HYDRODYNAMIC THEORY OF A MAGNETIC LIQUID

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Abstract

The generalized hydrodynamic equations for a magnetic liquid model in an inhomogeneous external field using Zubarev's method of nonequilibrium statistical operator are obtained. In this model the "liquid" subsystem is treated as classical one and the "magnetic" subsystem is described by means of quantum mechanics methods. The properties of linearized hydrodynamical equations are analysed for a weak nonequilibrium case. The equations for the time correlation functions and forthe collective mode spectrum are derived. We stud also some limiting cases when the dynamic variables of one of subsystems can be neglected.

1 Introduction

An actual problem of the modern theory of liquids is the problem of microscopic description of dynamic properties for a system of particles possessing localized magnetic moments. These systems are also of interest in the theory of magnetism since liquid magnets could have ideal soft-magnetic properties due to their isotropy. Though the existence of ferromagnetism in equilibrium liquid state have been di ussed from experimental point of view, such a possibility is supposed to be proved in principle [1] and the experiments with $Au_{37}Co_{27}$ [2,3] confirm availability of short-range spin order in a fluid-system. Another aspect of the investigations in this field is related to the study of magnetic colloid suspensions [4,5], namely their general dynamic properties and the influence caused by the "magnetic" subsystem, especially of an external magnetic field. The question about the mutual influence of the "magnetic" subsystem and the "liquid" one for liquid metals (in particular transitional 3d-metals [6,7]) can be also studied on a base of the approach we propose here.

Until recently, theoretical description of magnetic liquid dynamics was based in large part on phenomenological approaches or on the Green-function method for some simple models . The analysis of the ferromagnetism existence in equilibrium liquid state was done for some simplified models in papers [6,8]. Using functional integration method, the investigation of a free energy, a "liquid" equation of state and spin wave spectrum for liquid and amorphous magnets model was carried out by Vakarchuk, Rudavskij and Ponedilok [9–11]. In their papers the influence of magnetic interactions on the structure of liquid was also studied. Subsequently, the theory was generalized to the case of two-component liquid magnets by Vakarchuk and

Margolych [12,13]. Some dynamic properties of liquid ferromagnets, namely high-frequency behavior and oscillation spectrum were considered on basis of phenomenological equations of motion by I.O.Akhiezer and I.T.Akhiezer [14,15]. One should also recall the papers [16–18] where the equilibrium statistical theory for description of the phase behaviour of ferro- (dipolar-) fluids was developed.

A physical system where the proposing approach can be applied is a ferrofluid, i.e. a suspension of small (about 100 Å) particles carrying permanent magnetic (dipolar) moments [19]. There exist the voluminous literature on this subject but it is important to note some papers that are close to the discussed problem. The equilibrium properties of a ferrofluid were studied by Kalikmanov [20] and Groh and Dietrich [18]. The investigation of rotational diffusion on basis of a generalized Smoluchovski equation was performed by Felderhof and Jones [21]. Some problems of transport phenomena for the diluted suspension of ferroparticles were considered by Rubi and Miguel [22].

To ascertain the fields of application of macroscopic dynamic equations and the relations between various theoretical approaches one has consequently to derive the exact hydrodynamic equations and equations for the time correlation functions proceeding from microscopic description and then to analyse them in detail. This paper is just dedicated to solve the problem for a model of monoatomic magnetic liquid.

In the present paper the generalized hydrodynamic equations for a magnetic liquid in external inhomogeneous magnetic field are obtained [29,30] by Zubarev's method of nonequilibrium statistical operator [25,26]. We consider as an example the model proposed in [8] and generalized later in [9]. For the case of small deviations from equilibrium state the generalized hydrodynamic equations are derived, and their general analysis is carried out. It is shown that in the limiting cases when the dynamic variables of one of subsystems (liquid or magnetic) could be neglected, the well-known hydrodynamic equations (magnetic hydrodynamics [23,24] or molecular hydrodynamics [27,28]) are reproduced. The equations for the time correlation functions and the collective mode spectrum of the system are obtained as well.

2 Description of the model

Let's consider a system of N magnetic particles with spin $\mathbf{S}_j$ in a liquid state in an external inhomogeneous magnetic field. Hamiltonian of such system can be written in the form

$$\hat{H}(t) = H_L + \hat{H}_S(t), \tag{2.1}$$

where

$$H_L = \sum_{f=1}^{N} \frac{\mathbf{p}_f^2}{2m} + \frac{1}{2}\sum_{f\neq l} \Phi(|\mathbf{r}_{fl}|) \tag{2.2}$$

is a classical part of the Hamiltonian describing the "liquid" subsystem as simple classical liquid. For subsequent calculations the potential $\Phi(|\mathbf{r}_{fl}|)$ is to be specified and can be chosen as a Lennard-Jones potential, soft or hard sphere one, etc. Hamiltonian of the "magnetic" subsystem can be considered either as classical or quantum one. Here, as pointed out above, we choose for $\hat{H}_S(t)$ the Hamiltonian of a quantum Heisenberg model. The general results which will be obtained here concern the other forms of spin interactions as well. In particular, the spin interaction may be a dipolar one (Stockmayer model) or, for example, the Hamiltonian of "magnetic" subsystem may be chosen as Ising-like one, etc. For the Heisenberg model of spin interaction we have

$$\hat{H}_S(t) = -\frac{1}{2}\sum_{f\neq l} J(|\mathbf{r}_{fl}|)\mathbf{S}_f\mathbf{S}_l - \int d\mathbf{r}\ \hat{\mathbf{m}}(\mathbf{r})\mathbf{B}(\mathbf{r},t), \tag{2.3}$$

where $\mathbf{B}(\mathbf{r},t)$ is an external inhomogeneous magnetic field. It is assumed that field $\mathbf{B}(\mathbf{r},t)$ may vary in time but rather slowly, and the typical time of its variation is much larger than the other relaxation times of the system. The second term in the right-hand part of (2.3) describes interaction of spins with an external magnetic field $\mathbf{B}(\mathbf{r},t)$ where $\hat{\mathbf{m}}(\mathbf{r})$ is the density of magnetic moment

$$\hat{\mathbf{m}}(\mathbf{r}) = \sum_{f=1}^{N} \mathbf{S}_f\delta(\mathbf{r}-\mathbf{r}_f). \tag{2.4}$$

The Liouville operator of the system can be written as follows

$$i\hat{L} = iL^L + i\hat{L}^S(t), \tag{2.5}$$

where

$$iL^L = \sum_{f=1}^{N}\frac{\mathbf{p}_f}{m}\frac{\partial}{\partial\mathbf{r}_f} - \frac{1}{2}\sum_{f\neq l}\frac{\partial}{\partial\mathbf{r}_f}\{\Phi(r_{fl}) - J(r_{fl})\mathbf{S}_f\mathbf{S}_l\}\times$$
$$\times\left(\frac{\partial}{\partial\mathbf{p}_f} - \frac{\partial}{\partial\mathbf{p}_l}\right) + \sum_{f=1}^{N}\frac{\partial}{\partial\mathbf{r}_f}(\mathbf{B}(\mathbf{r}_f;t)\mathbf{S}_f)\frac{\partial}{\partial\mathbf{p}_f}$$

is a "liquid" part. In the second and third terms ofthe operator iL^L we have the contributions from a spin Hamiltonian, and $iL^S(t)$ is a "spin" part of the Liouville operator

$$i\hat{L}^S(t)\mathcal{A} = \frac{i}{\hbar}[\hat{H}_S(t),\mathcal{A}] = \frac{i}{\hbar}[\hat{H}_S(t)\mathcal{A} - \mathcal{A}\ \hat{H}_S(t)].$$

However, it should be stressed that for actual calculations it is often more convenient to consider a "liquid" subsystem as quantum one and to perform the classical limit only afterwards.

Among variables corresponding to an abbreviate description of hydrodynamic stage, the conservative quantities have to be considered, namely, the densities of particles' number $\hat{n}(\mathbf{r})$, momentum $\hat{\mathbf{p}}(\mathbf{r})$, total energy $\hat{\varepsilon}(\mathbf{r})$, and magnetic moment $\hat{\mathbf{m}}(\mathbf{r})$, where

$$\hat{n}(\mathbf{r}) = \sum_{f=1}^{N}\delta(\mathbf{r}-\mathbf{r}_f), \tag{2.6}$$

$$\hat{\mathbf{p}}(\mathbf{r}) = \sum_{f=1}^{N}\mathbf{p}_f\delta(\mathbf{r}-\mathbf{r}_f), \tag{2.7}$$

$$\hat{\varepsilon}(\mathbf{r}) = \sum_{f=1}^{N}\{\frac{p_f^2}{2m} + \frac{1}{2}\sum_{l(l\neq f)}^{N}[\Phi(r_{fl}) - J(r_{fl})\mathbf{S}_f\mathbf{S}_l]\}\delta(\mathbf{r}-\mathbf{r}_f), \tag{2.8}$$

and $\hat{\mathbf{m}}(\mathbf{r})$ is defined above. The Fourier transforms of dynamic variables can be found using the definition

$$\mathcal{A}(\mathbf{k}) = \int d\mathbf{r}\, \mathcal{A}(\mathbf{r}) \exp(i\mathbf{k}\mathbf{r}).$$

The set of so-called orthogonal dynamic variables $\{\hat{P}_\alpha\}$ with properties

$$(\hat{P}_\alpha, \hat{P}_\beta) = \delta_{\alpha\beta}(\hat{P}_\alpha, \hat{P}_\alpha)$$

can be introduced by a linear transformation. Here

$$(A, B) = \int_0^1 d\tau \langle A, \rho_0^\tau B \rho_0^{-\tau}\rangle_0 \tag{2.9}$$

is an equilibrium correlation function, and $\langle\ldots\rangle_0$ means averaging with equilibrium statistical operator ρ_0. One can realize the transition to such set of dynamic variables by Schmidt's orthogonalization procedure to initial operators with the definition for the scalar product in the form (2.9). As the results, the orthogonal set of dynamic variables $\hat{P}_\alpha = \{\hat{\aleph}, \hat{\mathbf{p}}, \hat{h}, \hat{\mathbf{m}}\}$ consists of the variables

$$\hat{\aleph}(k) = (1 - \mathcal{P}_m)\hat{n}(\mathbf{k}), \tag{2.10}$$

$$\hat{h}(k) = (1 - \mathcal{P}_\aleph)(1 - \mathcal{P}_m)\hat{\varepsilon}(\mathbf{k}), \tag{2.11}$$

where

$$\mathcal{P}_m \ldots = \sum_{\mathbf{k}}(\ldots, \hat{\mathbf{m}}(-\mathbf{k}))_0\, (\hat{\mathbf{m}}(\mathbf{k}), \hat{\mathbf{m}}(-\mathbf{k}))_0^{-1}\, \hat{\mathbf{m}}(\mathbf{k}) \tag{2.12}$$

and

$$\mathcal{P}_\aleph \ldots = \sum_{\mathbf{k}}(\ldots, \hat{\aleph}(-\mathbf{k}))_0\, (\hat{\aleph}(\mathbf{k}), \hat{\aleph}(-\mathbf{k}))_0^{-1}\, \hat{\aleph}(\mathbf{k}) \tag{2.13}$$

are the projection operators. For obtaining the last expressions we used the equalities $(\hat{\mathbf{p}}(\mathbf{k}), \hat{\alpha}(-\mathbf{k}))_0 = 0$ which are valid in the case of classic treatment of the "liquid" subsystem if $\hat{\alpha} = \{\hat{n}, \hat{\varepsilon}, \hat{\mathbf{m}}\}$.

Now, the matrix of static correlation functions $\tilde{\Phi}(\mathbf{k}) = \| \tilde{\Phi}_{\alpha\beta}(\mathbf{k}) \|$ constructed with the use of a vector-column $\{\hat{P}_\alpha\} = \{\hat{\aleph}, \hat{\mathbf{p}}, \hat{h}, \hat{\mathbf{m}}\}$ has the diagonal form

$$\tilde{\Phi}_{\alpha\beta}(\mathbf{k}) = (\hat{P}(\mathbf{k}), \hat{P}^{+}(-\mathbf{k}))_{\alpha\beta} = \delta_{\alpha\beta}(\hat{P}_\alpha(\mathbf{k}), \hat{P}_\alpha(-\mathbf{k})). \tag{2.14}$$

Here $\hat{P}^{+}(\mathbf{k})$ is a vector-line. The dependence of dynamic quantities on wave vector $\mathbf{k}$ will be further omitted and only when necessary it will be indicated obviously.

3 Nonequilibrium statistical operator

Nonequilibrium state of a system is described by the nonequilibrium statistical operator $\rho(\mathbf{x}^N;t)$ which is a solution of Liouville equation

$$\frac{\partial}{\partial t}\rho(\mathbf{x}^N;t) + i\hat{L}\rho(\mathbf{x}^N;t) = 0, \tag{3.1}$$

where $i\hat{L}$ is the Liouville operator, and $\mathbf{x}^N = \{\mathbf{p},\mathbf{r},\mathbf{S}\}^N$.

Nonequilibrium statistical operator $\rho(\mathbf{x}^N;t)$ is normalized to unity

$$\mathrm{Sp}\ \rho(\mathbf{x}^N;t) = 1, \tag{3.2}$$

where

$$\mathrm{Sp}\ (\ldots) = \int\ldots\int \frac{(d\mathbf{r}d\mathbf{p})^N}{N!(2\pi\hbar)^{3N}}\ Sp_{\{S_1,\ldots,S_N\}}(\ldots).$$

To determine the nonequilibrium statistical operator $\rho(\mathbf{x}^N;t)$ from the Liouville equation (3.1) one has to pose the boundary condition corresponding to physics of the system under consideration. Let us suppose the nonequilibrium statistical operator $\rho(\mathbf{x}^N;t)$ is equal to quasi-equilibrium statistical operator $\rho_q(\mathbf{x}^N;t)$ at initial moment of time $t=t_0$, i.e.

$$\rho(\mathbf{x}^N;t)|_{t=t_0} = \rho_q(\mathbf{x}^N;t_0). \tag{3.3}$$

The initial time t_0 tends to minus infinity at final stage of the calculation, the thermodynamic limit having been performed. For a construction of $\rho_q(\mathbf{x}^N;t)$ we have to restrict ourselves a priori to a set of physical quantities determining an abbreviate description of the nonequilibrium state. Using Zubarev's method [25,26] of nonequilibrium statistical operator, the retarded solutions of the Liouville equation (3.1) with the boundary condition (3.3) can be found. The same result can be obtained from the Liouville equation with an infinitesimal source ($\epsilon \to +0$) in the right-hand side

$$\frac{\partial}{\partial t}\rho(\mathbf{x}^N;t) + i\hat{L}\rho(\mathbf{x}^N;t) = -\epsilon(\rho(\mathbf{x}^N;t) - \rho_q(\mathbf{x}^N;t)), \tag{3.4}$$

which destroys the symmetry of the Liouville equation with respect to time inversion $t \to -t$.

The quasi-equilibrium statistical operator $\rho_q(\mathbf{x}^N;t)$ can be found from the condition of the informational entropy extremum under the additional conditions that the mean values of dynamic variables are fixed and with preservation of the normalization

$$\mathrm{Sp}\ \rho_q(\mathbf{x}^N;t) = 1. \tag{3.5}$$

At our case this means that the average values of densities of particles' number $\langle\hat{n}(\mathbf{r})\rangle^t$, momentum $\langle\hat{\mathbf{p}}(\mathbf{r})\rangle^t$, total energy $\langle\hat{\varepsilon}(\mathbf{r})\rangle^t$ and magnetic moment $\langle\hat{\mathbf{m}}(\mathbf{r})\rangle^t$ or the corresponding quantities for the orthogonal dynamic variables $\hat{\aleph}$, $\hat{\mathbf{p}}$, $\hat{h}$, and $\hat{\mathbf{m}}$ are fixed. As a result, for the quasi-equilibrium statistical operator $\rho_q(\mathbf{x}^N;t)$ we obtain the expression

$$\rho_q(\mathbf{x}^N;t) = \exp\{-\Phi(t) - \sum_\alpha \hat{P}_\alpha F_\alpha(t)\} = \exp\{-S(t)\}, \tag{3.6}$$

where

$$\Phi(t) = \ln \text{ Sp } \exp\{-\sum_{\alpha} \hat{P}_\alpha F_\alpha(t)\} \tag{3.7}$$

is so-called the Massieu-Planck functional. The set of intensive quantities $\{F_\alpha(t)\}$ can be determined from self-consistency conditions

$$\langle \hat{P}_\alpha \rangle^t = \langle \hat{P}_\alpha \rangle_q^t = \text{Sp } \hat{P}_\alpha \rho_q(\mathbf{x}^N; t). \tag{3.8}$$

The quantities $\{F_\alpha(t)\}$ are connected [29,30] with a local chemical potential, a local temperature, a mean hydrodynamic velocity and an internal magnetic field by the linear transformation providing transition from the initial set of dynamic variables $\{\hat{n}, \hat{\mathbf{p}}, \hat{\varepsilon}, \hat{\mathbf{m}}\}$ to the set of orthogonal dynamic variables $\{\hat{\aleph}, \hat{\mathbf{p}}, \hat{h}, \hat{\mathbf{m}}\}$. In the all above and next expressions it is supposed that integration over coordinate $\mathbf{r}$ (in the case of the space representation of dynamic variables) or the summation over wave-vector $\mathbf{k}$ (in the case of Fourier transforms of dynamic variables) are performed together with the summation over α.

The equation (3.4) can be rewritten in the form

$$\{\frac{\partial}{\partial t} + i\hat{L} + \epsilon\}\Delta\rho(\mathbf{x}^N; t) = -\{\frac{\partial}{\partial t} + i\hat{L}\}\ \rho_q(\mathbf{x}^N; t), \tag{3.9}$$

where

$$\Delta\rho(\mathbf{x}^N; t) = \rho(\mathbf{x}^N; t) - \rho_q(\mathbf{x}^N; t).$$

Since the operator $\rho_q(\mathbf{x}^N; t)$ depends on time only via $F_\alpha(t)$ (or $\langle \hat{P}_\alpha \rangle^t$), one can introduce the projection operator $\mathcal{P}_q(t)$ according the definition

$$\frac{\partial}{\partial t}\rho_q(\mathbf{x}^N; t) = -\mathcal{P}_q(t)\ i\hat{L}\rho(\mathbf{x}^N; t). \tag{3.10}$$

The operator $\mathcal{P}_q(t)$ is known as the Kawasaki-Guntion projection operator and acts only on the statistical operators. For $\rho_q(\mathbf{x}^N; t)$ in the form (3.6), $\mathcal{P}_q(t)$ has the following structure

$$\mathcal{P}_q(t)(\ldots) = \{\rho_q(\mathbf{x}^N; t) - \sum_{\alpha} \frac{\delta\rho_q(t)}{\delta\langle \hat{P}_\alpha \rangle^t}\langle \hat{P}_\alpha \rangle^t\}\text{Sp } (\ldots) + \\ + \sum_{\alpha} \frac{\delta\rho_q(t)}{\delta\langle \hat{P}_\alpha \rangle^t}\text{Sp } \hat{P}_\alpha(\ldots) \tag{3.11}$$

and possesses the following properties

$$\mathcal{P}_q(t)\rho(t) = \rho_q(t), \qquad \mathcal{P}_q(t)\rho_q(t) = \rho_q(t).$$

With consideration of (3.10) the Liouville equation takes the form

$$\{\frac{\partial}{\partial t} + (1 - \mathcal{P}_q(t))i\hat{L} + \epsilon\}\ \Delta\rho(\mathbf{x}^N; t) = -(1 - \mathcal{P}_q(t))i\hat{L}\rho_q(\mathbf{x}^N; t). \tag{3.12}$$

Hence, a formal solution for the nonequilibrium statistical operator is

$$\rho(\mathbf{x}^N; t) = \rho_q(\mathbf{x}^N; t) - \\ - \int_{-\infty}^{t} e^{\epsilon(t-t')}T(t,t')(1 - \mathcal{P}_q(t'))i\hat{L}(t')\rho_q(\mathbf{x}^N; t')dt', \tag{3.13}$$

where

$$T(t,t') = exp_+\{-\int_{t'}^{t} d\tau(1-\mathcal{P}_q(\tau))i\hat{L}(\tau)\} \tag{3.14}$$

is a generalized operator of time evolution with regard to projecting. Let us consider in (3.12) the action of $\mathcal{P}_q$ and $i\hat{L}$ operators on the quasi-equilibrium statistical operator $\rho_q(\mathbf{x}^N;t)$. Taking into account the structure of the Liouville operator (2.5), the result of its action can be written in the form

$$\mathcal{P}_q(t)i\hat{L}\rho_q(\mathbf{x}^N;t) = -\sum_\alpha F_\alpha(t)\times$$
$$\times\int_0^1 d\tau\ [\rho_q(\mathbf{x}^N;t)]^\tau \mathcal{P}(t)\dot{\hat{P}}_\alpha[\rho_q(\mathbf{x}^N;t)]^{1-\tau}, \tag{3.15}$$

where

$$\dot{\hat{P}}_\alpha \equiv i\hat{L}\hat{P}_\alpha. \tag{3.16}$$

The equality

$$\mathcal{P}_q\int_0^1 d\tau(\rho_q(\mathbf{x}^N;t))^\tau\ \hat{\mathcal{X}}\ (\rho_q(\mathbf{x}^N;t))^{1-\tau} =$$
$$=\int_0^1 d\tau(\rho_q(\mathbf{x}^N;t))^\tau\ \mathcal{P}\hat{\mathcal{X}}\ (\rho_q(\mathbf{x}^N;t))^{1-\tau} \tag{3.17}$$

has been used here, where $\hat{\mathcal{X}}$ is an arbitrary dynamic quantity depending on coordinates of phase space, and $\mathcal{P}$ is the Mori projection operator

$$\mathcal{P}(t)... = \langle...\rangle_q^t + \sum_\alpha \frac{\delta\langle...\rangle_q^t}{\delta\langle\hat{P}_\alpha\rangle^t}\{\hat{P}_\alpha - \langle\hat{P}_\alpha\rangle^t\} \tag{3.18}$$

with properties

$$\mathcal{P}(t)\ \mathcal{P}(t) = \mathcal{P}(t), \qquad \mathcal{P}(t)\ (1-\mathcal{P}(t)) = 0, \qquad \mathcal{P}(t)\hat{P}_\alpha = \hat{P}_\alpha.$$

By contrast to the Kawasaki-Guntion projection operator, the operator (3.18) acts only on the dynamic variables (or dynamic operators). Finally, taking into account (3.13) and (3.15), the nonequilibrium statistical operator can be written in the form

$$\rho(\mathbf{x}^N;t) = \rho_q(\mathbf{x}^N;t) + \int_{-\infty}^{t} dt' e^{\epsilon(t-t')}\sum_\alpha F_\alpha(t')\times$$
$$\times\int_0^1 d\tau\ [\rho_q(\mathbf{x}^N;t')]^\tau\ T(t,t')\hat{I}_\alpha(t')[\rho_q(\mathbf{x}^N;t')]^{1-\tau}, \tag{3.19}$$

where

$$\hat{I}_\alpha(t) = (1-\mathcal{P}(t))i\hat{L}(t)\hat{P}_\alpha \tag{3.20}$$

are the generalized fluxes. The operator of evolution $T(t,t')$ is now defined in terms of the Mori projection operator $\mathcal{P}(t)$

$$T(t,t') = exp_+\{-\int_{t'}^{t} d\tau(1-\mathcal{P}(\tau))\ i\hat{L}(\tau)\}. \tag{3.21}$$

The nonequilibrium statistical operator (3.19) describes the nonequilibrium hydrodynamic state of a magnetic liquid with the Hamiltonian (2.1) if the dynamic variables are chosen as discussed before. An external inhomogeneous magnetic field $\mathbf{B}(\mathbf{r},t)$ is present in non-explicit form in $\rho_q(\mathbf{x}^N;t)$ and also in the Liouville operator $\hat{L}(t)$. The nonequilibrium statistical operator is represented in the terms of generalized dissipative fluxes (3.20) describing transport phenomena in a magnetic liquid. In accordance with the hypothesis of an abbreviated description of hydrodynamic state, the nonequilibrium statistical operator is a functional of the observed quantities varying in time (the mean values of particles' number, momentum, and energy densities as well as the magnetic moment density). Hence, one should derive the transport equations for them, i.e. the generalized hydrodynamic equations for a magnetic liquid, in order to have self-consistent description of dynamic properties of the system.

4 Nonlinear transport equations

To obtain transport equations for average values $\langle\hat{P}_\alpha\rangle^t$, the equalities

$$\frac{\partial}{\partial t}\langle\hat{P}_\alpha\rangle^t = \langle\dot{\hat{P}}_\alpha\rangle^t = \langle\dot{\hat{P}}_\alpha\rangle^t_q + \langle(1-\mathcal{P}(t))\dot{\hat{P}}_\alpha\rangle^t \tag{4.1}$$

can be used. The equalities (4.1) follow directly from the definition (3.18) of Mori operator. Using the nonequilibrium statistical operator (3.19) in the right-hand side of (4.1), the generalized transport equations for a magnetic liquid can be obtained

$$\frac{\partial}{\partial t}\langle\hat{P}_\alpha\rangle^t = \langle\dot{\hat{P}}_\alpha\rangle^t_q + \sum_\beta \int\limits_{-\infty}^{t} dt' e^{\epsilon(t-t')}\phi_{\alpha\beta}(t,t')(\hat{P}_\beta,\hat{P}_\beta)F_\beta(t')dt', \tag{4.2}$$

where

$$\phi_{\alpha\beta}(t,t') = \int\limits_0^1 d\tau\ \langle\hat{I}_\alpha(t),\, (\rho_q(\mathbf{x}^N;t'))^\tau\ T(t,t')\hat{I}_\beta(t')(\rho_q(\mathbf{x}^N;t'))^{-\tau}\rangle^t_q \times$$
$$\times(\hat{P}_\beta,\hat{P}_\beta)^{-1} = (\hat{I}_\alpha(t),\ T(t,t')\hat{I}_\beta(t'))_q(\hat{P}_\beta,\hat{P}_\beta)^{-1} \tag{4.3}$$

are the so-called generalized memory functions of a system or the generalized transport kernels defined in terms of quasi-equilibrium statistical operator $\rho_q(t)$. It is important to note that the generalized transport coefficients of the system can be represented in terms of the generalized memory functions $\phi_{\alpha\beta}(t;t')$.

The transport equation system (4.2) for the chosen set of dynamic variables corresponds of an abbreviated description of nonequilibrium behaviour of a magnetic liquid and may be applied to describe both strong and weak nonequilibrium states of the system. In general, this is a system of nonlinear equations. The intensive quantities $F_\alpha(t)$ entering in the quasi-equilibrium statistical operator $\rho_q(\mathbf{x}^N;t)$ depend on the averages $\langle\hat{P}_\alpha\rangle^t$ via equations of self-consistency (3.8). The last ones have to be determined from the system (4.2). Besides, as the generalized memory functions $\phi_{\alpha\beta}(t;t')$ are unknown, the question about the solutions of the system (4.2) may be considered only under the following condition: approximations for these functions are to be based on analysis of the expression (4.3) and the corresponding equations for the higher-order memory functions. However, it is well-known that the restriction to the linear case is a good approximation for transport phenomena in a fluid. The nonlinear equations are to be used only for special problems of nonequilibrium physics, for example to describe the dynamical behaviour near the phase transition point, and this is not a main subject of this article.

For the weak non-equilibrium case, the transport equations (4.2) can be essentially simplified. Let us consider this case in more detail and derive the linear transport equations.

5 Generalized hydrodynamic equations

The behaviour of a system near the equilibrium may be described by set of the linear equations for deviations of macroscopic quantities $\langle\hat{P}_\alpha\rangle^t$ from the equilibrium values $\langle\hat{P}_\alpha\rangle_0 = \mathrm{Sp}\ \hat{P}_\alpha\rho_0(\mathrm{x}^N)$, where ρ_0 is an equilibrium statistical operator at temperature $1/\beta$. Assuming the deviations of intensive quantities $\delta F_n(t) = F_n(t) - F_n^0$ to be small, (here F_n^0 are their equilibrium values) the following expressions can be obtained from (3.6)-(3.7):

$$\Phi(t) = \Phi_0 - \sum_\alpha \langle\hat{P}_\alpha\rangle_0\ \delta F_\alpha(t), \tag{5.1}$$

$$S(t) = S_0 - \sum_\alpha \Delta\hat{P}_\alpha\ \delta F_\alpha(t), \tag{5.2}$$

where

$$\Phi_0 = \ln\ \mathrm{Sp}\ \exp\{-\sum_\alpha \hat{P}_\alpha\ F_\alpha^0\}, \tag{5.3}$$

$$S_0 = \Phi_0 + \sum_\alpha \hat{P}_\alpha\ F_\alpha^0, \tag{5.4}$$

and $\Delta\hat{P}_\alpha = \hat{P}_\alpha - \langle\hat{P}_\alpha\rangle_0$. The Gibbs equilibrium distribution reads

$$\rho_0 = \exp\{-\Phi_0 - \sum_n \hat{P}_n\ F_n^0\} = exp\{-S_0\}. \tag{5.5}$$

From the definition of quasi-equilibrium statistical operator (3.6), in linear approximation we obtain

$$\rho_q(t) = \{1 - \sum_\alpha \int_0^1 d\tau\ \Delta\hat{P}_\alpha(\tau)\ \delta F_\alpha(t)\}\rho_0, \tag{5.6}$$

where

$$\Delta\hat{P}_\alpha(\tau) = \rho_0^\tau \Delta\hat{P}_\alpha \rho_0^{-\tau}. \tag{5.7}$$

Using the conditions of self-consistency (3.8) and taking into account the orthogonality properties of the dynamic variables $\hat{P}_\alpha$, the relationship between the deviations of intensive and extensive quantities can be found

$$\delta\langle\hat{P}_\alpha\rangle^t = -(\Delta\hat{P}_\alpha, \Delta\hat{P}_\alpha)\delta F_\alpha(t). \tag{5.8}$$

The equilibrium correlation functions (A, B) have been defined above by the expression (2.9). As it follows from (3.19), the expression for linearized nonequilibrium statistical operator reads

$$\delta\rho(t) = \delta\rho_q(t) + \sum_\alpha \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \delta F_\alpha(t') \times$$
$$\times \int_0^1 d\tau\ \rho_0^\tau\ T_0(t-t')\ (1-\mathcal{P}) i\hat{L}\Delta\hat{P}_\alpha\ \rho_0^{1-\tau}. \tag{5.9}$$

Using Fourier transformation for the time-dependent functions

$$f(t) = \int d\omega\ \tilde{f}(\omega) exp(i\omega t),$$

the expressions (5.6) and (5.9) can be written in a matrix form:

$$\delta\tilde{\rho}_q(\omega) = -\int_0^1 d\tau\ \Delta\hat{P}^+(\tau)\delta\tilde{F}(\omega)\rho_0, \tag{5.10}$$

$$\delta\tilde{\rho}(\omega) = \delta\tilde{\rho}_q(\omega) + \int_0^1 d\tau\ \rho_0^\tau \frac{1}{i\omega + \epsilon + (1-\mathcal{P})i\hat{L}} (1-\mathcal{P})\dot{\hat{P}}^+ \rho_0^{1-\tau} \delta\tilde{F}(\omega), \tag{5.11}$$

where $\Delta\hat{P}^+$ is a vector-line with the elements $\{\Delta\hat{P}_\alpha\}$ and $\delta\tilde{F}(\omega)$ is a vector-column with elements $\{\delta\tilde{F}_\alpha(\omega)\}$. The linearized projection operator has the following structure

$$\mathcal{P}\ldots = (\ldots, \Delta\hat{P}^+)(\Delta\hat{P}, \Delta\hat{P}^+)^{-1}\Delta\hat{P}. \tag{5.12}$$

Respectively, the evolution operator has the following form:

$$T_0(t-t') = \exp\{-(t-t')(1-\mathcal{P})i\hat{L}_N\}. \tag{5.13}$$

Using the linearized solution of the Liouville equation (5.11), it is easy to obtain the set of the generalized hydrodynamic equations or, in the other words, the linearized transport equations for the macroscopic quantities $\langle\Delta\hat{P}\rangle^t$. From the definition

$$\frac{\partial}{\partial t}\langle\Delta\hat{P}\rangle^t = \langle i\hat{L}\Delta\hat{P}\rangle^t$$

it follows

$$\{i\omega - i\Omega_0 + \tilde{\phi}_\epsilon(\omega)\}\langle\Delta\hat{P}\rangle^\omega = 0 \tag{5.14}$$

where

$$i\Omega_0 = (\Delta\dot{\hat{P}}, \Delta\hat{P}^+)_0\ (\Delta\hat{P}, \Delta\hat{P}^+)_0^{-1} \tag{5.15}$$

is a frequency matrix, and

$$\tilde{\phi}_\epsilon(\omega) = ((1-\mathcal{P})\dot{\hat{P}}, \tfrac{1}{i\omega+\epsilon+(1-\mathcal{P})i\hat{L}}(1-\mathcal{P})\dot{\hat{P}}^+)\ (\Delta\hat{P}, \Delta\hat{P}^+)^{-1} \tag{5.16}$$

is a matrix of the memory functions. The matrix equation (5.14) is the set of the linearized equations of generalized hydrodynamics for a magnetic liquid in an external magnetic field $\mathbf{B}(\mathbf{r}; t)$.

It can be shown that the equations for the time correlation functions have the similar to (5.14) structure. Really, the fundamental solution of the Liouville equation (3.4) can be also written in the form

$$\rho(\mathbf{x}^N;t) = \rho_q(\mathbf{x}^N;t) - \int\limits_{-\infty}^{t} dt'\ e^{\epsilon(t'-t)-i\hat{L}(t-t')}\{\frac{\partial}{\partial t'} + i\hat{L}(t')\}\rho_q(\mathbf{x}^N;t'). \quad (5.17)$$

For the case of a weak nonequilibrium behaviour after Fourier-transformation for the time-dependent functions, we obtain

$$\delta\tilde{\rho}(\omega) = \delta\tilde{\rho}_q(\omega) + \int\limits_0^1 d\tau\ \rho_0^{\tau}\frac{1}{i\omega+\epsilon+i\hat{L}}\{\dot{\hat{P}}^{+} + i\omega\Delta\hat{P}^{+}\}\rho_0^{1-\tau}\delta\tilde{F}(\omega). \quad (5.18)$$

From the equations of self-consistency (3.8)

$$\mathrm{Sp}\ \{\Delta\hat{P}[\delta\tilde{\rho}(\omega) - \delta\tilde{\rho}_q(\omega)]\} = 0,$$

using the solution in the form (5.18), one finds

$$\begin{aligned} i\omega\delta\tilde{F}(\omega) &= -\frac{1}{(\Delta\hat{P},\Delta\hat{P}^{+})^{z}}(\Delta\hat{P},\Delta\dot{\hat{P}}^{+})^{z}\delta\tilde{F}(\omega) = \\ &= \{-\frac{1}{(\Delta\hat{P},\Delta\hat{P}^{+})^{z}}(\Delta\hat{P},\Delta\dot{\hat{P}}^{+}) + z\ \}\delta\tilde{F}(\omega), \end{aligned} \quad (5.19)$$

where

$$(A,B^{+})^{z} = (A,\frac{1}{z+iL}B^{+}) \quad (5.20)$$

with $A,B = \{\Delta\hat{P},\Delta\dot{\hat{P}}\}$ being the matrices of Laplace transforms of the corresponding time correlation functions and $z = i\omega + \epsilon$.

A comparison between (5.14) and (5.19) in view (5.8) shows that the matrix equation for the time correlation functions can be written in the form

$$\{\ z - i\Omega_0 + \tilde{\phi}(z)\}(\Delta\hat{P},\Delta\hat{P}^{+})^{z} = (\Delta\hat{P},\Delta\hat{P}^{+}), \quad (5.21)$$

where $(\Delta\hat{P},\Delta\hat{P}^{+})^{z}$ is the matrix of Laplace transforms of the time correlation functions. Another result that follows immediately from such mathematical treatment and can be useful for subsequent calculations is the expression for the matrix of memory functions

$$\tilde{\phi}(z) = \{(\dot{\hat{P}},\dot{\hat{P}}^{+})^{z} - (\dot{\hat{P}},\Delta\hat{P}^{+})^{z}\frac{1}{(\Delta\hat{P},\Delta\hat{P}^{+})^{z}}(\Delta\hat{P},\dot{\hat{P}}^{+})^{z}\}\frac{1}{(\Delta\hat{P},\Delta\hat{P}^{+})} \quad (5.22)$$

It is important to note that as it follows from the definition (5.20), the matrix $(\Delta\hat{P},\Delta\hat{P}^{+})^{z}$ can be expressed in terms of the retarded Green functions

$$G^{(r)}_{AB}(t-t') = -i\theta(t-t')\int\limits_0^1 d\tau\ \langle A(t)\rho_0^{\tau}B^{+}(t')\rho_0^{-\tau}\rangle_0, \quad (5.23)$$

where $\theta(t) = 1$ or 0 for $t > 0$ or $t > 0$ correspondingly.

The spectrum of collective modes can be determined from equation

$$Det \mid z - i\Omega_0 + \tilde{\phi}(z) \mid = 0, \tag{5.24}$$

giving the poles of the retarded Green functions (5.23) constructed on the set of dynamic variables $\{\Delta\hat{P}_\alpha\}$.

Now, for the calculations of the time correlation functions, the generalized transport coefficients connected with the memory functions $\tilde{\phi}(z)$, and the collective mode spectrum, one has to find elements of the frequency matrix $i\Omega_0$ and the matrix of memory functions $\tilde{\phi}(z)$. Using certain approximation for memory functions (Markovian, Gaussian, etc.) the problem can be reduced to the calculation of static correlation functions. The investigation of these questions will be presented elsewhere.

6 The limiting cases

The sets of equations (5.14) and (5.21), the structure of frequency matrix (5.15) and the matrix of memory functions (5.16), the matrix equation for collective modes (5.24) are represented in form allowing to study some limiting cases. Let us consider the cases when the typical time of one or several dynamic variables are much larger than the other ones. It means that the system can be considered as a "quasi" equilibrium one interms of the "fast" dynamic variables for the time scale of the "slow" ones. From view-point of general approach it is possible to eliminate the "fast" dynamical variables from the hydrodynamic description. In these cases the system of equations (5.14) simplifies.

(a) *A pure "liquid" dynamics.* The set of equations (5.14) transforms to the molecular hydrodynamic equations of a simple classic liquid [27,28], the variables of the "magnetic" subsystem having been formally neglected. Meanwhile, the matrices $i\Omega_0(\mathbf{k})$ and $\phi(\mathbf{k};t-t')$ transform to the matrices $i\Omega_0^L(\mathbf{k})$ and $\phi^L(\mathbf{k};t-t')$ well-known from fluid hydrodynamics, where

$$i\Omega_0^L(\mathbf{k}) = \begin{pmatrix} 0 & i\Omega_{np}^L(\mathbf{k}) & 0 \\ i\Omega_{pn}^L(\mathbf{k}) & 0 & i\Omega_{ph}^L(\mathbf{k}) \\ 0 & i\Omega_{hp}^L(\mathbf{k}) & 0 \end{pmatrix}$$

with matrix elements

$$i\Omega_{\alpha\beta}^L(\mathbf{k}) = \langle iL^L\hat{P}_\alpha(\mathbf{k})\Delta\hat{P}_\beta(-\mathbf{k})\rangle_0 \langle\Delta\hat{P}_\beta(\mathbf{k})\Delta\hat{P}_\beta(-\mathbf{k})\rangle_0^{-1}, \tag{6.1}$$

where $\hat{P}_\alpha(\mathbf{k}) = \{\hat{n}(\mathbf{k}), \hat{\mathbf{p}}(\mathbf{k}), \hat{h}_L(\mathbf{k})\}$,

$$\hat{h}_L(\mathbf{k}) = \hat{\varepsilon}_L(\mathbf{k}) - \langle\hat{\varepsilon}_L(\mathbf{k})\hat{n}(-\mathbf{k})\rangle_0\langle\hat{n}(\mathbf{k})\hat{n}(-\mathbf{k})\rangle_0^{-1}\hat{n}(\mathbf{k}) \tag{6.2}$$

and $\hat{\varepsilon}_L(\mathbf{k})$ are the "liquid" part of enthalpy and energy densities, respectively. The correlation function $\langle\hat{n}(\mathbf{k})\hat{n}(-\mathbf{k})\rangle_0/N = S(k)$ is a static structure factor. For the matrix of the memory functions in this case we have

$$\phi^L(\mathbf{k};t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \phi_{pp}^L(\mathbf{k};t) & \phi_{ph}^L(\mathbf{k};t) \\ 0 & \phi_{hp}^L(\mathbf{k};t) & \phi_{hh}^L(\mathbf{k};t) \end{pmatrix}$$

The nonzero elements $\phi^L_{\alpha\beta}(\mathbf{k};t,t')$

$$\phi^L_{pp}(\mathbf{k};t) = ((1-\mathcal{P}_L)\dot{\hat{\mathbf{p}}}(\mathbf{k}),\ T^L_0(t)(1-\mathcal{P}_L)\dot{\hat{\mathbf{p}}}(-\mathbf{k}))\ \langle\Delta\hat{\mathbf{p}}(\mathbf{k})\Delta\hat{\mathbf{p}}(-\mathbf{k})\rangle_0^{-1},$$

$$\phi^L_{ph}(\mathbf{k};t) = ((1-\mathcal{P}_L)\dot{\hat{\mathbf{p}}}(\mathbf{k}),\ T^L_0(t)(1-\mathcal{P}_L)\dot{\hat{h}}(-\mathbf{k}))\ \langle\Delta\hat{h}(\mathbf{k})\Delta\hat{h}(-\mathbf{k})\rangle_0^{-1},$$

$$\phi^L_{hp}(\mathbf{k};t) = ((1-\mathcal{P}_L)\dot{\hat{h}}(\mathbf{k}),\ T^L_0(t)(1-\mathcal{P}_L)\dot{\hat{\mathbf{p}}}(-\mathbf{k}))\ \langle\Delta\hat{\mathbf{p}}(\mathbf{k})\Delta\hat{\mathbf{p}}(-\mathbf{k})\rangle_0^{-1},$$

$$\phi^L_{hh}(\mathbf{k};t) = ((1-\mathcal{P}_L)\dot{\hat{h}}(\mathbf{k}),\ T^L_0(t)(1-\mathcal{P}_L)\dot{\hat{h}}(-\mathbf{k}))\ \langle\Delta\hat{h}(\mathbf{k})\Delta\hat{h}(-\mathbf{k})\rangle_0^{-1},$$

are connected with the generalized transport coefficients of a liquid [27,28]. The expressions for the Mori projection operator $\mathcal{P}_L$ and the evolution operator $T^L_0(t)$ can be found, the variables of "magnetic" subsystem having been cancelled in (5.12) and (5.13). It is important to note that the averaging in this limiting case has to be performed with the help of the full statistical operator and is therefore more complicated than in case of a simple liquid. On the other side, in our case the Liouville operator has additional terms due to "spin" subsystem. In the simplest case, considering the "magnetic" subsystem as equilibrium, the model of an "effective" liquid with the "effective" potential of interaction depending on the equilibrium intensive quantities of "magnetic" subsystem can be proposed. Then, the methods developed for a simple liquid can be used.

(b) *A pure "magnetic" dynamics.* The hydrodynamic equations for "magnetic" subsystem can be derived neglecting formally in the set of equations (5.14) the variables of "liquid" subsystem

$$\frac{\partial}{\partial t}\langle\Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^t - i\Omega^S_{mm}(\mathbf{k})\langle\Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^t - i\Omega^S_{mh}(\mathbf{k})\langle\Delta\hat{h}_S(\mathbf{k})\rangle^t +$$
$$+\int_{-\infty}^{t} dt'\phi^S_{mm}(\mathbf{k};t-t')\langle\Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^{t'} + \int_{-\infty}^{t} dt'\phi^S_{mh}(\mathbf{k};t-t')\langle\Delta\hat{h}_S(\mathbf{k})\rangle^{t'} = 0, \quad (6.3)$$

$$\frac{\partial}{\partial t}\langle\Delta\hat{h}_S(\mathbf{k})\rangle^t - i\Omega^S_{hm}(\mathbf{k})\langle\Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^t - i\Omega^S_{hh}(\mathbf{k})\langle\Delta\hat{h}_S(\mathbf{k})\rangle^t +$$
$$+\int_{-\infty}^{t} dt'\phi^S_{hm}(\mathbf{k};t-t')\langle\Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^{t'} + \int_{-\infty}^{t} dt'\phi^S_{hh}(\mathbf{k};t-t')\langle\Delta\hat{h}_S(\mathbf{k})\rangle^{t'} = 0, \quad (6.4)$$

where $\hat{h}_S(\mathbf{k})$ is a "spin" part of enthalpy density.

$$\hat{h}_S(\mathbf{k}) = (1-\mathcal{P}_M)\hat{\varepsilon}_S(\mathbf{k}).$$

The elements of frequency matrix $i\Omega^S(k)$ have the following structure

$$i\Omega^S_{mm}(\mathbf{k}) = (i\hat{L}\hat{\mathbf{m}}(\mathbf{k}),\Delta\hat{\mathbf{m}}(-\mathbf{k}))\ (\Delta\hat{\mathbf{m}}(\mathbf{k}),\Delta\hat{\mathbf{m}}(-\mathbf{k}))^{-1},$$

$$i\Omega^S_{hm}(\mathbf{k}) = (i\hat{L}\hat{h}_S(\mathbf{k}),\Delta\hat{\mathbf{m}}(-\mathbf{k}))\ (\Delta\hat{\mathbf{m}}(\mathbf{k}),\Delta\hat{\mathbf{m}}(-\mathbf{k}))^{-1},$$

$$i\Omega^S_{mh}(\mathbf{k}) = (i\hat{L}\hat{\mathbf{m}}(\mathbf{k}),\Delta\hat{h}_S(-\mathbf{k}))\ (\Delta\hat{h}_S(\mathbf{k}),\Delta\hat{h}_S(-\mathbf{k}))^{-1},$$

$$i\Omega^S_{hh}(\mathbf{k}) = (i\hat{L}\hat{h}_S(\mathbf{k}),\Delta\hat{h}_S(-\mathbf{k}))\ (\Delta\hat{h}_S(\mathbf{k}),\Delta\hat{h}_S(-\mathbf{k}))^{-1}.$$

Respectively, the expressions for the memory functions or, in the other words, for the transport kernels are

$$\phi^S_{\alpha\beta}(\mathbf{k};t) = ((1-\mathcal{P}_S)\dot{\hat{P}}_\alpha(\mathbf{k}), T^S_0(t)(1-\mathcal{P}_S)\dot{\hat{P}}_\beta(-\mathbf{k}))\,(\Delta\hat{P}_\beta(\mathbf{k}),\Delta\hat{P}_\beta(-\mathbf{k}))^{-1}$$

with $\hat{P}_\alpha(\mathbf{k}) = \{\hat{\mathbf{m}}(\mathbf{k}),\ \hat{h}_S(\mathbf{k})\}$. The functions $\phi^S_{mm}(\mathbf{k};t)$, $\phi^S_{hm}(\mathbf{k};t)$ (or $\phi^S_{mh}(\mathbf{k};t)$), and $\phi^S_{hh}(\mathbf{k};t)$ are connected with the generalized transport coefficients of the spin diffusion, the thermal diffusion, and the thermomagnetic diffusion, respectively. The projection operator $\mathcal{P}_S$ and the operator of evolution $T^S_0(\mathbf{k},t)$ for the "spin" system in external magnetic field can be obtained from (5.12) and (5.13), the variables of "liquid" subsystem having been canceled.

In fact the equations (6.3) and (6.4) can be also used for the description of a solid or an amorphous magnet dynamics. The question is only in which way the averaging is defined. For a solid magnet the averaging is to be performed with the help of the equilibrium statistical operator $\hat{\rho}^S_0$ with the Hamiltonian (2.3) under condition that the positions of the particles are fixed in lattice. Similar equations for a solid magnet have been derived by Schwabl and Michel [23]. For dynamics of an amorphous magnet, the averaging over the "spin" subsystem variables have to be on the base of $\hat{\rho}^S_0$ as well, but the averaging of the equations for observed quantities over positions of the particles is to be performed afterwards with the corresponding distribution function. In our case we must consider, as the definition, the averaging with the full equilibrium statistical operator $\hat{\rho}_0$ of a magnetic liquid, having the Hamiltonian (2.1).

As it follows from the expressions for $i\Omega^S_{\alpha\beta}(\mathbf{k})$ and $\phi^S_{\alpha\beta}(\mathbf{k};t)$, the influence of the "liquid" subsystem on the "magnetic" dynamics is connected with two reasons. First, the averaging is performed by the full equilibrium statistical operator $\hat{\rho}_0$. Second, the new terms arise in the coefficients of the equations (6.3) and (6.4) comparing with a pure solid magnet from the dynamics. It can be shown considering the expressions for the derivatives of $\hat{\mathbf{m}}(\mathbf{k})$ and $\hat{h}_S(\mathbf{k})$. We obtain from the definition:

$$\dot{\hat{\mathbf{m}}}(\mathbf{k}) = i\hat{L}\hat{\mathbf{m}}(\mathbf{k}) = -\sum_{f=1}^{N} i\frac{(\mathbf{k}\mathbf{p}_f)}{m}\,\mathbf{S}_f\exp(i\mathbf{k}\mathbf{r}_f) + \\ + \sum_{f=1}^{N} i\hat{L}_S\mathbf{S}_f\exp(i\mathbf{k}\mathbf{r}_f), \tag{6.5}$$

$$\dot{\hat{h}}_S(\mathbf{k}) = i\hat{L}\hat{h}_S(\mathbf{k}) = iL^L\hat{h}_S(\mathbf{k}) + i\hat{L}_S\hat{h}_S(\mathbf{k}). \tag{6.6}$$

Only the last terms in (6.5) and (6.6) have a pure magnetic origin.

(c) *A pure "magnetic" relaxation.* In the case of isothermal processes, i.e. when of the energy transport for the "magnetic" system is not so important, the equation for a mean magnetic moment $\langle\hat{\mathbf{m}}(\mathbf{k})\rangle^t$ can be obtained from the equations (6.3) and (6.4)

$$\frac{\partial}{\partial t}\langle\Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^t - i\Omega^S_{mm}(\mathbf{k})\langle\Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^t + \int_{-\infty}^{t} dt'\phi^S_{mm}(\mathbf{k};t-t')\langle\Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^{t'} = 0. \tag{6.7}$$

In the other words, it is the generalized Bloch's equation in an external magnetic field $\mathbf{B}(\mathbf{r};t)$. The expressions for $\phi^S_{mm}(\mathbf{k};t-t')$ are similar to the former one, but the projection operator is defined using only one dynamic variable, namely $\hat{\mathbf{m}}(\mathbf{k})$ (see (2.12)).

Taking into account that

$$\langle \Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^t = -(\Delta\hat{\mathbf{m}}(\mathbf{k}), \Delta\hat{\mathbf{m}}(-\mathbf{k}))\, \delta\mathbf{b}(\mathbf{k},t),$$

where $\delta\mathbf{b}(\mathbf{k},t) = \mathbf{b}(\mathbf{k},t) - \mathbf{B}(\mathbf{k},t)$, and $\mathbf{b}(\mathbf{k},t)$ is an internal magnetic field, the equation (6.7) can be rewritten in the other form in which an external magnetic field $\mathbf{B}(\mathbf{k},t)$ will be presented obviously. Such equation was studied by Robertson [31] and later by Kalashnikov and Auslender [24].

In the case when an external field can be represented as a sum of two terms describing respectively time-independent and a weakly time-dependent parts of an external field, the expression for the matrix of dynamic susceptibilities $\chi(\mathbf{k},\omega)$ can be derived on the basis of fluctuation-dissipative theorem and equation for the time correlation functions. After some mathematical manipulations, the result can be written as follows

$$\chi(\mathbf{k},\omega) = \frac{-i\Omega^S_{mm}(\mathbf{k}) + \tilde{\phi}^S_{mm}(\mathbf{k};\omega - i\epsilon)}{i\omega - i\Omega^S_{mm}(\mathbf{k}) + \tilde{\phi}^S_{mm}(\mathbf{k};\omega - i\epsilon)}\chi(\mathbf{k}), \qquad (6.8)$$

where $\chi(\mathbf{k}) = -\beta(\Delta\hat{\mathbf{m}}(\mathbf{k}), \Delta\hat{\mathbf{m}}(\mathbf{k}))$ is the tensor of differential static magnetic susceptibility, and $\epsilon \to +0$ as mentioned above.

As it follows from (6.8), in contrary to the solution of usual Bloch equation [32], the dynamic susceptibility has the proper static limit $\chi(\mathbf{k},\omega) \to \chi(\mathbf{k})$ when $\omega \to 0$. In the other limit when the relaxation term is very large and $\tilde{\phi}^S_{mm}(\mathbf{k};\omega - i\epsilon) \to \infty$, we also obtain the correct result $\chi(\mathbf{k},\omega) \to \chi(\mathbf{k})$. It is necessary to note that in our approach as it follows from (6.5), the relaxation term $\tilde{\phi}^S_{mm}(\mathbf{k};\omega - i\epsilon)$ has a contribution which is connected with the diffusion of particles. Such additional term for a weak inhomogeneous system was phenomenologically introduced by Torrey [33].

(d) *Simple case of a "mixed" dynamics.* We consider also one example of so-called "mixed" dynamics when the typical time scale of two dynamical variables are much larger than the other ones, but the first variables are connected with the "liquid" subsystem and the other are with the "magnetic" subsystem. In particular, the dynamical fluctuations of the density particles' number and the density of magnetic moment will be considered. For this case the hydrodynamic equations read:

$$\frac{\partial}{\partial t}\langle \Delta\hat{\aleph}(\mathbf{k})\rangle^t - i\Omega_{\aleph\aleph}(\mathbf{k})\langle \Delta\hat{\aleph}(\mathbf{k})\rangle^t - i\Omega_{\aleph m}(\mathbf{k})\langle \Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^t +$$
$$+\int_{-\infty}^{t} dt'\phi_{\aleph\aleph}(\mathbf{k};t-t')\langle \Delta\hat{\aleph}(\mathbf{k})\rangle^{t'} + \int_{-\infty}^{t} dt'\phi_{\aleph m}(\mathbf{k};t-t')\langle \Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^{t'} = 0, \quad (6.9)$$

$$\frac{\partial}{\partial t}\langle \Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^t - i\Omega_{m\aleph}(\mathbf{k})\langle \Delta\hat{\aleph}(\mathbf{k})\rangle^t - i\Omega_{mm}(\mathbf{k})\langle \Delta\hat{m}(\mathbf{k})\rangle^t +$$
$$+\int_{-\infty}^{t} dt'\phi_{m\aleph}(\mathbf{k};t-t')\langle \Delta\hat{\aleph}(\mathbf{k})\rangle^{t'} + \int_{-\infty}^{t} dt'\phi_{mm}(\mathbf{k};t-t')\langle \Delta\hat{\mathbf{m}}(\mathbf{k})\rangle^{t'} = 0. (6.10)$$

where $\hat{\aleph}(\mathbf{k})$ was defined before by the expression (2.10).

Solving these equations with respect to one of the variables, the expressions for the renormalized memory functions can be obtained

$$\tilde{\phi}^{(ren)}_{\aleph\aleph}(\mathbf{k};z) = \tilde{\phi}_{\aleph\aleph}(\mathbf{k};z) - (-i\Omega_{\aleph m}(\mathbf{k}) + \tilde{\phi}_{\aleph m}(\mathbf{k};z)) \times$$

$$\times \frac{1}{z - i\Omega_{mm}(\mathbf{k}) + \tilde{\phi}_{mm}(\mathbf{k};z)}(-i\Omega_{m\aleph}(\mathbf{k}) + \tilde{\phi}_{m\aleph}(\mathbf{k};z)), \quad (6.11)$$

$$\tilde{\phi}^{(ren)}_{mm}(\mathbf{k};z) = \tilde{\phi}_{mm}(\mathbf{k};z) - (-i\Omega_{m\aleph}(\mathbf{k}) + \tilde{\phi}_{m\aleph}(\mathbf{k};z)) \times$$

$$\times \frac{1}{z - i\Omega_{\aleph\aleph}(\mathbf{k}) + \tilde{\phi}_{\aleph\aleph}(\mathbf{k};z)}(-i\Omega_{\aleph m}(\mathbf{k}) + \tilde{\phi}_{\aleph m}(\mathbf{k};z)). \quad (6.12)$$

As it can be seen from (6.11), (6.12), the subsystems under consideration depend one on the other on the dynamical level. The expressions (6.11)-(6.12) may be analysed for the specified type of an external field and interparticle interactions. This problem will be studied elsewhere. However, from the general structure of (6.11) and (6.12) we can conclude that the spin dynamics will be described by more complicated equation than the Bloch equation, even though the Markovian approximation for the memory functions are used. Moreover, the another equation will describe the coupled dynamics of the "liquid" subsystem, namely, the density fluctuations in our case.

7 Conclusions

A general formalism of Zubarev's method of the nonequilibrium statistical operator is applied to derive the generalized hydrodynamic equations of a magnetic liquid. Our calculation starts from a rigorous equation for the statistical operator of a system containing particles with localized spins and with consideration both translational and magnetic degrees of freedom. This approach has permitted microscopic derivation of the equations for the time correlation functions (the retarded Green functions) and the equation for the collective mode spectrum. The microscopic expressions for the relaxation kernels of these equations were also obtained.

The limiting cases, where assumption of a large difference between the typical time scales of two groups of the considering dynamic variables is valid, were studied. In particular, the limiting cases for a pure "liquid", a pure "magnetic" and a "mixed" dynamics were analysed. For the first two cases the known hydrodynamic equations were found.

The sets of equations (5.14), (5.21) and the equation for the spectrum of collective modes (5.23) can be used for study of nonequilibrium behaviour for particular cases of a model Hamiltonian (2.1).

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