input" data (i.e. changes in numerical values of loop integrals for fixed d) on the "output" one (changes in the values of fixed points and critical exponents). This programme can be fulfilled in different ways. The way chosen here is as follows [33]. Let us denote the physical value μ which is calculated on the base of loop integrals which are accurate, say, to l digits as $\mu^{(l)}$. The same value calculated on the base of loop integrals accurate to $(l-1)$ digits will be denoted as $\mu^{(l-1)}$, correspondingly. Then function $\delta\mu = \mid \mu^{(l-1)} - \mu^{(l)} \mid$ will show how the changes in the lth digit of the loop integral do influence on the value of μ. In what follows below we'll be interested in the influence caused on the value of fixed point u^* and correlation length critical exponent ν.

On Fig.6 we have plotted changes in the value of fixed point u^* (calculated by formula (4.4) for $m = 1$) caused by changes in the values of loop integrals $i_1 - i_8$. Calculations of u^* as a function of d were performed with a step $\delta d = 0.1$ (dotes on the plot). Values of u^* calculated on the base of loop integrals accurate to 5 digits (and denoted $u^{*(5)}$) were taken to be the reference one. Curve 4 represents dependence on d of the function $\mid u^{*(4)} - u^{*(5)} \mid$, curve 3 is the function $\mid u^{*(3)} - u^{*(5)} \mid$ and so on. Dashed lines are the exponential approximations of the corresponding curves. The result of performing the same procedure in the case of critical exponent $\nu(m = 1)$ within the same resummation procedure is plotted on Fig.7. Graphs 6,7 gives one the answer about accuracy of determination of loop integrals which ensures the necessary accuracy in determination of u^* or ν, correspondingly.

One more interesting feature appears if one compares the plots for δu^* and $\delta\nu$. In the case of critical exponent ν function $\delta\nu$ decays with increase of d. Such a strong decay does not take place for the changes in the fixed point value δu^*. It means that one can evaluate loop integrals with lower accuracy for high d in order to ensure necessary accuracy in critical exponents. This fact essentially simplifies the problem of calculation of numerical values of loop integrals especially in the region $d > 3.8$.

The same behaviour of $\delta\nu$ and δu^* is observed for different m and we do not represent the corresponding graphs here.

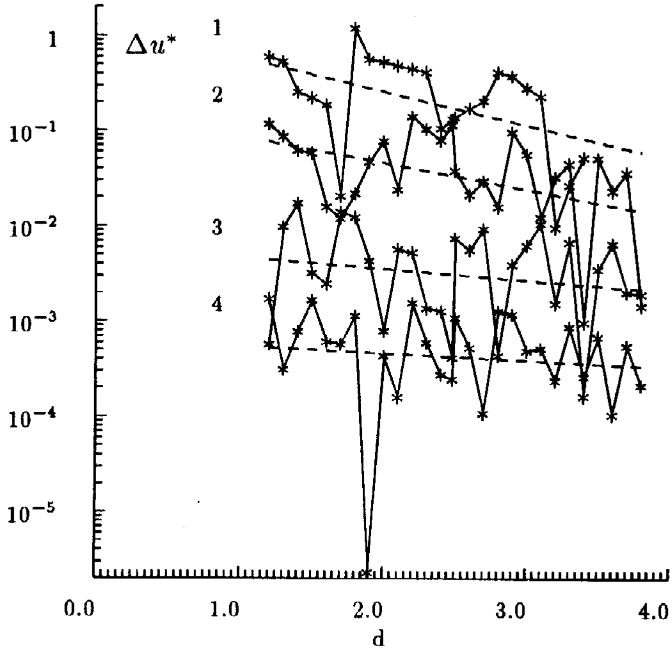

Figure 6: Changes in the value of the fixed point u^* for different accuracy of loop integrals versus d. $m = 1$. Three-loop approximation. Loop integrals are accurate to: 1 - 10^{-1}, 2 - 10^{-2}, 3 - 10^{-3}, 4 – 10^{-4}.

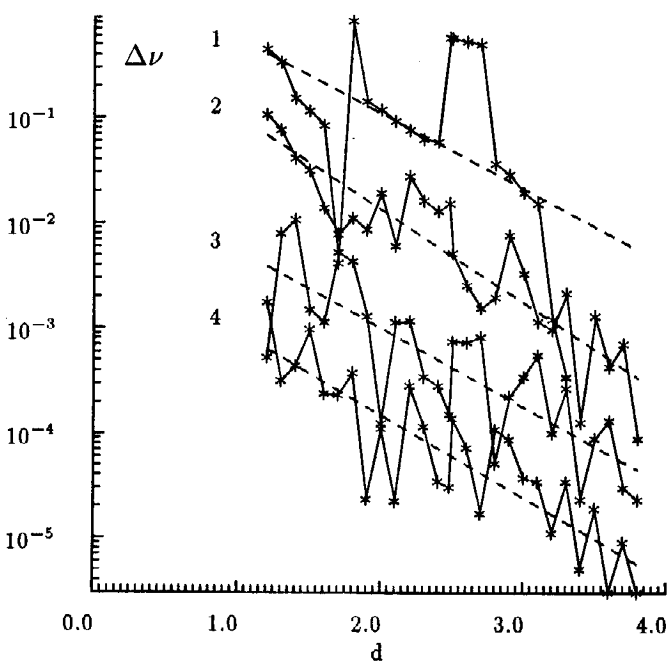

Figure 7: Changes in the value of the critical exponent ν for different accuracy of loop integrals versus d. $m = 1$. Three-loop approximation. Loop integrals are accurate to: 1 - 10^{-1}, 2 - 10^{-2}, 3 - 10^{-3}, 4 - 10^{-4}.

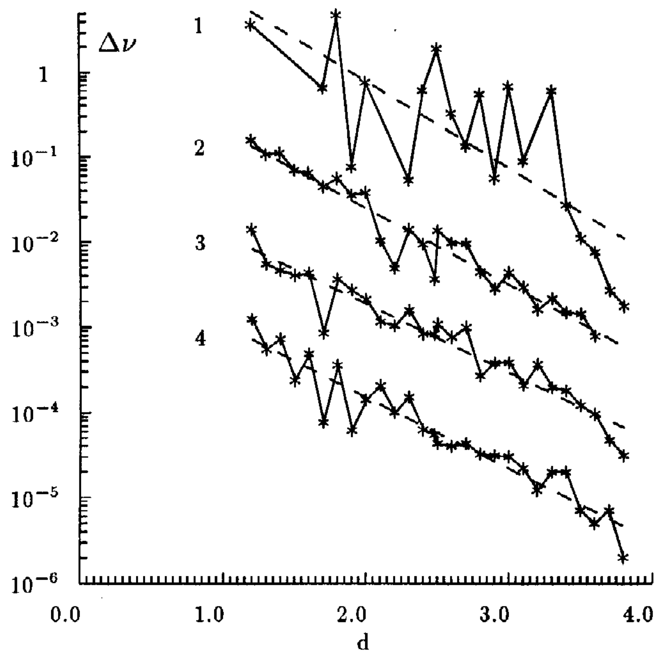

Figure 8: Changes in the values of critical exponent ν for different accuracy of loop integrals versus d. $m = 1$. Three-loop integrals are accurate to: 1 - 10^{-1}, 2 - 10^{-2}, 3 - 10^{-3}, 4 - 10^{-4}. Values of the two-loop integrals are fixed.

If one fixes the values of two-loop integrals and is interested in the influence, caused by three-loop ones, it appears that corresponding curves are of the same type (see Fig.8 for example). But the changes caused by three-loop integrals are slightly lower, then those caused by two-loop integrals. This is manifested in lower amplitude of deviation of curves for δu^* (or $\delta\nu$) from their exponential approximations (dashed curves).

The ways of resummation considered here do not make any essential influence on the behaviour described above. On Figs. 9-10 we have shown the changes in fixed point coordinate (Fig. 9) and in critical exponent ν (Fig.10) caused by changes in loop integrals when the resummation was performed on the base of formula (4.6).

4.3 Estimates of the critical exponents

a) The Ising Model.

In order to improve preliminary results obtained in Sec. 4.1 let us impose the expressions (2.10), (2.11) to yield the values of the critical exponents of the Ising model for the cases, where results are known exactly, i.e. for $d = 1$ ($\nu = \gamma = \infty$, $\eta = 1$) and for $d = 2$ ($\nu = 1$, $\gamma = 1.75$, $\eta = 0.25$). The simplest possible way to do this is to choose the highest-order term of the series under consideration as a free parameter and to move it in order to obtain the results requested. The corresponding value of the higher-order term for the β-function can be found from the condition of existence of

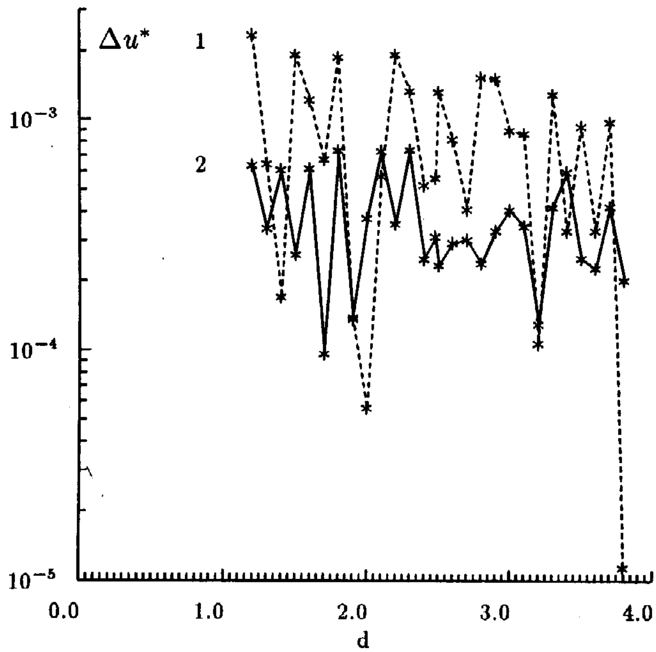

Figure 9: Changes in the value of the fixed point u^* for different accuracy of loop integrals versus d. $m = 1$. Three-loop approximation. 1 - change in the 5th digit of two-loop integrals, 2 - the same change in three-loop integrals.

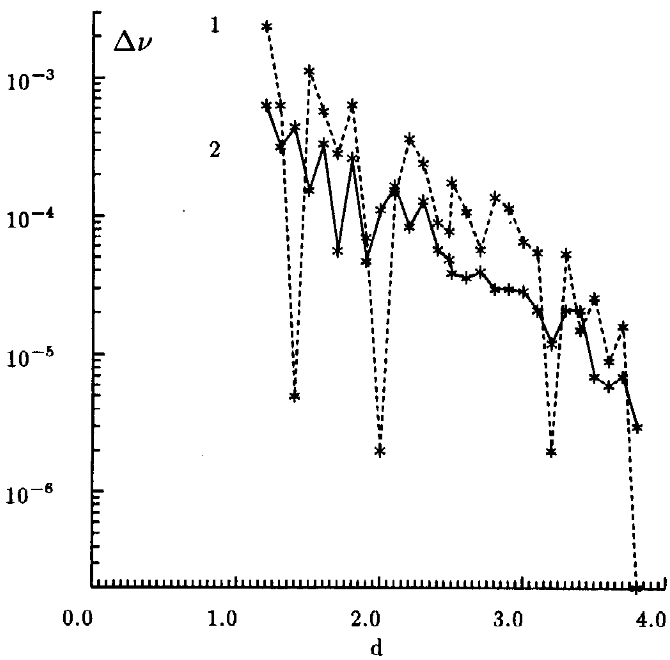

Figure 10: Changes in the value of of the critical exponent ν for different accuracy of loop integrals versus d. $m = 1$. Three-loop approximation. 1 - change in the 5th digit of two-loop integrals, 2 - the same change in three-loop integrals.

d	u^*	ν	η
1.0	-	∞	1.
1.2	3.6823	2.982	.777
1.4	2.9311	1.898	.595
1.6	2.5165	1.432	.451
1.8	2.2375	1.169	.338
2.0	2.0299	1.000	.250
2.2	1.8656	.882	.181
2.4	1.7295	.794	.126
2.6	1.6128	.727	.085
2.8	1.5097	.673	.052
3.0	1.4161	.630	.030
3.2	1.3291	.594	.013
3.4	1.2461	.564	.004
3.6	1.1651	.540	.000
3.8	1.0839	.518	.000

Table 4. The critical exponents ν, η and the stable fixed point coordinate u^* of the Ising model obtained from the fixed- dimension renormalization group approach [45] are tabulated as the functions of d.

infra-red fixed point u^* for $d > 1$ and its absence for $d < 1$, i.e. extremum of the resummed β-function β^R being at the point of tangency to u-axis.

In the case of ϵ-expansion such a procedure was considered in [44]. Let us note, however one essential difference between the application of such a scheme in the frames of ϵ-expansion and in the frames of fixed-dimension renormalization group approach. Dimension d being an expansion parameter in the ϵ-expansion techniques does not enter the coefficients of any series under consideration. So taking the highest-order term c_k as a free parameter in a series for some function $f(\epsilon)$ (for any critical exponent) $f(\epsilon)\ O(c_k\epsilon^k)$, one ensures the coincidence of results only for one chosen value of dimension d (in [44] it is $d = 2$). In spite of this in the scheme which is based on the application of renormalization group equations at fixed space dimension, coefficients of the series over coupling constant are dimensionally-dependent and the term of the series being of the form $c_k(d)u^k$. This allow one to impose the series under consideration to yield the requested/exact information for several values of d(by choosing appropriate values of $c_k(d)$ for integer $d = 1, 2, .., $ for instance). Then for the intermediate d the highest order term $c_k(d)$ can be approximated by some fair curve. Of course the reliability of such approximation to a great extend depends on the fact how smooth is the behaviour of the function we are approximating.

Table 4 gives our results for the fixed point coordinate u^*, correlation length critical exponent ν and pair correlation function critical exponent η between lower and upper critical dimension $1 < d \leq 4$. These results were obtained in two-loop approximation involving additional parameters into the resummation procedure [45]

In Table 5 we compare our results obtained for some values of d with those, obtained by the other methods: [5] - from the resummed ϵ-expansion, [21] - by a variational method derived from high-temperature series expansion (the error bounds are chosen to include all published results), [23] - from the application of finite-size scaling methods to numerical transfer-matrix data.

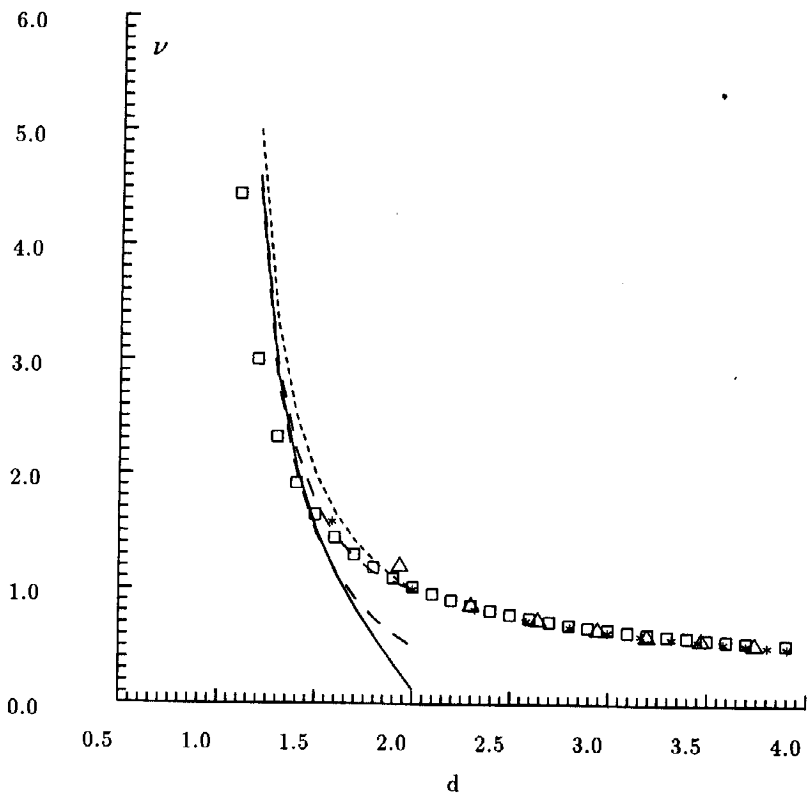

Figure 11: The Ising model correlation length critical exponent ν as a function of dimension d. Squares show our results, asterisks show data of [19], triangles show data of [29,30]. The lines are the results of the $\epsilon' = d - 1$ expansion for the near planar interface model to the orders of one loop (dotted line), two loops (dot-dash line), three loops (dashed line) and four loops (solid line) [12-14].

			From [5]	
d	ν	η	ν	η
1.250	2.593	.728	1.5-4.5	.30-1.00
1.375	1.983	.616	1.6-2.6	.30-0.80
1.500	1.627	.519	1.45-1.85	.35-0.65
1.650	1.353	.420	1.30-1.44	.30-0.50
1.750	1.223	.363	1.20-1.26	.30-0.40
1.875	1.098	.303	1.09-1.11	.27-0.33
2.000	1.000	.250	1.0	.25

	From [6]		From [22,23]	
d	ν	η	ν	η
1.250	1.7-3.4	.758-.83	3.6832-3.7750	.7532-.7544
1.375	1.6-2.9	.635-.679	2.3535-2.3747	.6362-.6398
1.500	1.49-1.84	.567-.631	1.7547-1.7596	.5312-.5328
1.650	1.27-1.38	.450-.507	1.3780-1.3795	.4226-.4238
1.750	1.18-1.26	.396-.425	1.2234-1.2255	.3630-.3634
1.875	1.11-1.13	.312-.322	1.0902-1.0906	.3006-.3010
2.000		.247-.253		

Table 5. The critical exponents of the Ising model ν, η obtained from the fixed-dimension renormalization group approach (columns 2,3) are compared with the data, obtained from the other approaches. See the text for a full description.

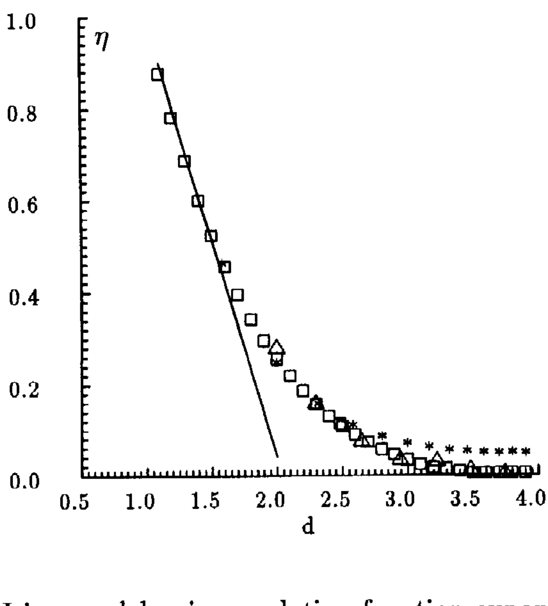

Figure 12: The Ising model pair correlation function exponent η as a function of dimension d. Squares show our results, asterisks show data of [19], triangles show data of [29,30]. The solid line is obtained from the result for the droplet model [15].

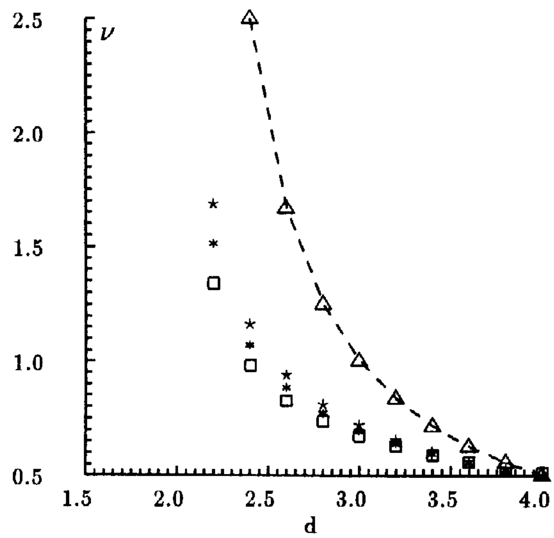

Figure 13: The $O(m)$-symmetric model correlation length critical exponent ν as a function of dimension d. Squares correspond to $m = 2$, asterisks correspond to $m = 3$, stars correspond to $m = 4$. In the case $m = \infty$ we re-reproduce the exact result for the spherical model (dashed curve) [41].

On Fig.11 we compare our results for critical exponent ν with $\epsilon' = d - 1$ expansion results for the near-planar interface model [2] (region of low values of d) and with data, obtained on the base of Kadanoff lower-bond renormalization transformation [19] and by study of the physical branch of the exact renormalization group equation solution [29,30]. Value of the critical exponent ν for the near-planar interface model:

$$\nu = \epsilon'^{-1} - \frac{1}{2} + \frac{\epsilon'}{2} - \frac{7\epsilon'^2}{8}$$

is shown in the first, second, third and fourth leading orders [12-14].

Fig.12 gives the comparison of our data for critical exponent η with the results, obtained from Kadanoff lower-bond renormalization transformation [19], from the exact renormalization group equation [29,30], and with the value of η for droplet model, expressed by the formula [15]:

$$d + \eta - 2 = 8\pi^{-1}\epsilon'^{-(2+\epsilon')/2}[1 + O(\epsilon')]\exp\{-1 - 2C - 2/\epsilon'\},$$

$C \simeq 0.577$ being Euler's constant.

b) $O(m)$-Symmetric Model.

As it was mentioned above in contrary to the Ising model $O(m)$-symmetric model up to now was not intensively studied in the case of arbitrary

[2]Strong arguments have been given about the correspondence of the critical behaviour of this model to that of the Ising model [12-14].

	m = 2			m = 3			m = 4		
d	u^*	ν	γ	u^*	ν	γ	u^*	ν	γ
2.0	-	∞	∞	-	∞	∞	-	∞	∞
2.2	2.0296	1.329	2.540	1.9895	1.512	2.884	1.9464	1.687	3.219
2.4	1.7907	.971	1.878	1.7604	1.072	2.072	1.7275	1.166	2.253
2.6	1.6285	.816	1.593	1.6043	.882	1.721	1.5779	.941	1.837
2.8	1.5041	.725	1.425	1.4841	.770	1.513	1.4623	.810	1.592
3.0	1.4024	.662	1.310	1.3854	.694	1.371	1.3671	.721	1.426
3.2	1.3349	.618	1.226	1.3210	.638	1.267	1.3061	.656	1.303
3.4	1.2644	.580	1.156	1.2534	.593	1.182	1.2419	.604	1.204
3.6	1.1888	.549	1.095	1.1808	.556	1.110	1.1726	.562	1.123
3.8	1.1069	.522	1.044	1.1022	.525	1.050	1.0975	.528	1.056

Table 6. The critical exponents ν, γ and fixed point coordinate u^* of $O(m)$-symmetric model obtained from the fixed-dimension renormalization group approach [48] are tabulated as the functions of d for m = 2, 3, 4.

space dimension d. One more reason for considering the case $m > 1$ is that in accordance with the Mermin-Wagner-Hohenberg theorem [42,43][3] continuous symmetry can be spontaneously broken only if the space dimension is greater then two[4], whereas for Ising model ($m = 1$) lower critical dimension is $d = 1$. So by means of continuous change of the space dimension d one can try to run down the mechanism of phase transition disappearance near lower critical dimension.

Starting from the expressions for β- and γ-functions (2.9)-(2.11) in the case of $m = 2,3,4$ and performing the resummation procedure analogous to that, applied to the Ising model in the previous section (now the additional parameters in the resummation procedure were chosen from the assumption that the lower critical dimension is two) one can obtain the information about the dimensional dependence of the critical exponents. On Fig.13 we plot the values of correlation length critical exponent ν as a function of d for $m = 2,3,4,\infty$[48]. Again in the case $m = \infty$ we re-reproduce the exact result for the spherical model [41]. Table 6 contains the three- loop results for the stable fixed point value u^* and the critical exponents ν and γ. Let us note that while in [28] the singular behaviour of critical exponents in the point $d = 2$ was found, one of the results of [29,30] show that critical exponents in this point still remain analytic. Our results, involving the information about the location of singularity at $d = 2$ are in accordance with [28] in the case of d close to $d = 2$ and for $d \simeq 3$ agree with those obtained in [29,30].

5 Conclusions

In this paper we have briefly described the method for study of the critical behaviour in non-integer space dimension. Following the idea of Parisi [31] to perform the calculations directly in 2 and 3 dimensions we proposed to consider the renormalization group functions directly at the arbitrary non-integer d. The main results are as follows.

[3]Let us mention here the generalization of the Mermin-Wagner- Hohenberg to the case of fractals of continuous symmetry [46].

[4]Exact solutions for $2d$ classical Heisenberg model for $m = 4$ and $m = 3$ also demonstrate the absence of magnetic ordering (see [47] and references therein).

1. In three-loop approximation we have obtained β- and γ -functions directly for arbitrary value of d. Being asymptotic such series are to be resummed in order to investigate the critical behaviour of the model under consideration.

2. In order to obtain the values of two- and three-loop integrals $i_1 - i_8$ entering the series for β- and γ-functions one can use the integral representations given in the Appendix. The tables of numerical values of loop integrals are given elsewhere [49].

3.Simple Padé-Borel resummation technique was chosen to extract the main features of the critical behaviour of $O(m)$-symmetric model. Two ways of representation of β-function were used. In two-loop approximation for the dimension of space $d > 3.4$ these two representations lead practically to the same values of the critical exponents, whereas in three-loop approximation such a coincidence takes place already starting from $d = 3$. The discrepancies in the fixed point coordinates calculated in such schemes are stronger.

4. Influence of the accuracy of determination of loop integrals on the fixed points and critical exponents values was studied. It appears that in frames of considered here resummation procedure influence caused by changes in loop integrals values for great d is rather more weaker as for small values of d. This enables one to take loop integrals with different accuracy in different regions of space dimension d ensuring necessary accuracy in the critical exponents values.

5. Critical behaviour of the Ising and $O(m)$-symmetric models was studied on the base of improved resummation procedure. The results are presented in Tables 4-5 and on Figs.11-13. One of the advantages of our approach is that it allows one to cover all the region of d between lower and upper critical dimensions $1 < d \leq 4$ in the case of the Ising model and $2 < d \leq 4$ for the model with continuous symmetry.

6. It should be noted that the most of papers about study of critical behaviour at non-integer d contain analytic continuation in terms of dimension d which is of purely formal character (this does not concern, of course, the critical phenomena on fractal lattices). In particular our approach (as well as the ϵ-expansion technique) involves formal analytic continuation of momentum integrals, e.g. $\int d^dk \Rightarrow \int k^{d-1}dk$. It has been questioned even whether performing such a continuation one can describe a ferromagnetic phase transition in non-integer d [40]. Thus, a good agreement of our results with those obtained by other methods (in the regions of d, where these methods can be applied) give one more indirect evidence of mutual correspondence between "space dimensions" which appear in different schemes.

7. The method considered in this paper as well as the resummation scheme can be applied to other lattice models in non-integer dimensions, such as model with quenched randomness, q-state Potts model. It would be interesting to repeat our calculations in the above mentioned context.

Acknowledgement

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Appendix

In this appendix we'll write the integral representations for the loop integrals, denoted in the article as $i_1 - i_8$. Numerical values of these integrals are tabulated elsewhere [49] for non-integer value of space dimension d for

$0 \le d \le 4$ with a step $\delta d = 0.1$. Combining Feynman parameterization and direct integration one can find the following representations for the integrals $i_1 - i_8$:

$$i_1 = \int_0^\infty \frac{q^{d-1}}{(q^2+1)^2} f_1(q) dq, \tag{A.1}$$

$$i_2 = -\gamma \int_0^\infty dq \frac{q^{d-1}}{(q^2+1)^3}\left[q^2 \frac{d-4}{d} + 1\right] f_1(q), \tag{A.2}$$

$$i_3 = \frac{4-d}{4}\gamma \int_0^\infty dq q^{d-1} f_1(q) \left[\frac{6-d}{2} f_4(q) - \frac{1}{q^2+1}\right], \tag{A.3}$$

$$i_4 = \frac{4-d}{4}\gamma \int_0^\infty dq \frac{q^{d-1}}{(q^2+1)} f_1(q) f_2(q), \tag{A.4}$$

$$i_5 = \gamma \int_0^\infty dq \frac{q^{d-1}}{(q^2+1)^2} f_1^2(q), \tag{A.5}$$

$$i_6 = \frac{(6-d)(4-d)}{4}\gamma \int_0^\infty dq q^{d-1} f_1(q) f_3(q) \tag{A.6}$$

$$\begin{aligned} i_7 = {} & \frac{\Gamma(5-d)\gamma}{[\Gamma(2-d/2)]^2} \int_0^\infty dq \frac{q^{d-1}}{q^2+1} \int_0^1 du u^{2-d/2}(1-u) \times \\ & \times \int_0^1 dy (1-y) \int_0^1 dx \int_0^1 dv \, [u(1-y)y + 1 - u]^{5-3d/2} \times \\ & \times \left[\delta q^2 + u(1-y)y + 1 - u\right]^{d-5}, \end{aligned} \tag{A.7}$$

$$i_8 = -\gamma \int_0^\infty dq \frac{q^{d-1}}{(q^2+1)^3}\left[q^2 \frac{d-4}{d} + 1\right] f_1^2(q). \tag{A.8}$$

Here:

$$\begin{aligned} & \gamma = 2/[\Gamma(d/2)\Gamma(2-d/2)], \\ & f_1(q) = \int_0^1 dx [1 + q^2 x(1-x)]^{d/2-1}, \\ & f_2(q) = \int_0^1 dx x [1 + q^2 x(1-x)]^{d/2-3}, \\ & f_3(q) = \int_0^1 dx x(1-x) [1 + q^2 x(1-x)]^{d/2-4}, \\ & f_4(q) = \int_0^1 dx x^2 [1 + q^2 x(1-x)]^{d/2-4}, \\ & \delta = (1-x)(1-y)\left\{yu(1-u) + u(1-u)\left[x(1-y) - vy\right] + \right. \\ & \left. +(1-y)yxu^2\right\} + xyvu(1-y)(1-u) + v(1-u)^2(1-v). \end{aligned}$$

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