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A CANONICAL FORMALISM OF DISSIPATIVE QUANTUM SYSTEMS. NON-EQUILIBRIUM THERMOFIELD DYNAMICS

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Introduced is a canonical formalism of quantum systems in far-from-equilibrium state, named Non-Equilibrium Thermo Field Dynamics (NETFD), which provides a unified viewpoint covering whole the aspects in non-equilibrium statistical mechanics, i.e. the Boltzmann, the Fokker-Planck, the Langevin and the stochastic Liouville equations.

It is shown how the semi-free time-evolution generator of the quantum Fokker-Planck equation for *non-stationary* situations is derived upon a couple of basic requirements which are extracted from the fundamental characteristics related to the Liouville equation. With the generator, it is demonstrated how to make a canonical theory for dissipative quantum systems. The annihilation and creation operators are introduced by means of a time-dependent Bogoliubov transformation.

It is shown that, within the formalism of NETFD, there are two possibilities in the introduction of an external field. One is by an hermitian hat-Hamiltonian, the other is by a non-hermitian hat-Hamiltonian. With the former hat-Hamiltonian, the $\hat{S}$-matrix and the generating functional method are introduced to give the relation between the method of NETFD with the one of Schwinger's closed-time path.

With the latter non-hermitian interaction hat-Hamiltonian, the general expression of the stochastic semi-free time-evolution generator is derived for a *non-stationary* Gaussian white quantum stochastic process. The correlation of the random force operators are also derived generally. With the generator, it is presented how a unified framework of quantum stochastic differential equations can be constructed. The stochastic Liouville equations and the Langevin equations of the system, both of Ito and Stratonovich types, are investigated in a unified manner.

Whole the framework of NETFD is mapped to a c-number space by means of the coherent state representation within NETFD.

The system of stochastic differential equations is constructed also upon the hermitian interaction hat-Hamiltonian. An interpretation of the Mori formula is given within the framework of NETFD. A mathematical reformulation of NETFD is performed, where the stochastic time-evolution generator is given in terms of a martingale. The Monte Carlo wave-function method, i.e. the quantum jump simulation, is reviewed in terms of NETFD.

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Contents

1. Introduction

In this paper, we will introduce a *canonical formalism* of quantum systems in far-from-equilibrium state, named Non-Equilibrium Thermo Field Dynamics (NETFD). This is a unified formalism which enables us to treat dissipative quantum systems (covering whole the aspects in non-equilibrium statistical mechanics listed in Table 1) by the method similar to the usual quantum mechanics and quantum field theory which accommodate the concept of the dual structure in the interpretation of nature, i.e. in terms of the *operator algebra* and the *representation space*. The representation space of NETFD (named *thermal space*) is composed of the direct product of two Hilbert spaces, the one for *non-tilde* fields and the other for *tilde* fields.[1] It was revealed that *dissipation* is taken into account by a *rotation* in whole the two Hilbert spaces. The terms constituted by the multiplication of tilde and non-tilde fields in the infinitesimal time-evolution generator take care of dissipative (i.e. irreversible) phenomena. This notion was discovered first when NETFD was constructed [1,2].[2]

Boltzmann tried to explain the irreversibility of nature based on the microscopic and reversible Newton's mechanics. It was revealed that he had introduced a stochastic manipulation, which is called the *molecular chaos*, without knowing it in the course of the derivation of the Boltzmann equation (see [4] for a brief review of the irreversibility in statistical mechanics). Besides the technical transparency of our new method, we expect that its dual structure, as a quantum theory of dissipative fields, may provide us with a breakthrough to realize Boltzmann's original dream. The duality was not recognized in Boltzmann's days.

It is known that one can divide the fundamental aspects in non-equilibrium statistical mechanics into four categories as shown in Table 1. In category *I*, we deal with a one-particle distribution function (in the μ-phase-space within classical statistical mechanics) with the assumption of *molecular chaos* or something similar which introduces an irreversibility. In category *II*, we handle a density operator which describes the *distribution* of the *ensemble* of a system under consideration. Within the terminology of

[1] In NETFD, any operator A is associated with its tilde field $\tilde{A}$ (see **Tool 1** in section 3.).

[2] This notion had not appeared in the formulation of the equilibrium thermo field dynamics (TFD) [3] which is an operator formalism of the Gibbs ensembles. This is one of the essential difference between NETFD and TFD.

Table 1: Fundamental Aspects in Non-Equilibrium Statistical Mechanics

	Founder	*Basic Equations*	*Key Words*
I	Boltzmann	Boltzmann eq. kinetic eq.	one-particle distr. func. molecular chaos
II	Gibbs	master eq. Fokker-Planck eq.	density operator ensembles
III	Einstein	Langevin eq.	random force dynamical variables
IV	Kubo	stochastic Liouville eq.	random force phase-space variables

classical statistical mechanics, we treat the assembly of points in the Γ-phase-space, each point of which describes a dynamical state of an element system of the ensemble. Irreversibility is introduced by a coarse graining in Γ-space. In category *III*, we study a path of a dynamical variable which is generated by a stochastic equation with a specified random process. The correlation of random forces introduces irreversible behavior of the system. In category *IV*, we treat a distribution of the bundle of paths in the phase-space [7,8]. For each time, one has patterns of flows in the phase-space corresponding to an element of a random force (stochastic process) at the time. Traversing the pattern, a point, which represents the dynamical state of a system, evolves in time just the same as is described by the corresponding Langevin equation.

The framework of NETFD was constructed first [1,2] by, so to speak, a *principle of correspondence* based upon the damping theoretical argument within the density operator formalism [9]-[11] (see Appendices A and B). It was reconstructed upon the seven axioms [12]. Then, the most general expression of the renormalized time-evolution generator in the interaction representation (the semi-free hat-Hamiltonian) was derived together with an equation for the one-particle distribution function [13,14]. Therefore, we see that it was started to build NETFD upon the fundamental aspects *I* and *II* in Table 1. Within these aspects, the canonical formalism of dissipative quantum fields in NETFD was formulated, and the close structural resemblance between NETFD and usual quantum field theories was revealed [15,16]. The generating functional within NETFD was derived [17]. Furthermore, the kinetic equation was derived within NETFD [21], and the relation between NETFD and the closed time-path methods [18]-[20] was shown. The extension of NETFD to the hydrodynamical region as well as the kinetic region was started [22,23].[3]

The framework of NETFD has been extended [26]-[37] to take account of the aspects *III* and *IV* as well as *I* and *II*. Here again NETFD allowed us to construct a unified canonical theory of quantum stochastic operators. The stochastic Liouville equations both of the Ito and of the Stratonovich types were introduced in the Schrödinger representation. Whereas, the Langevin equations both of the Ito and of the Stratonovich types were constructed as the Heisenberg equation of motion with the help of the time-evolution generator of corresponding stochastic Liouville equations. The Ito formula

[3] Zubarev and Tokarchuk admired the method of NETFD, and they also started to use it for the investigation of these regions [24] (see also [25] for the application to the problem of the quark-gluon plasma).

was derived for quantum systems.

In section 2., we review the density operator method of the Liouville equation. We will extract a couple of fundamental characteristics related to the Liouville equation. In section 3., technical basics and some fundamentals of NETFD are listed. The relation of the thermal vacuum to the density matrix is revealed. In section 4., we will show how the *semi-free* time-evolution generator of the quantum Fokker-Planck equation for non-stationary situations is derived upon the basic requirements. The semi-free generator is *bi-linear* and *globally gauge invariant*. With the generator, we will demonstrate how to make a canonical theory for dissipative quantum systems. The annihilation and creation operators are introduced by means of a time-dependent Bogoliubov transformation. The two-point function (propagator) is also derived. In section 5., we will show that there are two possibilities in the introduction of an external field. One is by an hermitian hat-Hamiltonian, the other is by a non-hermitian hat-Hamiltonian. In section 6., the $\hat{S}$-matrix within NETFD will be introduced. In section 7., the generating functional method is introduced, which gives us the relation between the method of NETFD with the one of Schwinger's closed-time path. In section 8., the general expression of the stochastic semi-free time-evolution generator is derived for a *non-stationary* Gaussian white quantum stochastic process (non-stationary quantum Wiener process) by means of the non-hermitian interaction hat-Hamiltonian. The correlation of the random force operators are also derived generally. With the generator, we will present how a unified framework of quantum stochastic differential equations can be constructed, i.e. the stochastic Liouville equations and the Langevin equations both of Ito and Stratonovich types of the system are investigated in a unified manner. The relation among the Fokker-Planck equation and the quantum stochastic differential equations is given in Fig. 1. In section 9., whole the framework of NETFD will be mapped to a c-number space by means of the coherent state representation within NETFD. In section 10., we will try to construct the system of stochastic differential equations upon the hermitian interaction hat-Hamiltonian investigated in section 5.. This approach may be intimately related to the one by mathematicians for the formulation of the quantum stochastic differential equations. In section 11., an interpretation of the Mori formula will be given within the framework of NETFD. In section 12., the formulation derived in previous sections will be applied to the case of *stationary* stochastic process which is equivalent to the model of a damped harmonic oscillator. The Fokker-Planck equation and the Heisenberg equation of motion for coarse grained operators are explicitly handled. The irreversibility of the system is investigated in terms of the Boltzmann entropy. In section 13., a mathematical reformulation of NETFD will be performed. The stochastic time-evolution generator will be given in terms of a martingale. In section 14., the Monte Carlo wave-function method, i.e. the quantum jump simulation, will be reviewed in terms of NETFD. Section 15. is devoted to discussions. The open problems and the prospect are also included. Appendices A–G are added in order to make the paper self-contained.

2. Liouville Equation

Let us remember that the system of the Liouville equation

$$\frac{\partial}{\partial t}\rho(t) = -iL\rho(t), \tag{2.1}$$

can be treated formally as one of a canonical theory.

The Liouville equation has the following general characteristics:

D1. The hermiticy of the Liouville operator iL:

$$(iL)^{\dagger} = iL. \tag{2.2}$$

D2. The conservation of probability (tr $\rho = 1$):

$$\mathrm{tr}\ LX = 0. \tag{2.3}$$

D3. The hermiticy of the density operator:

$$\rho^{\dagger}(t) = \rho(t). \tag{2.4}$$

If the above characteristics were violated, the system of the Liouville equation may not describe the nature which should be treated by statistical mechanics.

The expectation value of an operator A is given by

$$\langle A \rangle_t = \mathrm{tr}\ A\rho(t). \tag{2.5}$$

Substituting the formal solution

$$\rho(t) = \mathrm{e}^{-iLt}\rho(0), \tag{2.6}$$

of (2.1) into (2.5), and using the property (2.3) of **D2**, we see that

$$\begin{aligned} \langle A \rangle_t &= \mathrm{tr}\ A\mathrm{e}^{-iLt}\rho \\ &= \mathrm{tr}\ \mathrm{e}^{iLt}A\mathrm{e}^{-iLt}\rho \\ &= \mathrm{tr}\ A(t)\rho, \end{aligned} \tag{2.7}$$

where we put $\rho = \rho(0)$, and introduced a Heisenberg operator

$$A(t) = \mathrm{e}^{iLt}A\mathrm{e}^{-iLt}. \tag{2.8}$$

This procedure of introducing a Heisenberg operator is similar to the one in quantum mechanics and in quantum field theory.

It is easy to show that the Heisenberg operator $A(t)$ satisfies the Heisenberg equation:

$$\frac{dA(t)}{dt} = i[L,\ A(t)]. \tag{2.9}$$

¿From the above inspection, we understand that the system of the Liouville equation can be phrased as if it had the structure of canonical theory. Quite a lot of discussions related to various methods in non-equilibrium statistical mechanics were formulated upon the above mentioned canonical theory, e.g. the projector formulation of the damping theory [38]-[40],[9,10], the Mori formalism [41] and so on. With the help of the canonical theory of the Liouville space, one can indeed write down his formulation and/or method transparently. However, at the moment when people tries to apply the formulation to solve a model, he usually knows that the method is far from easy to use in contrast to the cases in quantum mechanics. For example, people are forced to deal with complicated and nontransparent calculations under tr.

3. Technical Basics of NETFD

The canonical theory for dissipative quantum systems, will be presented in the following, has been constructed essentially on the same fundamental requirements given in the previous section, and has the merits overcoming the above mentioned disadvantages in the application of the method. Among the merits of NETFD are a straightforward and comprehensible treatment of transient phenomena and a transparent algebraic structure (see, for example, [42]–[47]). Furthermore, as will be shown in the following, this merits of the method unified whole the aspects in non-equilibrium statistical mechanics, i.e. the Boltzmann, the Fokker-Planck, the Langevin and the stochastic Liouville equations. It admits us to deal with dissipative systems by algebraic manipulations similar to the usual quantum mechanics.

The formalism of NETFD is constructed upon the following fundamental basics.

Tool 1. Any operator A in NETFD is accompanied by its partner (tilde) operator $\tilde{A}$. The *tilde conjugation* $\sim$ is defined by:

$$(A_1 A_2)^{\sim} = \tilde{A}_1 \tilde{A}_2, \tag{3.1}$$

$$(c_1 A_1 + c_2 A_2)^{\sim} = c_1^* \tilde{A}_1 + c_2^* \tilde{A}_2, \tag{3.2}$$

$$(\tilde{A})^{\sim} = A, \tag{3.3}$$

$$(A^\dagger)^{\sim} = \tilde{A}^\dagger, \tag{3.4}$$

where c_1 and c_2 are c-numbers.

Tool 2. The tilde and non-tilde operators in the Schrödinger representation are mutually commutative:

$$[A,\ \tilde{B}] = 0. \tag{3.5}$$

Tool 3. The tilde and non-tilde operators are related with each other through the relation

$$\langle 1|A^\dagger = \langle 1|\tilde{A}, \tag{3.6}$$

where $\langle 1|$ is the thermal bra-vacuum (see (3.10) below).

Tool 4. The expectation value of an operator A is given by $\langle 1|A|0\rangle$ where $|0\rangle$ is the thermal ket-vacuum (see (3.10) below). Observable operators consist only of non-tilde operators.

Within the framework of NETFD, the dynamical evolution of systems is described by the *Schrödinger equation* ($\hbar = 1$)

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}\ |0(t)\rangle, \tag{3.7}$$

which is related with the Liouville equation (2.1).

The basics **D1**, **D2** and **D3** in the previous section, related to the Liouville equation, are termed respectively as follows:

B1. The *hat-Hamiltonians* $\hat{H}$, an infinitesimal time-evolution generator, satisfies

$$\left(i\hat{H}\right)^{\sim} = i\hat{H}. \tag{3.8}$$

This characteristics is named *tildian*. The tildian hat-Hamiltonian is not necessarily hermitian operator.

B2. The hat-Hamiltonian has zero eigenvalue for the thermal bra-vacuum:

$$\langle 1|\hat{H} = 0. \tag{3.9}$$

This is the manifestation of the conservation of probability, i.e. $\langle 1|0(t)\rangle = 1$.

B3. The thermal vacuums $\langle 1|$ and $|0\rangle$ are *tilde invariant*:

$$\langle 1|^{\sim} = \langle 1|, \quad |0\rangle^{\sim} = |0\rangle, \tag{3.10}$$

and are normalized as $\langle 1|0\rangle = 1$.

The Heisenberg equation within NETFD for an operator A is given by

$$\frac{d}{dt}A = i[\hat{H},\ A], \tag{3.11}$$

(cf. the Heisenberg equation (2.9)).

Now, we introduce a set of states [48]

$$|m, \tilde{n}\rangle = |m\rangle|\tilde{n}\rangle \tag{3.12}$$

where $|m\rangle$ and $|\tilde{n}\rangle$ satisfy

$$a^{\dagger}a|m\rangle = m|m\rangle, \qquad \tilde{a}^{\dagger}\tilde{a}|\tilde{n}\rangle = n|\tilde{n}\rangle, \tag{3.13}$$

$$\langle m|a^{\dagger}a = \langle m|m, \qquad \langle\tilde{n}|\tilde{a}^{\dagger}\tilde{a} = \langle\tilde{n}|n, \tag{3.14}$$

the ortho-normality

$$\langle m|m'\rangle = \delta_{m,m'}, \qquad \langle\tilde{n}|\tilde{n}'\rangle = \delta_{n,n'}, \tag{3.15}$$

and the completeness

$$\sum_m |m\rangle\langle m| = 1, \qquad \sum_n |\tilde{n}\rangle\langle\tilde{n}| = 1. \tag{3.16}$$

We see that the ortho-normality and the completeness for $|m, \tilde{n}\rangle$ are given respectively by

$$\langle m, \tilde{n}|m', \tilde{n}'\rangle = \delta_{m,m'}\delta_{n,n'}, \tag{3.17}$$

$$\sum_{m,n} |m, \tilde{n}\rangle\langle m, \tilde{n}| = 1. \tag{3.18}$$

The matrix elements $\langle k, \tilde{\ell}|A|m, \tilde{n}\rangle$ and $\langle k, \tilde{\ell}|\tilde{A}|m, \tilde{n}\rangle$ with the operator A, consist only of non-tilde operators, reduce respectively to

$$\begin{aligned}\langle k, \tilde{\ell}|A|m, \tilde{n}\rangle &= \langle k|A|m\rangle\ \langle\tilde{\ell}|\tilde{n}\rangle \\ &= \langle k|A|m\rangle\delta_{\ell,n},\end{aligned} \tag{3.19}$$

$$\begin{aligned}\langle k, \tilde{\ell}|\tilde{A}|m, \tilde{n}\rangle &= \langle k|m\rangle\ \langle\tilde{\ell}|\tilde{A}|\tilde{n}\rangle \\ &= \delta_{k,m}\langle\ell|A|n\rangle^{*} \\ &= \delta_{k,m}\langle n|A^{\dagger}|\ell\rangle,\end{aligned} \tag{3.20}$$

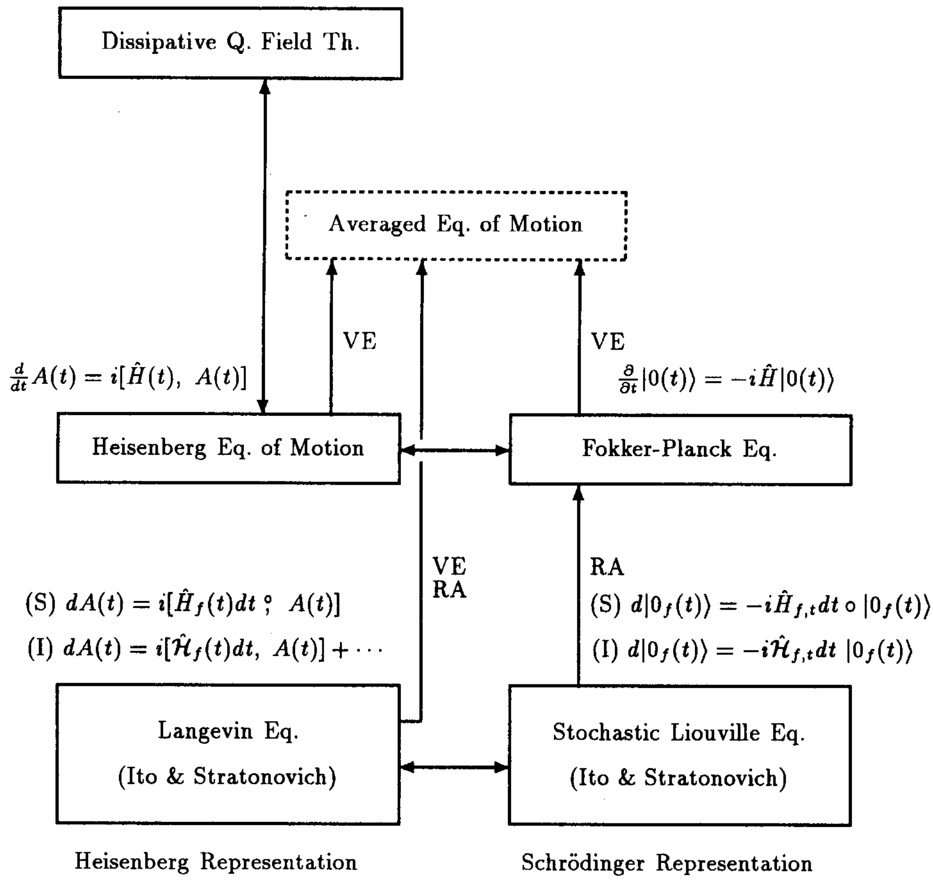

Figure 1: Structure of the Formalism. RA stands for the random average. VE stands for the vacuum expectation. (I) and (S) indicate Ito and Stratonovich types, respectively.

where we used the property

$$|\tilde{n}\rangle = |n\rangle^{\sim}. \tag{3.21}$$

Note that the state $|m, \tilde{n}\rangle$ satisfies

$$|m, \tilde{n}\rangle^{\sim} = |n, \tilde{m}\rangle. \tag{3.22}$$

We can represent the thermal vacuums as

$$|0(t)\rangle = \sum_{n,m} P_{n,m}(t)|n, \tilde{m}\rangle, \tag{3.23}$$

$$\langle 1| = \sum_{n} \langle n, \tilde{n}|. \tag{3.24}$$

The normalization of $\langle 1|0(t)\rangle$ reduces then to

$$\begin{aligned} 1 &= \langle 1|0(t)\rangle = \sum_{k}\sum_{n,m} P_{n,m}(t)\langle k, \tilde{k}|n, \tilde{m}\rangle \\ &= \sum_{k} P_{k,k}(t), \end{aligned} \tag{3.25}$$

where we used the ortho-normality (3.17). With the help of (3.22), we see that the tilde-invariance of the thermal vacuum $|0(t)\rangle$ leads to

$$P^{*}_{m,n}(t) = P_{n,m}(t), \tag{3.26}$$

as follows:

$$\begin{aligned} |0(t)\rangle^{\sim} &= \sum_{n,m} P^{*}_{n,m}(t)|n, \tilde{m}\rangle^{\sim} \\ &= \sum_{n,m} P^{*}_{n,m}(t)|m, \tilde{n}\rangle \\ &= \sum_{m,n} P^{*}_{m,n}(t)|n, \tilde{m}\rangle \\ &= |0(t)\rangle. \end{aligned} \tag{3.27}$$

4. Semi-Free Hat-Hamiltonian

4.1. A Derivation of the Semi-Free Hat-Hamiltonian

The hat-Hamiltonian of the semi-free field is bi-linear in $(a, \tilde{a}, a^{\dagger}, \tilde{a}^{\dagger})$, and is invariant under the phase transformation $a \to ae^{i\theta}$:

$$\hat{H}_t = g_1(t)a^{\dagger}a + g_2(t)\tilde{a}^{\dagger}\tilde{a} + g_3(t)a\tilde{a} + g_4(t)a^{\dagger}\tilde{a}^{\dagger} + g_0(t), \tag{4.1}$$

where $g(t)$'s are time-dependent c-number complex functions.

The operators a, $\tilde{a}^{\dagger}$, etc. satisfy the canonical commutation relation:

$$[a_{\mathbf{k}},\ a^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}, \qquad [\tilde{a}_{\mathbf{k}},\ \tilde{a}^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}. \tag{4.2}$$

The tilde and non-tilde operators are mutually commutative. Throughout this paper, we do not label explicitly the operators a, $\tilde{a}^\dagger$, etc. with a subscript $\mathbf{k}$ for specifying a momentum and/or other degrees of freedom. However, remember that we are dealing with a *dissipative quantum field.*

B1 in section 3. makes (4.1) tildian:

$$\hat{H}_t = \omega(t)(a^\dagger a - \tilde{a}^\dagger \tilde{a}) + i\hat{\Pi}_t, \tag{4.3}$$

with

$$\hat{\Pi}_t = c_1(t)(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + c_2(t) a\tilde{a} + c_3(t) a^\dagger \tilde{a}^\dagger + c_4(t), \tag{4.4}$$

where $\omega(t) = \Re\text{e}\ g_1(t) = -\Re\text{e}\ g_2(t)$, $c_1(t) = \Im\text{m}\ g_1(t) = \Im\text{m}\ g_2(t)$, $c_2(t) = \Im\text{m}\ g_3(t)$, $c_3(t) = \Im\text{m}\ g_4(t)$ and $c_4(t) = \Im\text{m}\ g_0(t)$.

With the help of **Tool 3** in section 3. for $A = a$:

$$\langle 1| a^\dagger = \langle 1| \tilde{a}, \tag{4.5}$$

B2 gives us relations

$$2c_1(t) + c_2(t) + c_3(t) = 0, \qquad c_3(t) + c_4(t) = 0. \tag{4.6}$$

Then, (4.4) reduces to

$$\hat{\Pi}_t = c_1(t)(a^\dagger a + \tilde{a}^\dagger \tilde{a}) + c_2(t) a\tilde{a} - [2c_1(t) + c_2(t)]\, a^\dagger \tilde{a}^\dagger + [2c_1(t) + c_2(t)]\,. \tag{4.7}$$

Let us write down here the Heisenberg equations for a and $a^\dagger$ (see (3.11)):

$$\frac{d}{dt} a(t) = -i\omega(t) a(t) + c_1(t) a(t) - (2c_1(t) + c_2(t))\, \tilde{a}^{\dagger\!\dagger}(t), \tag{4.8}$$

$$\frac{d}{dt} a^{\dagger\!\dagger}(t) = i\omega(t) a^{\dagger\!\dagger}(t) - c_1(t) a^{\dagger\!\dagger}(t) - c_2(t)\tilde{a}(t). \tag{4.9}$$

Since the semi-free hat-Hamiltonian $\hat{H}_t$ is not necessarily hermite, we introduced the symbol $\dagger\!\dagger$ in order to distinguish it from the hermite conjugation $\dagger$. However in the following, we will use $\dagger$ instead of $\dagger\!\dagger$, for simplicity, unless it is confusing. By making use of the Heisenberg equations (4.8) and (4.9), we obtain the equation of motion for a vector $\langle 1| a^\dagger(t) a(t)$ in the form

$$\frac{d}{dt}\langle 1| a^\dagger(t) a(t) = -2\kappa(t) \langle 1| a^\dagger(t) a(t) + i\Sigma^<(t)\langle 1|, \tag{4.10}$$

where we introduced $\kappa(t)$ and $\Sigma^<(t)$ respectively by

$$\kappa(t) = c_1(t) + c_2(t), \tag{4.11}$$

$$\Sigma^<(t) = i[2c_1(t) + c_2(t)]. \tag{4.12}$$

In deriving (4.10), we used **Tool 3** in order to eliminate tilde operators.

Applying the thermal ket vacuum to (4.10), we obtain the equation of motion for the *one-particle distribution function*

$$n(t) = \langle 1| a^{\dagger\!\dagger}(t) a(t) |0\rangle = \langle 1| a^\dagger a |0(t)\rangle, \tag{4.13}$$

as

$$\frac{d}{dt}n(t) = -2\kappa(t)n(t) + i\Sigma^{<}(t). \tag{4.14}$$

The equation (4.14) is the Boltzmann equation of the system. The function $\Sigma^{<}(t)$ is given when the interaction hat-Hamiltonian is specified.

The initial ket-vacuum $|0\rangle = |0(t=0)\rangle$ is specified by

$$a|0\rangle = f\tilde{a}^{\dagger}|0\rangle, \tag{4.15}$$

with a real quantity f. Here, we are neglecting the *initial correlation* [49]. The initial condition of the one-particle distribution function $n = n(t=0)$ can be derived by treating $\langle 1|a\tilde{a}|0\rangle$ as follows. In the first place,

$$\begin{aligned}\langle 1|a\tilde{a}|1\rangle &= \langle 1|afa^{\dagger}|0\rangle \\ &= f\left(\langle 1|a^{\dagger}a|0\rangle + \langle 1|0\rangle\right) \\ &= f(n+1),\end{aligned} \tag{4.16}$$

where we used the tilde conjugate of (4.15) for the first equality, and the canonical commutation relation (4.2) for the second. On the other hand,

$$\begin{aligned}\langle 1|a\tilde{a}|0\rangle &= \langle 1|\tilde{a}a|0\rangle \\ &= \langle 1|a^{\dagger}a|0\rangle \\ &= n.\end{aligned} \tag{4.17}$$

Here, for the first equality, we used **Tool 2**, i.e., the commutativity between the tilde and non-tilde operators, and, for the second equality, **Tool 3** or equivalently (4.5). Equating (4.16) and (4.17), we see that

$$n = \frac{f}{1-f}, \qquad \left(f = \frac{n}{1+n}\right). \tag{4.18}$$

Now let us return to the derivation of the semi-free hat-Hamiltonian. Solving (4.11) and (4.12) with respect to $c_1(t)$ and $c_2(t)$, and substituting (4.14) for $\Sigma^{<}(t)$ into (4.4), we finally arrive at the most general form of the renormalized hat-Hamiltonian $\hat{H}_t$ in the interaction representation [13,14]:[4]

$$\begin{aligned}\hat{H}_t &= \hat{H}_{S,t} - i\kappa(t)\left\{[1+2n(t)]\left(a^{\dagger}a + \tilde{a}^{\dagger}\tilde{a}\right) - 2[1+n(t)]\,a\tilde{a} - 2n(t)a^{\dagger}\tilde{a}^{\dagger}\right\} \\ &\quad -i\frac{d}{dt}n(t)\bar{a}^{\mu}\tau^{\mu\nu}a^{\nu} - i2\kappa(t)n(t) \\ &= [\omega(t) - i\kappa(t)]\,\bar{a}^{\mu}a^{\mu} - i\left[\frac{d}{dt} + 2\kappa(t)\right]\bar{a}^{\mu}n(t)^{\mu\nu}a^{\nu} + \omega(t) + i\kappa(t),\end{aligned} \tag{4.19}$$

where[5]

$$\hat{H}_{S,t} = \omega(t)\left(a^{\dagger}a - \tilde{a}^{\dagger}\tilde{a}\right), \tag{4.20}$$

[4]Throughout this paper, we confine ourselves to the case of boson fields, for simplicity. The extension to the case of fermion fields are rather straightforward.

[5]The following formulation is valid for the cases where $H_{S,t}$ has non-linear terms within the conventional treatment of damping operators. For their non-conventional treatment, refer to [50]-[55],[43],[44].

and $\frac{d}{dt}n(t)$ is given by (4.14). Here, we introduced the thermal doublet notation: $a^{\mu=1} = a$, $a^{\mu=2} = \tilde{a}^{\dagger}$ and $\bar{a}^{\mu=1} = a^{\dagger}$, $\bar{a}^{\mu=2} = -\tilde{a}$, and the matrices $\tau^{\mu\nu}$: $\tau^{11} = \tau^{21} = 1$, $\tau^{12} = \tau^{22} = -1$, and

$$n(t)^{\mu\nu} = \langle 1|\bar{a}(t)^{\nu}a(t)^{\mu}|0\rangle = \begin{pmatrix} n(t) & -n(t) \\ 1+n(t) & -[1+n(t)] \end{pmatrix}. \quad (4.21)$$

The thermal doublet notation in the interaction representation was introduced by $a(t)^{\mu=1} = a(t)$, $a(t)^{\mu=2} = \tilde{a}^{\dagger\dagger}(t)$ and $\bar{a}(t)^{\mu=1} = a^{\dagger\dagger}(t)$, $\bar{a}(t)^{\mu=2} = -\tilde{a}(t)$.

4.2. Fokker-Planck Equation

We will call in the following the Schrödinger equation

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}_t^{tot}|0(t)\rangle, \quad (4.22)$$

the Fokker-Planck equation for coarse grained systems. The hat-Hamiltonian $\hat{H}_t^{tot}$ consists of the semi-free hat-Hamiltonian (4.19) and an interaction hat-Hamiltonian. Some general remarks on the interaction hat-Hamiltonian will be given in section 5..

The equation of motion for the averaged quantity $\langle 1|A|0(t)\rangle$ is derived with the help of the Fokker-Planck equation as

$$\frac{d}{dt}\langle 1|A|0(t)\rangle = -i\langle 1|A\hat{H}_t^{tot}|0(t)\rangle. \quad (4.23)$$

The same equation can be also derived by means of the Heisenberg equation

$$\frac{d}{dt}A(t) = i[\hat{H}^{tot}(t),\ A(t)], \quad (4.24)$$

by taking its vacuum expectation:

$$\frac{d}{dt}\langle 1|A(t)|0\rangle = i\langle 1|[\hat{H}^{tot}(t),\ A(t)]|0\rangle. \quad (4.25)$$

Here,

$$A(t) = V^{tot}(t)^{-1}AV^{tot}(t), \quad (4.26)$$
$$\hat{H}^{tot}(t) = V^{tot}(t)^{-1}\hat{H}_t^{tot}V^{tot}(t), \quad (4.27)$$

are the operators in the Heisenberg representation, and $\hat{V}^{tot}(t)$ is defined by

$$\frac{d}{dt}\hat{V}^{tot}(t) = -i\hat{H}_t^{tot}\hat{V}^{tot}(t), \qquad \left(i\hat{H}_t^{tot}\right)^{\sim} = i\hat{H}_t^{tot}, \quad (4.28)$$

with $\hat{V}^{tot}(0) = 1$.

We would like to emphasize here that the existence of the Heisenberg equation of motion (4.24) for coarse grained operators is one of the notable features of NETFD. This enabled us to construct a *canonical formalism of the dissipative quantum field theory*, where the coarse grained operators

$a(t)$ etc. in the Heisenberg representation preserve the equal-time canonical commutation relation

$$[a(t),\ a^\dagger(t)] = 1, \qquad [\tilde{a}(t),\ \tilde{a}^\dagger(t)] = 1. \tag{4.29}$$

Note that we have an equation of motion for a vector $\langle 1|A(t)$:

$$\frac{d}{dt}\langle 1|A(t) = i\langle 1|[\hat{H}^{tot}(t),\ A(t)], \tag{4.30}$$

in terms of only non-tilde operators with the help of the condition (4.5). This equation will have some important meanings later on.

4.3. Operators in the Interaction Representation

The operators in the interaction representation, appeared in the Heisenberg equations (4.8) and (4.9):

$$\frac{d}{dt}a(t) = -\left[i\omega(t)+\kappa(t)\right]a(t) + i\Sigma^<(t)\left[\tilde{a}^{\dagger\!\dagger}(t) - a(t)\right], \tag{4.31}$$

$$\frac{d}{dt}a^{\dagger\!\dagger}(t) = \left[i\omega(t)+\kappa(t)\right]a^{\dagger\!\dagger}(t) - 2\kappa(t)\tilde{a}(t) + i\Sigma^<(t)\left[a^{\dagger\!\dagger}(t) - \tilde{a}(t)\right] \tag{4.32}$$

are defined by

$$a(t) = \hat{V}^{-1}(t)a\hat{V}(t), \qquad \tilde{a}^{\dagger\!\dagger}(t) = \hat{V}^{-1}(t)\tilde{a}^\dagger\hat{V}(t), \tag{4.33}$$

where

$$\frac{d}{dt}\hat{V}(t) = -i\hat{H}_t\hat{V}(t), \qquad \left(i\hat{H}_t\right)^\sim = i\hat{H}_t, \tag{4.34}$$

with $\hat{V}(0)=1$. Since the semi-free hat-Hamiltonian $\hat{H}_t$ satisfies

$$\langle 1|\hat{H}_t = 0, \tag{4.35}$$

(see **B2** in section 3.), we understand that the semi-free operators satisfy

$$\langle 1|a^\dagger(t) = \langle 1|\tilde{a}(t), \qquad a(t)|0\rangle = \frac{n(t)}{1+n(t)}\tilde{a}^\dagger(t)|0\rangle, \tag{4.36}$$

(see (4.5) for the former, and (4.15) with (4.18) for the latter).

4.4. Annihilation and Creation Operators

Let us introduce the annihilation and creation operators, $\gamma(t)^{\mu=1} = \gamma(t)$, $\gamma(t)^{\mu=2} = \tilde{\gamma}^{\ddagger}(t)$ and $\bar{\gamma}(t)^{\mu=1} = \gamma^{\ddagger}(t)$, $\bar{\gamma}(t)^{\mu=2} = -\tilde{\gamma}(t)$, by

$$\gamma(t)^\mu = B(t)^{\mu\nu}a(t)^\nu, \qquad \bar{\gamma}(t)^\mu = \bar{a}(t)^\nu B^{-1}(t)^{\nu\mu}, \tag{4.37}$$

with the *time-dependent Bogoliubov transformation:*[6]

$$B(t)^{\mu\nu} = \begin{pmatrix} 1+n(t) & -n(t) \\ -1 & 1 \end{pmatrix}. \tag{4.38}$$

[6]There is a minor change in the normalization of the time-dependent Bogoliubov transformation compared with the original definition given in [1,2], [12]-[14]. This change makes the expression $\mathcal{G}(t,t')^{\mu\nu}$ simpler, and is essential in the formulation of the stochastic Liouville equation introduced below.

The annihilation and creation operators have the properties (cf. (4.36))

$$\gamma(t)|0\rangle = 0, \qquad \langle 1|\tilde{\gamma}^{\dagger}(t) = 0. \tag{4.39}$$

The equation of motion for the thermal doublet $\gamma(t)^{\mu}$ is derived as

$$\begin{aligned} \frac{d}{dt}\gamma(t)^{\mu} &= \frac{dB(t)^{\mu\nu}}{dt}a(t)^{\nu} + B(t)^{\mu\nu}\frac{d}{dt}a(t)^{\nu} \\ &= -i\left[\omega(t)\delta^{\mu\nu} - i\kappa(t)\tau_3^{\mu\nu}\right]\gamma(t)^{\nu}, \end{aligned} \tag{4.40}$$

where $\delta^{\mu\nu}$ is the Kronecker delta, and the matrix $\tau_3^{\mu\nu}$ is defined by $\tau_3^{11} = -\tau_3^{22} = 1$, $\tau_3^{12} = \tau_3^{21} = 0$. The solution of (4.40) has the property

$$\gamma(t)^{\mu} = \exp\left\{\int_{t'}^{t} ds\left[-i\omega(s)\delta^{\mu\nu} - \kappa(s)\tau_3^{\mu\nu}\right]\right\}\gamma(t')^{\nu}. \tag{4.41}$$

4.5. Two-Point Function

The time-ordered two-point function $G(t,t')^{\mu\nu}$ has the form

$$\begin{aligned} G(t,t')^{\mu\nu} &= -i\langle 1|T\left[a(t)^{\mu}\bar{a}(t')^{\nu}\right]|0\rangle \\ &= \left[B^{-1}(t)\mathcal{G}(t,t')B(t')\right]^{\mu\nu}, \end{aligned} \tag{4.42}$$

where

$$\mathcal{G}(t,t')^{\mu\nu} = -i\langle 1|T\left[\gamma(t)^{\mu}\bar{\gamma}(t')^{\nu}\right]|0\rangle = \begin{pmatrix} G^R(t,t') & 0 \\ 0 & G^A(t,t') \end{pmatrix}, \tag{4.43}$$

with

$$G^R(t,t') = -i\theta(t-t')\exp\left\{\int_{t'}^{t} ds\left[-i\omega(s) - \kappa(s)\right]\right\}, \tag{4.44}$$

$$G^A(t,t') = i\theta(t'-t)\exp\left\{\int_{t'}^{t} ds\left[-i\omega(s) + \kappa(s)\right]\right\}. \tag{4.45}$$

In deriving the above expression, we used the elements of the solution (4.41) with some algebraic manipulations.

4.6. Miscellaneous

The representation space (the thermal space) of NETFD is the vector space spanned by the set of bra and ket state vectors which are generated, respectively, by cyclic operations of the annihilation operators $\gamma(t)$ and $\tilde{\gamma}(t)$ on $\langle 1|$, and of the creation operators $\gamma^{\dagger}(t)$ and $\tilde{\gamma}^{\dagger}(t)$ on $|0\rangle$.

The normal product is defined by means of the annihilation and the creation operators, i.e. $\gamma^{\dagger}(t)$, $\tilde{\gamma}^{\dagger}(t)$ stand to the left of $\gamma(t)$, $\tilde{\gamma}(t)$. The process, rewriting physical operators in terms of the annihilation and creation operators, leads to a Wick-type formula, which in turn leads to Feynman-type diagrams for multi-point functions in the renormalized interaction representation. The internal line in the Feynman-type diagrams is the unperturbed two-point function (4.42).

5. Inclusion of External Fields

5.1. Hermitian Interaction Hat-Hamiltonian

The simplest interaction hat-Hamiltonian may be

$$\hat{H}'_t = H'_t - \tilde{H}'_t, \tag{5.1}$$

with a hermitian interaction Hamiltonian

$$H'_t = i\left[a^\dagger b(t) - b^\dagger(t) a\right], \tag{5.2}$$

where $b(t)$ and $b^\dagger(t)$ are operators in the external system and are assumed to commute with the operators a, $a^\dagger$ etc. of the relevant system. Note that the hat-Hamiltonian (5.1) is hermite. The tilde and non-tilde operators of the external system are related with each other by

$$\langle | b^\dagger(t) = \langle | \tilde{b}(t). \tag{5.3}$$

Applying the bra-vacuum $\langle 1|$ for the relevant system on (5.1), we have

$$\langle 1|\hat{H}'_t = -i\langle 1| \left[a\beta^\ddagger(t) + a^\dagger \tilde{\beta}^\ddagger(t)\right]. \tag{5.4}$$

Here we introduced a new operator

$$\beta^\ddagger(t) = b^\dagger(t) - \tilde{b}(t), \tag{5.5}$$

which annihilates the bra-vacuum $\langle |$ for the external system (cf. (5.3)):

$$\langle |\beta^\ddagger = 0. \tag{5.6}$$

If we apply the bra-vacuum $\langle |$ on (5.2) in addition to $\langle 1|$, we observe that

$$\langle\!\langle 1|\hat{H}'_t = 0, \tag{5.7}$$

where we introduced

$$\langle\!\langle 1| = \langle | \cdot \langle 1|. \tag{5.8}$$

The above investigation shows that a simple introduction of an interaction hat-Hamiltonian of the form (5.1) violates the conservation of probability within the relevant system. It can be understood by considering the Schrödinger equation

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\left(\hat{H}_t + \hat{H}'_t\right)|0(t)\rangle, \tag{5.9}$$

and apply $\langle 1|$. Note that the conservation of probability is satisfied for the total system, i.e., the relevant system and the external system, as can be seen by applying $\langle\!\langle 1|$ on (5.9).

5.2. Non-Hermitian Interaction Hat-Hamiltonian

Let us consider if we can have an interaction hat-Hamiltonian which satisfies the conservation of probability *within* the relevant system. We assume that the interaction hat-Hamiltonian is globally gauge invariant and bilinear:

$$\begin{aligned}\hat{H}_t'' &= i\Big\{h_1 a^\dagger b(t) + h_2 a^\dagger \tilde{b}^\dagger(t) + h_3 \tilde{a}\, b(t) + h_4 \tilde{a}\, \tilde{b}^\dagger(t) \\ &\quad + h_5 \tilde{a}^\dagger \tilde{b}(t) + h_6 \tilde{a}^\dagger b^\dagger(t) + h_7 a\, \tilde{b}(t) + h_8 a\, b^\dagger(t)\Big\}, \qquad (5.10)\end{aligned}$$

where the quantities h's are time-independent complex c-numbers.

The tildian (**B1** in section 3.):

$$\left(i\hat{H}_t''\right)^\sim = i\hat{H}_t'', \qquad (5.11)$$

gives us

$$h_1^* = h_5, \quad h_2^* = h_6, \quad h_3^* = h_7, \quad h_4^* = h_8. \qquad (5.12)$$

The requirement that the Schrödinger equation

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\left(\hat{H}_t + \hat{H}_t''\right)|0(t)\rangle, \qquad (5.13)$$

has the characteristics of the conservation of probability within the relevant system:

$$\langle 1|\hat{H}_t'' = 0, \qquad (5.14)$$

leads us to the relations

$$h_1 + h_3 = 0, \quad h_2 + h_4 = 0. \qquad (5.15)$$

With (5.12) and (5.15), (5.10) reduces to

$$\hat{H}_t'' = i\left[\alpha^{\ddagger}\beta(t) + \text{t.c.}\right], \qquad (5.16)$$

where we introduced

$$\begin{aligned}\alpha^{\ddagger} &= a^\dagger - \tilde{a}, &(5.17)\\ \beta(t) &= h_1 b(t) + h_2 \tilde{b}^\dagger(t). &(5.18)\end{aligned}$$

Let us consider the moments

$$\begin{aligned}\left\langle \beta(t)\tilde{\beta}(t)\right\rangle &= (h_1 + h_2)\left\{h_1^*\left\langle b^\dagger(t)b(t)\right\rangle + h_2^*\left\langle b(t)b^\dagger(t)\right\rangle\right\}, &(5.19)\\ \left\langle \tilde{\beta}(t)\beta(t)\right\rangle &= (h_1^* + h_2^*)\left\{h_1\left\langle b^\dagger(t)b(t)\right\rangle + h_2\left\langle b(t)b^\dagger(t)\right\rangle\right\}, &(5.20)\end{aligned}$$

where we used **Tool 2** and **Tool 3** in section 3. for $b(t)$, $b^\dagger(t)$ etc.. We are using the symbol $\langle\cdots\rangle = \langle|\cdots|t\rangle$ without specifying the dynamics which determines the ket-vacuum $|t\rangle$ of the external system. For the present purpose, the details of its dynamics are not required. With the further use of the property **Tool 2** of the commutativity, $\left\langle\beta(t)\tilde{\beta}(t)\right\rangle = \left\langle\tilde{\beta}(t)\beta(t)\right\rangle$, gives us the relations

$$(h_1 + h_2)h_1^* = (h_1^* + h_2^*)h_1, \qquad (h_1 + h_2)h_2^* = (h_1^* + h_2^*)h_2. \qquad (5.21)$$

which reduce to

$$h_1^* h_2 = h_1 h_2^* = (h_1^* h_2)^*, \tag{5.22}$$

and allow us to put

$$h_1 = \mu e^{i\theta}, \quad h_2 = \nu e^{i\theta}, \tag{5.23}$$

where $\mu = |h_1|$ and $\nu = |h_2|$.

The vector $\langle|\beta(t)$ is calculated as

$$\langle|\beta(t) = (\mu + \nu) e^{i\theta} \langle|b(t). \tag{5.24}$$

The further requirement that the *norm* of $\langle|\beta(t)$ should be equal to that of $\langle|b(t)$, i.e.

$$\|\langle|\beta(t)\| = \|\langle|b(t)\|, \tag{5.25}$$

leads us to the relation

$$\mu + \nu = 1. \tag{5.26}$$

This requirement indicates that the *intensities* of the external operators $\beta(t)$ and $b(t)$ are same. Putting the phase factor $e^{i\theta}$ on $b(t)$ and $\tilde{b}^\dagger(t)$, we have the non-hermitian interaction hat-Hamiltonian (5.16) with $\alpha^\ddagger$ defined by (5.17), and

$$\beta(t) = \mu b(t) + \nu \tilde{b}^\dagger(t), \tag{5.27}$$

with the real numbers μ and ν satisfying (5.26). The creation operator $\alpha^\ddagger$ annihilate the ket-vacuum $\langle 1|$:

$$\langle 1|\alpha^\ddagger = 0. \tag{5.28}$$

The above requirement for the norm makes the operators $\beta^\ddagger(t)$ defined by (5.5) and $\beta(t)$ in (5.27) canonical operators:

$$[\beta(t),\ \beta^\ddagger(t)] = 1. \tag{5.29}$$

5.3. Relation between the Two Interaction Hat-Hamiltonian

Note that the hermitian hat-Hamiltonian $\hat{H}_t'$ of (5.1) and the non-hermitian one $\hat{H}_t''$ of (5.16) are related each other by

$$\hat{H}_t' = \hat{H}_t'' - i\left[\alpha\beta^\ddagger(t) + \text{t.c.}\right], \tag{5.30}$$

where we introduced

$$\alpha = \mu a + \nu \tilde{a}^\dagger, \tag{5.31}$$

which forms a canonical set with $\alpha^\ddagger$ in (5.17):

$$[\alpha,\ \alpha^\ddagger] = 1. \tag{5.32}$$

6. $\hat{S}$-matrix

Let us consider the Fokker-Planck equation (4.22):

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}_t^{tot}|0(t)\rangle. \tag{6.1}$$

When the hat-Hamiltonian $\hat{H}_t^{tot}$ in (6.1) can be divided into two parts as

$$\hat{H}_t^{tot} = \hat{H}_t + \hat{H}_t', \tag{6.2}$$

we can introduce the thermal vacuum ket-vector in the interaction representation as

$$|0(t)\rangle_I = \hat{V}^{-1}(t)|0(t)\rangle, \tag{6.3}$$

with $\hat{V}(t)$ defined by (4.34). The Fokker-Planck equation (6.1) then reduces to

$$\frac{\partial}{\partial t}|0(t)\rangle_I = -i\hat{H}'(t)|0(t)\rangle_I, \tag{6.4}$$

where we introduced

$$\hat{H}'(t) = \hat{V}^{-1}(t)\hat{H}_t'\hat{V}(t). \tag{6.5}$$

This can be formally solved in terms of the state of the system at an initial time t_0 as

$$|0(t)\rangle_I = \hat{S}(t,t_0)|0(t_0)\rangle_I, \tag{6.6}$$

with

$$\hat{S}(t,t_0) = \hat{S}(t)\hat{S}^{-1}(t_0), \tag{6.7}$$

where $\hat{S}(t)$ is specified by

$$\frac{d}{dt}\hat{S}(t) = -i\hat{H}'(t)\hat{S}(t), \tag{6.8}$$

with the initial condition $\hat{S}(t_0) = 1$. The thermal vacuum $|0(t)\rangle$ in the Schrödinger representation can be expressed by means of $\hat{S}(t,t_0)$ as

$$|0(t)\rangle = \hat{V}(t)\hat{S}(t,t_0)\hat{V}^{-1}(t_0)|0(t_0)\rangle. \tag{6.9}$$

Since $\hat{H}_t$ should satisfy

$$\langle 1|\hat{H}_t = 0, \tag{6.10}$$

the interaction Hamiltonian $\hat{H}'(t)$ in the interaction representation has the property

$$\langle 1|\hat{H}'(t) = 0. \tag{6.11}$$

Here, in this section, the thermal bra-vacuum $\langle 1|$ is assumed to be of whole the system (cf. subsection 5.1.). Then, (6.8) gives us

$$\langle 1|\hat{S}(t) = \langle 1|\hat{S}(t_0), \tag{6.12}$$

leading to

$$\langle 1|\hat{S}(t,t_0) = \langle 1|. \tag{6.13}$$

This is a manifestation of the conservation of probability, $\langle 1|0(t)\rangle = 1$. Note that the thermal bra vacuum in the interaction representation $_I\langle 1|$ becomes the same as the one in the Schrödinger representation:

$$_I\langle 1| = \langle 1|\hat{V}(t) = \langle 1|. \tag{6.14}$$

The overlap $\langle \ell, \tilde{\ell}|0(t)\rangle$ is given by

$$\langle \ell, \tilde{\ell}|0(t)\rangle = \sum_n \langle \ell, \tilde{\ell}|\hat{V}(t)\hat{S}(t,t_0)\hat{V}^{-1}(t_0)|n,\tilde{n}\rangle P_{n,n}(t_0), \tag{6.15}$$

where we put for the initial state

$$|0(t_0)\rangle = \sum_n P_{n,n}(t_0)|n,\tilde{n}\rangle, \tag{6.16}$$

with

$$\sum_n P_{n,n}(t_0) = 1, \tag{6.17}$$

which is consistent with the normalization $\langle 1|0(t_0)\rangle = 1$. We see that

$$\begin{aligned} \sum_\ell \langle \ell, \tilde{\ell}|0(t)\rangle &= \sum_\ell \sum_n \langle \ell, \tilde{\ell}|\hat{V}(t)\hat{S}(t,t_0)\hat{V}^{-1}(t_0)|n,\tilde{n}\rangle P_{n,n}(t_0) \\ &= \sum_n \langle 1|\hat{S}(t,t_0)\hat{V}^{-1}(t_0)|n,\tilde{n}\rangle P_{n,n}(t_0) \\ &= \sum_n P_{n,n}(t_0) \\ &= 1, \end{aligned} \tag{6.18}$$

where we used (6.10), (6.13) and

$$\langle 1|m,\tilde{n}\rangle = \sum_\ell \langle \ell, \tilde{\ell}|m,\tilde{n}\rangle = \sum_\ell \delta_{\ell,m}\delta_{\ell,n} = \delta_{m,n}. \tag{6.19}$$

Although the interaction hat-Hamiltonian $\hat{H}'_t$ has the structure (5.1):

$$\hat{H}'_t = H'_t - \tilde{H}'_t, \tag{6.20}$$

the hat-Hamiltonian $\hat{H}_t$ does *not*, in general. Therefore, one needs to calculate the matrix elements

$$\langle \ell, \tilde{\ell}|\hat{V}(t)\hat{S}(t,t_0)\hat{V}^{-1}(t_0)|n,\tilde{n}\rangle, \tag{6.21}$$

in order to obtain the overlap (6.15). Expanding the $\hat{S}$-*matrix* $\hat{S}(t,t_0)$ with respect to the order of $\hat{H}'_t$ as

$$\hat{S}(t,t_0) = \sum_{n=0}^{\infty} \hat{S}^{(n)}(t,t_0), \tag{6.22}$$

we can deal with any order of processes induced by $\hat{H}'_t$. See Appendix C for the first order process (the linear response) as a simplest example. For more complicated process such as the second order transient resonant light scattering, see [47].

Note that when the hat-Hamiltonian $\hat{H}_t$ is independent of time, and has the structure

$$\hat{H} = H - \tilde{H}, \tag{6.23}$$

in addition to $\hat{H}'_t$, the overlap (6.15) becomes the well-known form:

$$\begin{aligned} \langle \ell, \tilde{\ell} | 0(t) \rangle &= \sum_n \langle \ell | S(t,t_0) | n \rangle \langle \tilde{\ell} | \tilde{S}(t,t_0) | \tilde{n} \rangle P_{n,n}(t_0) \\ &= \sum_n \left| \langle \ell | S(t,t_0) | n \rangle \right|^2 P_{n,n}(t_0), \end{aligned} \tag{6.24}$$

where we assumed that $|n, \tilde{n}\rangle$ is an eigen-function of H with an real eigen-value E_n:

$$H|n,\tilde{n}\rangle = E_n|n,\tilde{n}\rangle, \qquad \tilde{H}|n,\tilde{n}\rangle = E_n|n,\tilde{n}\rangle. \tag{6.25}$$

In the case of (6.23),

$$\hat{S}(t,t_0) = S(t,t_0)\tilde{S}(t,t_0), \tag{6.26}$$

where $S(t,t_0)$ contains only non-tilde operators and is an unitary operator.

7. Generating Functional Method

Let us treat further the Fokker-Planck equation (4.22) [17]:

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}_t^{tot}|0(t)\rangle, \tag{7.1}$$

with

$$\hat{H}_t^{tot} = \hat{H}_t + \hat{H}'_t, \tag{7.2}$$

where $\hat{H}_t$ is given by (4.19), and $\hat{H}'_t$ is defined by

$$\hat{H}'_t = \bar{K}(t)^\mu a^\mu + \bar{a}^\mu K(t)^\mu = \bar{K}_\gamma(t)^\mu \gamma^\mu + \bar{\gamma}^\mu K_\gamma(t)^\mu, \tag{7.3}$$

(cf. (5.1)). The operators γ^μ and $\bar{\gamma}^\mu$ are defined through

$$\gamma^\mu = \hat{V}(t)\gamma(t)^\mu \hat{V}^{-1}(t), \quad \bar{\gamma}^\mu = \hat{V}(t)\bar{\gamma}(t)^\mu \hat{V}^{-1}(t), \tag{7.4}$$

(see (4.37)). The thermal doublet notation for the c-number external fields has been introduced by $K(t)^{\mu=1} = K(t)$, $K(t)^{\mu=2} = \tilde{K}^*(t)$ and $\bar{K}(t)^{\mu=1} = K(t)^*$, $\bar{K}(t)^{\mu=2} = -\tilde{K}(t)$. We see the relation

$$K_\gamma(t)^\mu = B(t)^{\mu\nu}K(t)^\nu, \qquad \bar{K}_\gamma(t) = \bar{K}(t)^\nu B^{-1}(t)^{\nu\mu}, \tag{7.5}$$

with (4.38) for $B(t)^{\mu\nu}$.

The generating functional for the system is defined by

$$Z[K,\bar{K}] = \langle 1|\hat{S}(\bar{t})|0\rangle, \tag{7.6}$$

where $\hat{S}(t)$ satisfies (6.8) with the initial condition $\hat{S}(0) = 1$.

Taking the functional derivative of the generating functional (7.6), we have

$$\delta \ln Z[K, \tilde{K}] = -i \int_0^{\bar{t}} dt \left[\delta \bar{K}_\gamma(t)^\mu \langle \gamma(t)^\mu \rangle + \langle \bar{\gamma}(t)^\mu \rangle \delta K_\gamma(t)^\mu\right], \tag{7.7}$$

where $\langle \gamma(t)^\mu \rangle$ and $\langle \bar{\gamma}(t)^\mu \rangle$ are defined by

$$\langle \gamma(t)^\mu \rangle = i \frac{\delta}{\delta \bar{K}_\gamma(t)^\mu} \ln Z[K, \tilde{K}] = \langle 1 | T \left[\hat{S}(\bar{t}) \gamma(t)^\mu\right] |0\rangle, \tag{7.8}$$

$$\langle \bar{\gamma}(t)^\mu \rangle = i \frac{\delta}{\delta K_\gamma(t)^\mu} \ln Z[K, \tilde{K}] = \langle 1 | T \left[\hat{S}(\bar{t}) \bar{\gamma}(t)^\mu\right] |0\rangle. \tag{7.9}$$

The equation of motion for $\langle \gamma(t)^\mu \rangle$ [17]:

$$\frac{d}{dt} \langle \gamma(t)^\mu \rangle = -\left[i\omega(t)\delta^{\mu\nu} + \kappa(t)\tau_3^{\mu\nu}\right] \langle \gamma(t)^\nu \rangle - iK_\gamma(t)^\mu, \tag{7.10}$$

with the boundary conditions

$$\begin{aligned} \langle \gamma(0)^{\mu=1} \rangle &= \langle \gamma(0) \rangle = 0, & \langle \gamma(\bar{t})^{\mu=2} \rangle &= \langle \tilde{\gamma}^{\dagger}(\bar{t}) \rangle = 0, \\ \langle \bar{\gamma}(\bar{t})^{\mu=1} \rangle &= \langle \gamma^{\dagger}(\bar{t}) \rangle = 0, & \langle \bar{\gamma}(0)^{\mu=2} \rangle &= -\langle \tilde{\gamma}(0) \rangle = 0, \end{aligned} \tag{7.11}$$

can be solved in the form

$$\langle \gamma(t)^\mu \rangle = \int_0^{\bar{t}} dt' \; \mathcal{G}(t, t')^{\mu\nu} K_\gamma(t')^\nu, \tag{7.12}$$

where $\mathcal{G}(t, t')^{\mu\nu}$ is given by (4.43). The boundary conditions in (7.11) are derived by the thermal state conditions (4.39).

Substituting (7.12) into (7.7), we finally obtain [17]

$$\begin{aligned} Z[K, \tilde{K}] &= \exp\left[-i \int_0^{\bar{t}} dt \int_0^{\bar{t}} dt' \bar{K}_\gamma(t)^\mu \mathcal{G}(t, t')^{\mu\nu} K_\gamma(t')^\nu\right] \\ &= \exp\left[-i \int_0^{\bar{t}} dt \int_0^{\bar{t}} dt' \bar{K}(t)^\mu G(t, t')^{\mu\nu} K(t')^\nu\right]. \end{aligned} \tag{7.13}$$

This expression for an open system was derived first by Schwinger by means of the closed-time path method [18] (see also [19,20]).

The derivation of the generating functional shown in this section reveals the relation between the quantum operator formalism of dissipative fields (NETFD) and their path integral formalism [18]. Note that the existence of a quantum operator formalism for dissipative fields had never been realized before NETFD was constructed.

8. Stochastic Semi-Free Hat-Hamiltonian

8.1. Quantum Stochastic Liouville Equations

8.1.1. Ito Type

Let us derive the general form of the semi-free hat-Hamiltonian $\hat{\mathcal{H}}_{f,t}$ for a stochastic Liouville equation

$$d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t}dt\ |0_f(t)\rangle, \tag{8.1}$$

which will turn out to be an Ito type [56] stochastic differential equation later. The hat-Hamiltonian for the *stochastic semi-free* field is bi-linear in a, $a^\dagger$, $dF(t)$, $dF^\dagger(t)$ and their tilde conjugates, and is invariant under the phase transformation $a \to ae^{i\theta}$, and $dF(t) \to dF(t)\ e^{i\theta}$. Here, a, $a^\dagger$ and their tilde conjugates are stochastic operators of a relevant system satisfying the canonical commutation relation

$$[a,\ a^\dagger] = 1, \qquad [\tilde{a},\ \tilde{a}^\dagger] = 1, \tag{8.2}$$

whereas $dF(t)$, $dF^\dagger(t)$ and their conjugates are random force operators. The tilde and non-tilde operators are related with each other by the relations

$$\langle 1|a^\dagger = \langle 1|\tilde{a}, \tag{8.3}$$

$$\langle|dF^\dagger(t) = \langle|d\tilde{F}(t), \tag{8.4}$$

where $\langle 1|$ and $\langle|$ are respectively the thermal bra-vacuum of the relevant system and of the random force.

We will employ the characteristics of the stochastic Liouville equation [5]-[8] of classical systems to quantum cases, i.e., the stochastic distribution function satisfies the conservation of probability within the phase space of a relevant system. This means in NETFD that

$$\langle 1|0_f(t)\rangle = 1, \tag{8.5}$$

leading to

$$\langle 1|\hat{\mathcal{H}}_{f,t}dt = 0. \tag{8.6}$$

Here the thermal bra-vacuum $\langle 1|$ is of the relevant system.

¿From the investigation in subsection 5.2. (cf. (5.16)), we know that the required form of the hat-Hamiltonian should be

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}_t dt + i\left\{\alpha^{\natural}dW(t) + \text{t.c.}\right\}, \tag{8.7}$$

where $\hat{H}_t$ is given by (4.19), i.e.

$$\hat{H}_t = \hat{H}_{S,t} + i\hat{\Pi}_t, \tag{8.8}$$

with

$$\hat{\Pi}_t = -\kappa(t)\left(\alpha^{\natural}\alpha + \text{t.c.}\right) + \left\{2\kappa(t)\left[n(t)+\nu\right] + \frac{d}{dt}n(t)\right\}\alpha^{\natural}\tilde{\alpha}^{\natural}. \tag{8.9}$$

We introduced a set of canonical stochastic operators[7]

$$\alpha = \mu a + \nu \tilde{a}^\dagger, \qquad \alpha^{\ddagger} = a^\dagger - \tilde{a}, \tag{8.10}$$

with $\mu + \nu = 1$, which satisfy the commutation relation

$$[\alpha,\ \alpha^{\ddagger}] = 1. \tag{8.11}$$

The stochastic operators a, $a^\dagger$, $\tilde{a}$ and $\tilde{a}^\dagger$ are of the Schrödinger representation satisfying[8]

$$[a,\ a^\dagger] = 1, \qquad [\tilde{a},\ \tilde{a}^\dagger] = 1. \tag{8.12}$$

The random force operators $dW(t), d\tilde{W}(t)$ are of the quantum stochastic Wiener process satisfying

$$\begin{aligned}
\left\langle dW(t)\right\rangle &= \left\langle d\tilde{W}(t)\right\rangle = 0, && (8.13)\\
\left\langle dW(t)dW(s)\right\rangle &= \left\langle d\tilde{W}(t)d\tilde{W}(s)\right\rangle = 0, && (8.14)\\
\left\langle dW(t)d\tilde{W}(s)\right\rangle &= \left\langle d\tilde{W}(s)dW(t)\right\rangle \\
&= \left\{2\kappa(t)\left[n(t)+\nu\right] + \frac{d}{dt}n(t)\right\}\delta(t-s)dtds, && (8.15)
\end{aligned}$$

with $\langle\cdots\rangle = \langle|\cdots|\rangle$, where the random force operator $dW(t)$ is defined by

$$dW(t) = \mu dF(t) + \nu d\tilde{F}^\dagger(t), \tag{8.16}$$

with $\mu + \nu = 1$. The *original* random force operators $dF(t)$ and $dF^\dagger(t)$ are of the *non-stationary* Gaussian white process, which is defined by

$$\begin{aligned}
\left\langle dF(t)\right\rangle &= \left\langle d\tilde{F}(t)\right\rangle = \left\langle dF^\dagger(t)\right\rangle = \left\langle d\tilde{F}^\dagger(t)\right\rangle = 0, && (8.17)\\
\left\langle dF^\dagger(t)dF(s)\right\rangle &= \left[2\kappa(t)n(t) + \frac{d}{dt}n(t)\right]\delta(t-s)dtds, && (8.18)\\
\left\langle dF(t)dF^\dagger(s)\right\rangle &= \left\{2\kappa(t)\left[n(t)+1\right] + \frac{d}{dt}n(t)\right\}\delta(t-s)dtds, && (8.19)
\end{aligned}$$

and zero for other combinations (see Appendix E for derivation). The one-particle distribution function $n(t)$ satisfies the Boltzmann equation (4.14). Within the stochastic convergence, these correlations reduce to[9]

$$dW(t) = d\tilde{W}(t) = 0, \tag{8.21}$$

[7] The expression of $\hat{\Pi}_t$ was given here by means of a set of canonical stochastic operators α, $\alpha^{\ddagger}$ and their tilde conjugates.

[8] We use the same notation a etc. for the stochastic semi-free operators as those for the coarse grained semi-free operators. We expect that there will be no confusion between them.

[9] For equal time $t = s$, (8.23) reads

$$dW(t)d\tilde{W}(t) = d\tilde{W}(t)dW(t) = \left[i\Sigma^{<}(t) + 2\nu\kappa(t)\right]dt. \tag{8.20}$$

$$
\begin{aligned}
dW(t)dW(s) &= d\tilde{W}(t)d\tilde{W}(s) = 0, &(8.22)\\
dW(t)d\tilde{W}(s) &= d\tilde{W}(s)dW(t)\\
&= \left\{2\kappa(t)\left[n(t)+\nu\right] + \frac{d}{dt}n(t)\right\}\delta(t-s)dtds\\
&= \left[i\Sigma^{<}(t) + 2\nu\kappa(t)\right]\delta(t-s)dtds. &(8.23)
\end{aligned}
$$

Taking the random average of the stochastic Liouville equation (8.1), we see that it reduces to the Fokker-Planck equation (4.22):

$$
\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}_t|0(t)\rangle, \tag{8.24}
$$

with $|0(t)\rangle = \left\langle |0_f(t)\rangle \right\rangle$, if the condition

$$
\left\langle \left\{\alpha^{\dagger}dW(t) + \text{t.c.}\right\}|0_f(t)\rangle \right\rangle = 0, \tag{8.25}
$$

is satisfied. This indicates that the multiplication should be of the Ito type [56] (see Appendix D). The random force operator $dW(t)$ does not correlate with the quantities at time t, i.e., $|0_f(t)\rangle$ in the case of (8.25).

8.1.2. Stratonovich Type

By making use of the relation between the Ito and the Stratonovich stochastic multiplications (see Appendix D), we can rewrite the Ito type stochastic Liouville equation (8.1) into the Stratonovich type as follows.

The relation (D.8) makes the term containing the random force operators in the right hand side of (8.1)

$$
\begin{aligned}
\left\{\alpha^{\dagger}dW(t) + \text{t.c.}\right\}|0_f(t)\rangle &= \left\{\alpha^{\dagger}dW(t) + \text{t.c.}\right\}\circ|0_f(t)\rangle\\
&\quad -\frac{1}{2}\left\{\alpha^{\dagger}dW(t) + \text{t.c.}\right\}d|0_f(t)\rangle. &(8.26)
\end{aligned}
$$

Substituting (8.1) into the last term for $d|0_f(t)\rangle$ and using the relations (8.21)–(8.23) for the multiplications among the random force operators, we finally arrived at the stochastic Liouville equation of Stratonovich type in the form

$$
d|0_f(t)\rangle = -i\hat{H}_{f,t}dt\circ|0_f(t)\rangle, \tag{8.27}
$$

with

$$
\begin{aligned}
\hat{H}_{f,t}dt &= \hat{H}_{S,t}dt - i\kappa(t)\left(\alpha^{\dagger}\alpha + \text{t.c.}\right)dt + i\left[\alpha^{\dagger}dW(t) + \text{t.c.}\right] &(8.28)\\
&= \hat{H}_{S,t}dt + \left[\alpha^{\dagger}\left(id\alpha + [\hat{H}_{S,t}dt,\ \alpha]\right) - \text{t.c.}\right], &(8.29)
\end{aligned}
$$

where the flow operators $d\alpha$ and $d\tilde{\alpha}$ are specified respectively by

$$
d\alpha = i[\hat{H}_{S,t}dt,\ \alpha] - \kappa(t)\alpha dt + dW(t), \tag{8.30}
$$

and its tilde conjugate. We introduced the symbol $\circ$ in order to indicate the Stratonovich stochastic multiplication [57] (see Appendix D).

We can derive the Fokker-Planck equation (8.24) by taking the random average of the Stratonovich stochastic Liouville equation (8.27) (see Appendix F).

8.2. Stochastic Semi-Free Operators

The stochastic semi-free operators are defined by

$$a(t) = \hat{V}_f^{-1}(t) a \hat{V}_f(t), \qquad \tilde{a}^{\dagger\dagger}(t) = \hat{V}_f^{-1}(t) \tilde{a}^{\dagger} \hat{V}_f(t), \tag{8.31}$$

where

$$d\hat{V}_f(t) = -i\hat{\mathcal{H}}_{f,t} dt\ \hat{V}_f(t), \tag{8.32}$$

with $\hat{V}_f(0) = 1$. Here, it is assumed that, at $t = 0$, the relevant system starts to contact with the irrelevant system representing the stochastic process described by the random force operators $dF(t)$, etc. defined in (8.17)–(8.19).[10]

The semi-free operators (8.31) keep the equal-time canonical commutation relation:

$$[a(t),\ a^{\dagger\dagger}(t)] = 1, \qquad [\tilde{a}(t),\ \tilde{a}^{\dagger\dagger}(t)] = 1, \tag{8.33}$$

and satisfy **Tool 3.** in section 3.:

$$\langle 1| a^{\dagger\dagger}(t) = \langle 1| \tilde{a}(t). \tag{8.34}$$

The tildian nature

$$\left(i\hat{\mathcal{H}}_{f,t} dt\right)^{\sim} = i\hat{\mathcal{H}}_{f,t} dt, \tag{8.35}$$

(see **B2** in section 3.) is consistent with the definition (8.31) of the semi-free operators. Since the tildian hat-Hamiltonian $\hat{\mathcal{H}}_{f,t} dt$ is not necessarily hermite, we introduced the symbol $\dagger\dagger$ in order to distinguish it from the hermite conjugation $\dagger$. However, we will use $\dagger$ instead of $\dagger\dagger$, for simplicity, unless it is confusing.

The stochastic semi-free operators and the random force operators satisfy the orthogonality

$$\left\langle a(t) d\mathcal{F}^{\dagger}(t) \right\rangle = 0, \quad \text{etc.}, \tag{8.36}$$

where the random force operator $d\mathcal{F}^{\dagger}(t)$ in the *Heisenberg* representation[11] is defined by

$$d\mathcal{F}^{\dagger}(t) = \hat{V}_f^{-1}(t) dF^{\dagger}(t) \hat{V}_f(t). \tag{8.37}$$

8.3. Quantum Langevin Equations

8.3.1. Stratonovich type

For the dynamical quantity

$$A(t) = \hat{V}_f^{-1}(t) A \hat{V}_f(t), \tag{8.38}$$

---

[10] Within the formalism, the random force operators $dF(t)$ and $dF^{\dagger}(t)$ are assumed to commute with any relevant system operator A in the Schrödinger representation: $[A,\ dF(t)] = [A,\ dF^{\dagger}(t)] = 0$.

[11] It can be the interaction representation when one includes non-linear terms in the hat-Hamiltonian, and performs a perturbational calculation. As we are dealing with only the semi-free case in this section, we call the representation as the Heisenberg one.

the quantum Langevin equation of the Stratonovich type is given by the stochastic Heisenberg equation as [28,30]

$$dA(t) = i[\hat{H}_f(t)dt \,\overset{\circ}{,}\, A(t)] \tag{8.39}$$

$$\begin{aligned} &= i[\hat{H}_S(t),\ A(t)]dt \\ &\quad + \kappa(t)\left\{[\alpha^{\natural}(t)\alpha(t),\ A(t)] + [\tilde{\alpha}^{\natural}(t)\tilde{\alpha}(t),\ A(t)]\right\} dt \\ &\quad - \left\{[\alpha^{\natural}(t),\ A(t)] \circ dW(t) + [\tilde{\alpha}^{\natural}(t),\ A(t)] \circ d\tilde{W}(t)\right\}, \end{aligned} \tag{8.40}$$

where

$$\hat{H}_f(t) = \hat{V}_f^{-1}(t)\hat{H}_{f,t}\hat{V}_f(t), \qquad \hat{H}_S(t) = \hat{V}_f^{-1}(t)\hat{H}_{S,t}\hat{V}_f(t), \tag{8.41}$$

$$[X(t) \,\overset{\circ}{,}\, Y(t)] = X(t) \circ Y(t) - Y(t) \circ X(t), \tag{8.42}$$

for arbitrary operators $X(t)$ and $Y(t)$. Use has been made of the fact that

$$\hat{V}_f^{-1}(t)dW(t)\hat{V}_f(t) = dW(t), \tag{8.43}$$

since the random force operator $dW(t)$ is commutative with $\hat{V}_f(t)$ due to the properties (8.22) and (8.23). Note that, using (8.40), we can readily verify that

$$d[A(t)B(t)] = dA(t) \circ B(t) + A(t) \circ dB(t), \tag{8.44}$$

for arbitrary relevant system operators A and B. This fact proves that the quantum stochastic differential equation (8.40) is indeed of the Stratonovich type.

The quantum Langevin equation of the Stratonovich type (8.40) is also derived by the algebraic identity

$$dA(t) = d\hat{V}_f^{-1}(t) \circ A\hat{V}_f(t) + \hat{V}_f^{-1}(t)A \circ d\hat{V}_f(t), \tag{8.45}$$

with the help of

$$d\hat{V}_f(t) = -i\hat{H}_{f,t}dt \circ \hat{V}_f(t), \qquad d\hat{V}_f^{-1}(t) = i\hat{V}_f^{-1}(t) \circ \hat{H}_{f,t}dt. \tag{8.46}$$

8.3.2. Ito type

When $dY(t)$ is $dW(t)$, and $X(t)$ is constituted by the relevant operators satisfying the quantum Langevin equation (8.40) of the Stratonovich type, the connection formula (D.5) reduces to

$$X(t) \circ dW(t) = X(t)dW(t) - \frac{1}{2}\left[i\Sigma^{<}(t) + 2\nu\kappa(t)\right]\left[\tilde{a}^{\dagger}(t) - a(t),\ X(t)\right] dt. \tag{8.47}$$

In deriving (8.47), we used the properties (8.23), and the fact that $dW(t)dt$ etc. can be neglected as higher orders.

By means of the connection formula (8.47) between the Ito and the Stratonovich products, we can derive the quantum Langevin equation of the Ito type from that of the Stratonovich type (8.40) as

$$\begin{aligned} dA(t) &= i[\hat{\mathcal{H}}_f(t)dt,\ A(t)] \\ &\quad + \left\{\alpha^{\natural}(t)[\tilde{\alpha}^{\natural}(t),\ A(t)] + \tilde{\alpha}^{\natural}(t)[\alpha^{\natural}(t),\ A(t)]\right\} dW(t)d\tilde{W}(t) \\ &= i[\hat{H}_S(t),\ A(t)]dt \\ &\quad +\kappa(t)\left\{[\alpha^{\natural}(t)\alpha(t),\ A(t)] + [\tilde{\alpha}^{\natural}(t)\tilde{\alpha}(t),\ A(t)]\right\} dt \\ &\quad + \left[i\Sigma^{<}(t) + 2\nu\kappa(t)\right][\tilde{\alpha}^{\natural}(t),\ [\alpha^{\natural}(t),\ A(t)]]dt \\ &\quad - \left\{[\alpha^{\natural}(t),\ A(t)]dW(t) + [\tilde{\alpha}^{\natural}(t),\ A(t)]d\tilde{W}(t)\right\}, \end{aligned} \tag{8.48, 8.49}$$

where

$$\hat{\mathcal{H}}_f(t)dt = \hat{V}_f^{-1}(t)\hat{\mathcal{H}}_{f,t}dt\hat{V}_f(t). \tag{8.50}$$

With the help of (8.49), we can find that the product $dA(t)dB(t)$ has the expression

$$\begin{aligned} dA(t)\cdot dB(t) = &\left[i\Sigma^{<}(t) + 2\nu\kappa(t)\right]\left\{[\alpha^{\natural}(t),\ A(t)][\tilde{\alpha}^{\natural}(t),\ B(t)]\right. \\ &\left. + [\tilde{\alpha}^{\natural}(t),\ A(t)][\alpha^{\natural}(t),\ B(t)]\right\} dt, \end{aligned} \tag{8.51}$$

which leads to the calculus rule of the Ito type

$$d[A(t)B(t)] = dA(t)\cdot B(t) + A(t)\cdot dB(t) + dA(t)\cdot dB(t), \tag{8.52}$$

for arbitrary relevant stochastic operators A and B. This proves that the quantum stochastic differential equation (8.49) is in fact of the Ito type. Furthermore, since (8.49) is the time-evolution equation for any relevant stochastic operator $A(t)$, it is *Ito's formula* for quantum systems as will be proven in section 9..

Putting a and $\tilde{a}^{\dagger}$ for A, we see that both (8.40) and (8.49) reduce to

$$d\alpha(t) = i[\hat{H}_S(t)dt,\ \alpha(t)] - \kappa(t)\alpha(t)dt + dW(t), \tag{8.53}$$

$$d\alpha^{\natural}(t) = i[\hat{H}_S(t)dt,\ \alpha^{\natural}(t)] + \kappa(t)\alpha^{\natural}(t)dt, \tag{8.54}$$

which are written in terms of the *original* operators in the form

$$da(t) = i[\hat{H}_S(t)dt,\ a(t)] - \kappa(t)[(\mu-\nu)a(t) + 2\nu\tilde{a}^{\dagger}(t)]dt + dW(t), \tag{8.55}$$

$$d\tilde{a}^{\dagger}(t) = i[\hat{H}_S(t)dt,\ \tilde{a}^{\dagger}(t)] - \kappa(t)[2\mu a(t) - (\mu-\nu)\tilde{a}^{\dagger}(t)]dt + dW(t). \tag{8.56}$$

The formal structures of (8.53) is the same as the flow operator (8.30) appeared in $\hat{H}_{f,t}$ of (8.29).

In the Langevin equation approach, the dynamical behavior of systems is specified when one characterizes the correlations of random forces. The quantum Langevin equation is the equation in the Heisenberg representation, therefore the characterization of random force operators should be performed in this representation. This cannot be done in terms of $d\mathcal{F}(t)$ etc., since the information of the stochastic process is masked by the dynamics generated by $\hat{H}_f(t)$ in these operators. Whereas, the specification of the correlation between $dW(t)$ etc. directly characterizes the stochastic process owing to the relations in (8.43).

8.4. Averaged Equation of Motion

Applying the random force bra-vacuum $\langle 1|$ to the Ito type Langevin equation (8.49), we have

$$
\begin{aligned}
d\langle 1|A(t) = \; & i\langle 1|[H_S(t),\; A(t)]dt + \kappa(t)\langle 1|A(t)\left[\alpha^{\text{♀}}\alpha + \text{t.c.}\right]dt \\
& + i\Sigma^{<}(t)\langle 1|A(t)\alpha^{\text{♀}}(t)\tilde{\alpha}^{\text{♀}}(t)dt \\
& + \langle 1|A(t)\left[\alpha^{\text{♀}}(t)dW(t) + \text{t.c.}\right]. \qquad (8.57)
\end{aligned}
$$

Applying further the bra-vacuum $\langle |$ to (8.57), we can derive the stochastic equation of motion of Ito type for the bra-vector state $\langle\!\langle 1|A(t)$ in the form

$$
\begin{aligned}
d\langle\!\langle 1|A(t) = \; & i\langle\!\langle 1|[H_S(t),\; A(t)]dt \\
& + \kappa(t)\left\{\langle\!\langle 1|[a^{\dagger}(t),\; A(t)]a(t) + \langle\!\langle 1|a^{\dagger}(t)[A(t),\; a(t)]\right\}dt \\
& + i\Sigma^{<}(t)\langle\!\langle 1|[a(t),\; [A(t),\; a^{\dagger}(t)]]dt \\
& + \langle\!\langle 1|[A(t),\; a^{\dagger}(t)]dF(t) + \langle\!\langle 1|[a(t),\; A(t)]dF^{\dagger}(t), \qquad (8.58)
\end{aligned}
$$

where we used the property $\langle |dW(t) = \langle |dF(t)$ and $\langle |d\tilde{W}(t) = \langle |dF^{\dagger}(t)$.

Whereas, the stochastic equation of motion of Stratonovich type for the bra-vector state $\langle\!\langle 1|A(t)$ is derived similarly in the form

$$
\begin{aligned}
d\langle\!\langle 1|A(t) = \; & i\langle\!\langle 1|[H_S(t),\; A(t)]dt \\
& + \kappa(t)\left\{\langle\!\langle 1|[a^{\dagger}(t),\; A(t)]a(t) + \langle\!\langle 1|a^{\dagger}(t)[A(t),\; a(t)]\right\}dt \\
& + \langle\!\langle 1|[A(t),\; a^{\dagger}(t)] \circ dF(t) + \langle\!\langle 1|[a(t),\; A(t)] \circ dF^{\dagger}(t). \qquad (8.59)
\end{aligned}
$$

These equations of motion for the bra-vector state may be intimately related with the Langevin equations given by Gardiner and Collett [58].

Putting the random force ket-vacuum $|\rangle$ and the ket-vacuum $|0\rangle$ of the relevant system to (8.58), we obtain the equation of motion for the expectation value of an arbitrary operator $A(t)$ of the relevant system as

$$
\begin{aligned}
\frac{d}{dt}\langle\!\langle A(t)\rangle\!\rangle = \; & i\,\langle\!\langle [H_S(t),\; A(t)]\rangle\!\rangle \\
& + \kappa(t)\left(\langle\!\langle [a^{\dagger}(t),\; A(t)]a(t)\rangle\!\rangle + \langle\!\langle a^{\dagger}(t)[A(t),\; a(t)]\rangle\!\rangle\right) \\
& + i\Sigma^{<}(t)\langle\!\langle [a(t),\; [A(t),\; a(t)^{\dagger}]]\rangle\!\rangle, \qquad (8.60)
\end{aligned}
$$

where $\langle\!\langle \cdots \rangle\!\rangle = \langle|\langle 1| \cdots |0\rangle|\rangle$, which means to take both random average and vacuum expectation. This is the exact equation of motion for systems with linear-dissipative coupling to reservoir, which can be also derived by means of Fokker-Planck equation (8.24). Here, we used the property

$$
\langle a(t)dW(t)\rangle = 0, \quad \text{etc.}, \qquad (8.61)
$$

which are the characteristics of the Ito multiplication [56]. Note that (8.60) was derived for general $\hat{H}_S$ including non-linear interaction terms.

9. Phase-Space Method

Mapping (8.24) to the one in phase-space by means of the coherent state representation for coarse grained operators, which is constructed just the same process as (G.1)–(G.13) with respect to $|0(t)\rangle$ and $P^{(\mu,\nu)}(z,t)$, we obtain the Fokker-Planck equation in phase-space [34]

$$\frac{d}{dt}P^{(\mu,\nu)}(z,t) = -i\Omega^{(\mu,\nu)}(z,t)\ P^{(\mu,\nu)}(z,t), \tag{9.1}$$

with

$$P^{(\mu,\nu)}(z,t) = \left\langle P_f^{(\mu,\nu)}(z,t)\right\rangle, \tag{9.2}$$

and the coarse-grained generator

$$\begin{aligned}\Omega^{(\mu,\nu)}(z,t) &= (-\partial z + \partial_* z^*)\,\hat{E}^{(\mu,\nu)}(z,\partial,t) + i\kappa(t)\,(\partial z + \partial_* z^*) \\ &\quad + i\left[i\Sigma^{<}(t) + 2\nu\kappa(t)\right]\partial\partial_*,\end{aligned} \tag{9.3}$$

where $\partial = \partial/\partial z$, $\partial_* = \partial/\partial z^*$. Note that the expression (9.1) for $\mu = 1$, $\nu = 0$ is the same as (A.9) obtained by mapping the master equation (A.1) in the density operator method by means of the coherent-state representation in the Liouville space [59]-[61] (see e.g. [9]).

The quantum stochastic Liouville equation (8.27) of the Stratonovich type is mapped as [34]

$$dP_f^{(\mu,\nu)}(z,t) = -i\Omega_f^{(\mu,\nu)}(z,t)dt \circ P_f^{(\mu,\nu)}(z,t), \tag{9.4}$$

with the stochastic time-evolution generator

$$\begin{aligned}\Omega_f^{(\mu,\nu)}(z,t)dt &= (-\partial z + \partial_* z^*)\,\hat{E}^{(\mu,\nu)}(z,\partial,t)dt \\ &\quad + i\kappa(t)\,(\partial z + \partial_* z^*)\,dt - i\left[\partial \circ dW(t) + \partial_* \circ dW^*(t)\right],\end{aligned} \tag{9.5}$$

mapped from $\hat{H}_{f,t}dt$, where $(-\partial z + \partial_* z^*)\hat{E}^{(\mu,\nu)}(z,\partial,t)$ is defined for $\hat{H}_{S,t}$ by means of (G.10) and (G.11) with the property

$$-z\hat{E}^{(\mu,\nu)}(z,\partial,t)\partial + z^*\hat{E}^{(\mu,\nu)}(z,\partial,t)\partial_* = (-\partial z + \partial_* z^*)\,\hat{E}^{(\mu,\nu)}(z,\partial,t). \tag{9.6}$$

Here, we are confining ourselves to the case where $H_{S,t}$ has the structure like $\sum_n g_n(t)\left(a^\dagger\right)^n a^n$, which leads us to

$$\hat{E}^{(\mu,\nu)}(z,\partial,t) = \sum_{\substack{p,q,m,n \\ p+q=m+n}} g_{p,q,m,n}(t)\left[z^p (z^*)^q\, \partial^m \partial_*^n + (z^*)^p z^q \partial_*^m \partial^n\right], \tag{9.7}$$

with real quantities $g_{p,q,m,n}(t)$. For a harmonic oscillator with *frequency* $\omega(t)$, $\hat{E}^{(\mu,\nu)}(z,\partial,t) = \omega(t)$.

The quantum stochastic Liouville equation (8.1) of the Ito type is mapped to the one in phase-space as

$$dP_f^{(\mu,\nu)}(z,t) = -i\Omega_f^{(\mu,\nu)}(z,t)dt\ P_f^{(\mu,\nu)}(z,t), \tag{9.8}$$

with

$$\Omega_f^{(\mu,\nu)}(z,t)dt = \Omega^{(\mu,\nu)}(z,t)dt - i\left[\partial\, dW(t) + \partial_*\, dW^*(t)\right]. \tag{9.9}$$

It is easily seen that (9.8) reduces to the Fokker-Planck equation (9.1) when the random average is taken.

The quantum Langevin equation (8.40) of the Stratonovich type is transformed to [34]

$$\begin{aligned} dA^{(\nu,\mu)}(t) &= i\Big[-\check{E}^{(\mu,\nu)}(z(t),\partial(t),t)z(t)\partial(t) \\ &\quad + \check{E}^{(\mu,\nu)}(z(t),\partial(t),t)z^*(t)\partial_*(t)\Big] A^{(\nu,\mu)}(t)dt \\ &\quad -\kappa(t)\left[z(t)\partial(t) + z^*(t)\partial_*(t)\right] A^{(\nu,\mu)}(t)dt \\ &\quad + \Big\{\Big[\partial(t)A^{(\nu,\mu)}(t)\Big] \circ dW(t) + \Big[\partial_*(t)A^{(\nu,\mu)}(t)\Big] \circ dW^*(t)\Big\}, \end{aligned} \tag{9.10}$$

where $\partial(t) = \partial/\partial z(t)$ and $\partial_*(t) = \partial/\partial z^*(t)$, and $\check{E}^{(\mu,\nu)}(z,\partial,t)$ is the adjoint differential operator function defined by

$$\begin{aligned} &\int_z f_1(z)\left[-z\hat{E}^{(\mu,\nu)}(z,\partial,t)\partial + z^*\hat{E}^{(\mu,\nu)}(z,\partial,t)\partial_*\right] f_2(z) \\ &\quad = \int_z f_2(z)\left[\partial\check{E}^{(\mu,\nu)}(z,\partial,t)z - \partial_*\check{E}^{(\mu,\nu)}(z,\partial,t)z^*\right] f_1(z), \end{aligned} \tag{9.11}$$

and use has been made of the property

$$\begin{aligned} &-\partial\check{E}^{(\mu,\nu)}(z,\partial,t)z + \partial_*\check{E}^{(\mu,\nu)}(z,\partial,t)z^* = \\ &= -\check{E}^{(\mu,\nu)}(z,\partial,t)z\partial + \check{E}^{(\mu,\nu)}(z,\partial,t)z^*\partial_*. \end{aligned} \tag{9.12}$$

Using the connection formula between the Ito and Stratonovich products in phase-space which has the same structure as (8.47) for quantum stochastic operators, we can derive the Langevin equation of the Ito type as

$$\begin{aligned} dA^{(\nu,\mu)}(t) &= i\Big[-\check{E}^{(\mu,\nu)}(z(t),\partial(t),t)z(t)\partial(t) \\ &\quad + \check{E}^{(\mu,\nu)}(z(t),\partial(t),t)z^*(t)\partial_*(t)\Big] A^{(\nu,\mu)}(t)dt \\ &\quad -\kappa(t)\left[z(t)\partial(t) + z^*(t)\partial_*(t)\right] A^{(\nu,\mu)}(t)dt \\ &\quad + \left[i\Sigma^<(t) + 2\nu\kappa(t)\right] \partial(t)\partial_*(t)A^{(\nu,\mu)}(t)dt \\ &\quad + \Big\{\Big[\partial(t)A^{(\nu,\mu)}(t)\Big] dW(t) + \Big[\partial_*(t)A^{(\nu,\mu)}(t)\Big] dW^*(t)\Big\}. \end{aligned} \tag{9.13}$$

This can be obtained also by mapping the quantum Langevin equation (8.49) of the Ito type into the one in phase-space.

By making use of (9.10) or (9.13) for $z(t)$, we have

$$dz(t) = -i\check{E}^{(\mu,\nu)}(z(t),\partial(t),t)z(t)dt - \kappa(t)z(t)dt + dW(t). \tag{9.14}$$

With the help of (9.14), we can rewrite (9.13) in the form

$$dA^{(\nu,\mu)}(t) = dz(t)\partial(t)A^{(\nu,\mu)}(t) + dz^*(t)\partial_*(t)A^{(\nu,\mu)}(t)$$

$$+ dz(t)dz^*(t)\partial(t)\partial_*(t)A^{(\nu,\mu)}(t), \tag{9.15}$$

where we used the relation

$$dz(t)dz^*(t) = dW(t)dW^*(t) = \left[i\Sigma^<(t) + 2\nu\kappa(t)\right] dt, \tag{9.16}$$

which is proven within the stochastic convergence with (9.14) and the properties (G.15)–(G.17). The equation (9.15) is nothing but the well known *Ito's formula* for complex stochastic variable $z(t)$.

It is worthy to note that, with the definition of flow:

$$dz_t = -i\check{E}^{(\mu,\nu)}(z,\partial,t)zdt - \kappa(t)zdt + dW(t), \tag{9.17}$$

being in the same structure as (9.14), the stochastic time-evolution generator (9.5) can be expressed in the form [34]

$$\begin{aligned}\Omega_f^{(\mu,\nu)}(z,t)dt &= -i\,(\partial\; dz_t + \partial_*\; dz_t^*) \\ &\quad -\check{E}^{(\mu,\nu)}(z,\partial,t)\,(-\partial z + \partial_* z^*)\,dt \\ &\quad +(-\partial z + \partial_* z^*)\,\hat{E}^{(\mu,\nu)}(z,\partial,t)dt.\end{aligned} \tag{9.18}$$

The latter two terms on the right hand side represent quantum effects. This is an extension of Kubo's generator for the stochastic Liouville equation [8] to quantum systems.

Taking average of (9.13) with respect to both the initial distribution $P_f^{(\mu,\nu)}(z)$ and the random forces, we obtain the equation of motion for the expectation value of an arbitrary observable operator $A(t)$ of the relevant system as

$$\begin{aligned}\frac{d}{dt}\langle\!\langle A(t)\rangle\!\rangle &= \Big\langle \int_z A^{(\nu,\mu)}(z)\Big[-i\,(-\partial z + \partial_* z^*)\,\hat{E}^{(\mu,\nu)}(z,\partial,t) \\ &\quad +\kappa(t)\,(\partial z + \partial_* z^*) + \left[i\Sigma^<(t) + 2\nu\kappa(t)\right]\partial\partial_*\Big] P_f^{(\mu,\nu)}(z,t)\Big\rangle \\ &= \int_z A^{(\nu,\mu)}(z)\Big[-i\,(-\partial z + \partial_* z^*)\,\hat{E}^{(\mu,\nu)}(z,\partial) \\ &\quad +\kappa(t)\,(\partial z + \partial_* z^*) + \left[i\Sigma^<(t) + 2\nu\kappa(t)\right]\partial\partial_*\Big] P^{(\mu,\nu)}(z,t),\end{aligned} \tag{9.19}$$

where

$$\langle\!\langle A(t)\rangle\!\rangle = \Big\langle \int_z A^{(\nu,\mu)}(z)P_f^{(\mu,\nu)}(z,t)\Big\rangle = \int_z A^{(\nu,\mu)}(z)P^{(\mu,\nu)}(z,t), \tag{9.20}$$

(see (G.13)). Here, we used the properties

$$\Big\langle z(t)dW(t)\Big\rangle = 0, \quad \text{etc.}, \tag{9.21}$$

which are the characteristics of the Ito multiplication [56] (see Appendix D). The averaged equation of motion (9.19) can also be derived by making use of the Fokker-Planck equation (9.1), as can be seen in the second expression of (9.19).

We showed that the framework, including both the quantum Fokker-Planck equation and the quantum stochastic differential equations constructed within NETFD, is compatible with the one of the classical Fokker-Planck equation and of the classical stochastic differential equations. It was done by mapping the entire framework of NETFD to the c-number phase-space by means of the phase-space method in thermal space [62]. Note that the mapped framework in phase-space keeps the information of quantum effects.

10. Unitary Time-Generation

Let us investigate what happens if we adopt the *hermitian* hat-Hamiltonian of the form

$$\hat{H}_{f,t}dt = \hat{H}_{S,t}dt + i\left[\alpha^{\natural}dW(t) + \text{t.c.}\right] - i\left[\alpha dW^{\natural}(t) + \text{t.c.}\right]. \tag{10.1}$$

for the stochastic Liouville equation (8.27) of the Stratonovich type where we used (5.1), i.e. (5.30), as the interaction hat-Hamiltonian between the relevant system and the random force system. In addition to the random force operators $dW(t)$ and its tilde conjugate, we introduced

$$dW^{\natural}(t) = dF^{\dagger}(t) - d\tilde{F}(t), \tag{10.2}$$

and its tilde conjugate which annihilate the ket-vacuum $\langle|$:

$$\langle|dW^{\natural}(t) = 0, \qquad \langle|d\tilde{W}^{\natural}(t) = 0, \tag{10.3}$$

(see (8.4)).

Since (10.1) is hermite:

$$\left(\hat{H}_{f,t}dt\right)^{\dagger} = \hat{H}_{f,t}dt. \tag{10.4}$$

the generator $\hat{V}_f(t)$ defined by

$$d\hat{V}_f(t) = -i\hat{H}_{f,t} \circ \hat{V}_f(t), \tag{10.5}$$

looks like a unitary operator satisfying

$$\hat{V}_f^{\dagger}(t) = \hat{V}_f^{-1}(t), \tag{10.6}$$

within the Stratonovich calculation.

With the help of (8.17)–(8.19), we can find the new correlations among the operators $dW(t)$ and $dW^{\natural}(t)$ and their tilde conjugate as

$$\left\langle dW^{\natural}(t)\right\rangle = \left\langle d\tilde{W}^{\natural}(t)\right\rangle = 0, \tag{10.7}$$

$$\left\langle dW^{\natural}(t)dW(s)\right\rangle = \left\langle d\tilde{W}^{\natural}(t)d\tilde{W}(s)\right\rangle = 0 \tag{10.8}$$

$$\left\langle dW(t)dW^{\natural}(s)\right\rangle = \left\langle d\tilde{W}(t)d\tilde{W}^{\natural}(s)\right\rangle = 2\kappa(t)\delta(t-s)dt\, ds. \tag{10.9}$$

Therefore, in addition to the relations (8.21)–(8.23), we have

$$dW(t)dW^{\dagger}(s) = d\tilde{W}(t)d\tilde{W}^{\dagger}(s) = 2\kappa(t)\delta(t-s)dt\, ds, \quad (10.10)$$
$$dW^{\dagger}(t)dW(s) = d\tilde{W}^{\dagger}(t)d\tilde{W}(s) = 0. \quad (10.11)$$

By making use of the relation (D.8) among the Ito and Stratonovich multiplication, we can derive the hat-Hamiltonian for the stochastic Liouville equation (8.1) of the Ito type in the form

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}_t dt + i\left[\alpha^{\dagger}dW(t) + \text{t.c.}\right] - i\left[\alpha dW^{\dagger}(t) + \text{t.c.}\right]. \quad (10.12)$$

Note that (10.12) is not hermitian, since $\hat{H}_t dt$ is not. Note also that (10.1) and (10.12) annihilate the ket-vacuum $\langle\!\langle 1| = \langle|\langle 1|$ of the whole system, i.e. the relevant system plus the random force system, but not the relevant vacuum $\langle 1|$ only (see the discussion in subsection 5.1.).

With the help of (8.21)–(8.23), (10.10) and (10.11), we can derive the Fokker-Planck equation (8.24) by taking the random average of the stochastic Liouville equation (8.1) of the Ito type with the generator (10.12).

The Langevin equation of the Stratonovich type is given by

$$\begin{aligned} dA(t) &= i[\hat{H}_f(t)dt \,\circ\, A(t)] \\ &= i\left[\hat{H}_S(t),\ A(t)\right]dt \\ &\quad - \left\{[\alpha^{\dagger}(t),\ A(t)]\circ d\mathcal{W}(t) + [\tilde{\alpha}^{\dagger}(t),\ A(t)]\circ d\tilde{\mathcal{W}}(t)\right\} \\ &\quad + \left\{d\mathcal{W}^{\dagger}(t)\circ[\alpha(t),\ A(t)] + d\tilde{\mathcal{W}}^{\dagger}(t)\circ[\tilde{\alpha}(t),\ A(t)]\right\}, \end{aligned} \quad (10.13)$$

where we defined the operators

$$d\mathcal{W}(t) = \hat{V}_f^{-1}(t)\circ dW(t)\circ \hat{V}_f(t), \quad (10.14)$$
$$d\mathcal{W}^{\dagger}(t) = \hat{V}_f^{-1}(t)\circ dW^{\dagger}(t)\circ \hat{V}_f(t). \quad (10.15)$$

With the help of the connection formulae (D.7) and (D.8), we can calculate the commutation relation between $\hat{V}_f(t)$ and $dW(t)$, $dW^{\dagger}(t)$ etc. of the Stratonovich type as

$$\begin{aligned} \left[\hat{V}_f(t) \,\circ\, dW(t)\right] &= \left[\hat{V}_f(t),\ dW(t)\right] + \frac{1}{2}\left[d\hat{V}_f(t),\ dW(t)\right] \\ &= \kappa(t)\alpha dt\ \hat{V}_f(t), \end{aligned} \quad (10.16)$$
$$\left[\hat{V}_f(t) \,\circ\, dW^{\dagger}(t)\right] = \kappa(t)\alpha^{\dagger}dt\hat{V}_f(t), \quad (10.17)$$

and their tilde conjugates, where we used the property

$$\left[\hat{V}_f(t),\ dW(t)\right] = \left[\hat{V}_f(t),\ dW^{\dagger}(t)\right] = 0, \quad (10.18)$$

and its tilde conjugates within the stochastic convergence, which comes from the characteristics of the Ito multiplication defined by (D.1) and (D.2):

$$\left\langle \hat{V}_f(t)dW(t)\right\rangle = \left\langle \hat{V}_f(t)dW^{\dagger}(t)\right\rangle = 0. \quad (10.19)$$

Using (10.16) and (10.17) for (10.14) and (10.15), we can derive

$$d\mathcal{W}(t) = dW(t) - \kappa(t)\alpha(t)dt \tag{10.20}$$
$$d\mathcal{W}^{\natural}(t) = dW^{\natural}(t) - \kappa(t)\alpha^{\natural}(t)dt. \tag{10.21}$$

By the substitution of (10.20), (10.21) and their tilde conjugates, (10.13) reads

$$\begin{aligned} dA(t) = {}& i[\hat{H}_S(t),\ A(t)]dt \\ & +\kappa(t)\Big\{[\alpha^{\natural}(t),\ A(t)]\alpha(t) - \alpha^{\natural}(t)[\alpha(t),\ A(t)] \\ & +[\tilde{\alpha}^{\natural}(t),\ A(t)]\tilde{\alpha}(t) - \tilde{\alpha}^{\natural}(t)[\tilde{\alpha}(t),\ A(t)]\Big\}dt \\ & -\Big\{[\alpha^{\natural}(t),\ A(t)] \circ dW(t) + [\tilde{\alpha}^{\natural}(t),\ A(t)] \circ d\tilde{W}(t)\Big\} \\ & +\Big\{dW^{\natural}(t) \circ [\alpha(t),\ A(t)] + d\tilde{W}^{\natural}(t) \circ [\tilde{\alpha}(t),\ A(t)]\Big\}. \end{aligned} \tag{10.22}$$

Note that $dW(t)$, $dW^{\natural}(t)$ etc. are not commutative with the operators at time t.

Use of the relation (D.5) and (D.6) to (10.22) leads us to the Langevin equation of the Ito type in the form

$$\begin{aligned} dA(t) = {}& i[\hat{H}_S(t),\ A(t)]dt \\ & +\kappa(t)\Big\{[\alpha^{\natural}(t),\ A(t)]\alpha(t) - \alpha^{\natural}[\alpha(t),\ A(t)] \\ & +[\tilde{\alpha}^{\natural}(t),\ A(t)]\tilde{\alpha}^{\natural}(t) - \tilde{\alpha}^{\natural}(t)[\tilde{\alpha}(t),\ A(t)]\Big\}dt \\ & +\Big[\Sigma^{<}(t) + 2\nu\kappa(t)\Big][\tilde{\alpha}^{\natural}(t),\ [\alpha^{\natural}(t),\ A(t)]]dt \\ & -\Big\{[\alpha^{\natural}(t),\ A(t)]dW(t) + [\tilde{\alpha}^{\natural}(t),\ A(t)]d\tilde{W}(t)\Big\} \\ & +\Big\{dW^{\natural}(t)[\alpha(t),\ A(t)] + d\tilde{W}^{\natural}(t)[\tilde{\alpha}(t),\ A(t)]\Big\}. \end{aligned} \tag{10.23}$$

¿From (10.22) and (10.23), we see that the equations of motion for the ket-vector $\langle\!\langle 1|A(t)$ reduce respectively to (8.59) and (8.58). Therefore, the averaged equation of motion reduces also to (8.60) in both cases.

11. An Interpretation of the Mori Formula

Let us consider a column vector

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ A_n \end{pmatrix}, \tag{11.1}$$

of a set of operators $\{A_i\ (i = 1, 2, \cdots, n)\}$ corresponding to gross variables. It satisfies the Heisenberg equation within NETFD:

$$\frac{d}{dt}A(t) = i[\hat{H},\ A(t)], \tag{11.2}$$

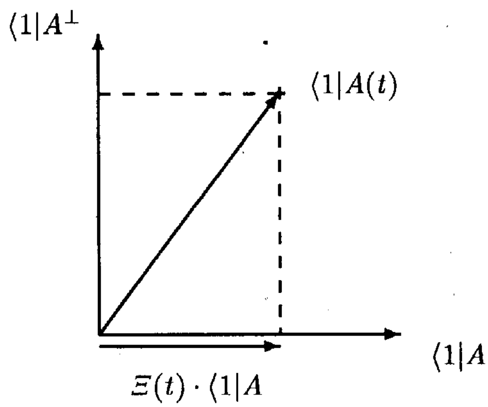

Figure 2: Schematic diagram of the thermal space decomposing the bra-vector state $\langle 1|A(t)$ into two sub-spaces $\langle 1|A$ and $\langle 1|A^{\perp}$.

where

$$\hat{H} = H - \tilde{H}, \tag{11.3}$$

with H being a Hamiltonian of the system under consideration. The hat-Hamiltonian annihilates the bra-vacuum (**B2** in section 3.):

$$\langle 1|\hat{H} = 0. \tag{11.4}$$

The Heisenberg equation is formally solved to give

$$A(t) = \mathrm{e}^{i\hat{H}t} A \mathrm{e}^{-i\hat{H}t}. \tag{11.5}$$

We will decompose the bra-vector $\langle 1|A(t)$ in the thermal space into two sub-spaces (see Fig. 2). One is the space spanned by $\{\langle 1|A_i\,(i = 1, 2, \cdots, n)\}$ and the other is the space perpendicular to it. We will denotes the latter space by $\langle 1|A^{\perp}$. Let us take

$$P = A^{\dagger}|0\rangle \cdot \langle 1|AA^{\dagger}|0\rangle^{-1} \cdot \langle 1|A, \tag{11.6}$$

as the projector onto the sub-space specified by the bra-vector $\langle 1|A$. We assume that the ket-vacuum $|0\rangle$ satisfies

$$\hat{H}|0\rangle = 0. \tag{11.7}$$

The equation of motion for the ket-vector $\langle 1|A(t)$ can be rewritten as

$$\frac{d}{dt}\langle 1|A(t) = i\langle 1|[\hat{H},\, A(t)]$$

$$
\begin{aligned}
&= -i\langle 1|A(t)\hat{H} \\
&= -i\langle 1|A\hat{H}\mathrm{e}^{-i\hat{H}t} \\
&= i\langle 1|[\hat{H},\ A]\mathrm{e}^{-i\hat{H}t} \\
&= \langle 1|\dot{A}\mathrm{e}^{-i\hat{H}t} \\
&\doteq i\Omega\langle 1|A(t) + \langle 1|F(t),
\end{aligned}
\tag{11.8}
$$

where we introduced

$$F(t) = \mathrm{e}^{i\hat{H}t}F\mathrm{e}^{-i\hat{H}t}, \tag{11.9}$$

with

$$F = \dot{A} - i\Omega A, \qquad \dot{A} = i[\hat{H},\ A], \tag{11.10}$$

$$i\Omega = \langle 1|\dot{A}A^{\dagger}|0\rangle \cdot \langle 1|AA^{\dagger}|0\rangle^{-1}. \tag{11.11}$$

In deriving (11.8), we used (11.4). It is easily checked that F satisfies

$$\langle 1|FA^{\dagger}|0\rangle = 0. \tag{11.12}$$

Let us introduce

$$R(t) = \mathrm{e}^{i\hat{H}t(1-P)}R(0)\mathrm{e}^{-i\hat{H}t(1-P)}, \tag{11.13}$$

with

$$R(0) = F. \tag{11.14}$$

Since

$$\langle 1|X(1-P)A^{\dagger}|0\rangle = 0, \tag{11.15}$$

we see that

$$\langle 1|R(t)A^{\dagger}|0\rangle = 0. \tag{11.16}$$

This shows that the vector $\langle 1|R(t)$ belongs to the sub-space $\langle 1|A^{\perp}$. By making use of the formula

$$\mathrm{e}^{-i\hat{H}t(1-P)} = \mathrm{e}^{-\hat{H}t} + \int_0^{\infty} ds \mathrm{e}^{i\hat{H}(s-t)(1-P)} i\hat{H}P\mathrm{e}^{-i\hat{H}s}, \tag{11.17}$$

we can derive the relation between two ket-vectors, $\langle 1|R(t)$ and $\langle 1|F(t)$, in the form

$$\langle 1|R(t) = \langle 1|F(t) + \int_0^{\infty} ds \Gamma(t-s)\langle 1|A(s), \tag{11.18}$$

with

$$\Gamma(t) = \langle 1|R(t)R^{\dagger}(0)|0\rangle \cdot \langle 1|AA^{\dagger}|0\rangle^{-1}. \tag{11.19}$$

In deriving (11.18), we used (11.4) and (11.7).

Substituting (11.18) into (11.8), we obtain

$$\frac{d}{dt}\langle 1|A(t) = i\Omega\langle 1|A(t) - \int_0^{\infty} ds \Gamma(t-s)\langle 1|A(s) + \langle 1|R(t), \tag{11.20}$$

where $i\Omega$ and $\Gamma(t)$ are defined respectively by (11.11) and (11.19). Note that $R(t)$ satisfies the orthogonality (11.16). The equation (11.20) may be intimately related to the Mori formula [41], and gives its reasonable interpretation.

With the help of the orthogonality (11.16) between the vectors $\langle 1|R(t)$ and $A^\dagger|0\rangle$, we have the equation of motion for the correlation matrix

$$\Xi(t) = \langle 1|A(t)A^\dagger|0\rangle \cdot \langle 1|AA^\dagger|0\rangle^{-1}, \tag{11.21}$$

in the form

$$\frac{d}{dt}\Xi(t) = i\Omega\Xi(t) + \int_0^\infty ds\Gamma(t-s)\Xi(s). \tag{11.22}$$

We see that $i\Omega$ is related with $\Xi(t)$ by the relation

$$i\Omega = \frac{d}{dt}\Xi(t)\Big|_{t=0}. \tag{11.23}$$

Note that since

$$\langle 1|A(t)|0\rangle = \langle 1|A|0\rangle, \tag{11.24}$$

(see (11.4) and (11.7)), we can make $\Xi(t)$ a second cumulant matrix by putting $\langle 1|A|0\rangle = 0$.

The *Langevin equation* (11.20) of the ket-vector states $\langle 1|A(t)$ may have a deeper meaning in the sense of the rigged Hilbert space [63] where the bra-vector states belong to the space conjugate to the ket-vector space which is spanned only by a set of observable states generated on the ket-vacuum $|0\rangle$. The conjugate space is wider than the observable ket-vector space, therefore the sub-space $\langle 1|A^\perp$ can have a rich variety. This variety may take care of the jump in the re-interpretation of the Mori formula from an ordinary differential equation to a stochastic differential equation. Its detailed investigation will be published elsewhere.

12. Semi-Free System with a Stationary Process

For the cases of *stationary* quantum stochastic processes, we just need to make the substitutions

$$i\Sigma^<(t) = 2\kappa\bar{n}, \quad \omega(t) = \omega, \quad \kappa(t) = \kappa, \tag{12.1}$$

in the formulae derived in previous sections.

Then, the Boltzmann equation (4.14) reads

$$\frac{d}{dt}n(t) = -2\kappa\left[n(t) - \bar{n}\right], \tag{12.2}$$

with

$$\bar{n} = \frac{1}{e^{\beta\omega} - 1}, \tag{12.3}$$

where β is the inverse of the temperature T of the environment, i.e. $\beta = 1/T$. The Boltzmann constant has been put to equal to 1. The Boltzmann equation (12.2) describes the system of a damped harmonic oscillator.

Substituting the Boltzmann equation (12.2) into the semi-free hat-Hamiltonian (4.19), we have [1,2,12]

$$\begin{aligned} \hat{H} &= \omega\left(a^\dagger a - \tilde{a}^\dagger \tilde{a}\right) \\ &\quad -i\kappa\left[(1+2\bar{n})\left(a^\dagger a + \tilde{a}^\dagger\tilde{a}\right) - 2(1+\bar{n})a\tilde{a} - 2\bar{n}a^\dagger\tilde{a}^\dagger\right] - i2\kappa\bar{n}. \end{aligned} \tag{12.4}$$

This hat-Hamiltonian is the same expression as the one derived by means of the principle of correspondence when NETFD was constructed first based upon the projection operator formalism of the damping theory [1,2] (see Appendix B).

The Fokker-Planck equation of the model is given by

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{12.5}$$

with (12.4), which is solved as

$$|0(t)\rangle = \exp\left[[n(t) - n(0)]\gamma^{\natural}\tilde{\gamma}^{\natural}\right]|0\rangle, \tag{12.6}$$

where $\gamma^{\natural}$ and $\tilde{\gamma}^{\natural}$ are defined by (12.12) below. The ket-thermal vacuum, $|0\rangle = |0(0)\rangle$, is specified by (4.15) which can be expressed in terms of d and $\tilde{d}^{\dagger}$, which are introduced in (12.10) below, as

$$d|0\rangle = (n - \bar{n})\, \tilde{d}^{\dagger}|0\rangle. \tag{12.7}$$

The attractive expression (12.6), which was obtained first in [64], led us to the notion of a mechanism named the *spontaneous creation of dissipation* [13,14], [65]-[67]. We can obtain the result (12.6) only by algebraic manipulations. This technical convenience of the operator algebra in NETFD, which is very much similar to that of the usual quantum mechanics, enables us to treat open systems in far-from-equilibrium state simpler and more transparent [42]-[47].

The hat-Hamiltonian (12.4) can be also written in the form

$$\hat{H} = \omega\left(d^{\dagger}d - \tilde{d}^{\dagger}\tilde{d}\right) - i\kappa\left(d^{\dagger}d + \tilde{d}^{\dagger}\tilde{d}\right) \tag{12.8}$$

$$= \omega\left(\gamma^{\natural}\gamma_t - \tilde{\gamma}^{\natural}\tilde{\gamma}_t\right) - i\kappa\left(\gamma^{\natural}\gamma_t + \tilde{\gamma}^{\natural}\tilde{\gamma}_t + 2\,[n(t) - \bar{n}]\,\gamma^{\natural}\tilde{\gamma}^{\natural}\right), \tag{12.9}$$

where $d^{\mu=1} = d$, $d^{\mu=2} = \tilde{d}^{\dagger}$ and $\bar{d}^{\mu=1} = d^{\dagger}$, $\bar{d}^{\mu=2} = -\tilde{d}$ are defined by

$$d^{\mu} = Q^{-1\mu\nu}a^{\nu}, \qquad \bar{d}^{\mu} = \bar{a}^{\nu}Q^{\nu\mu}, \tag{12.10}$$

with

$$Q^{\mu\nu} = \begin{pmatrix} 1 & \bar{n} \\ 1 & 1 + \bar{n} \end{pmatrix}. \tag{12.11}$$

The annihilation and creation operators, $\gamma^{\mu=1} = \gamma_t$, $\gamma^{\mu=2} = \tilde{\gamma}^{\natural}$ and $\bar{\gamma}^{\mu=1} = \gamma^{\natural}$, $\bar{\gamma}^{\mu=2} = -\tilde{\gamma}_t$, are defined through the relation

$$\gamma(t)^{\mu} = \hat{V}^{-1}(t)\gamma^{\mu}\hat{V}(t), \qquad \bar{\gamma}(t)^{\mu} = \hat{V}^{-1}(t)\bar{\gamma}^{\mu}\hat{V}(t), \tag{12.12}$$

where $\hat{V}(t)$ is specified by

$$\frac{\partial}{\partial t}\hat{V}(t) = -i\hat{H}\hat{V}(t), \tag{12.13}$$

with the initial condition $\hat{V}(0) = 1$.

It is easy to see from the diagonalized form (12.8) of $\hat{H}$ that

$$\begin{aligned} d(t) &= \hat{V}^{-1}(t)\, d\, \hat{V}(t) = d\, e^{-(i\omega+\kappa)t}, \\ \tilde{d}^{\dagger\dagger}(t) &= \hat{V}^{-1}(t)\, \tilde{d}^{\dagger}\, \hat{V}(t) = \tilde{d}^{\dagger}\, e^{-(i\omega-\kappa)t}. \end{aligned} \tag{12.14}$$

On the other hand, it is easy to see from the normal product form (12.9) of $\hat{H}$ that it satisfies **B2** in section 3., since the annihilation and creation operators satisfy

$$\gamma_t|0(t)\rangle = 0, \qquad \langle 1|\tilde{\gamma}^{\natural} = 0. \tag{12.15}$$

The difference between the operators which diagonalize $\hat{H}$ and the ones which make $\hat{H}$ in the form of normal product is one of the features of NETFD, and shows the point that the formalism is quite different from usual quantum mechanics and quantum field theory. This is a manifestation of the fact that the hat-Hamiltonian is a time-evolution generator for irreversible processes.

Let us check here the irreversibility of the system. The entropy of the system is given by

$$\mathcal{S}(t) = -\left\{n(t)\ln n(t) - [1+n(t)]\ln[1+n(t)]\right\}, \tag{12.16}$$

whereas the heat change of the system is given by

$$d'Q = \omega dn. \tag{12.17}$$

Thermodynamics tells us that

$$d\mathcal{S} = d\mathcal{S}_e + d\mathcal{S}_i, \qquad d\mathcal{S}_e = d'Q/T_R, \tag{12.18}$$

$$d\mathcal{S}_i \geq 0. \tag{12.19}$$

The latter inequality (12.19) is the second law of thermodynamics. Putting (12.16) and (12.17) into (12.18), for $d\mathcal{S}$ and $d\mathcal{S}_e$, respectively, we have a relation for the entropy production rate [4]

$$\frac{d\mathcal{S}_i}{dt} = \frac{d\mathcal{S}}{dt} - \frac{d\mathcal{S}_e}{dt} = 2\kappa\,[n(t)-\bar{n}]\ln\frac{n(t)[1+\bar{n}]}{\bar{n}[1+n(t)]} \geq 0. \tag{12.20}$$

It is easy to check that the expression on the right-hand side of the second equality satisfies the last inequality which is consistent with (12.19). The equality realizes either for the thermal equilibrium state, $n(t) = \bar{n}$, or for the quasi-stationary process, $\kappa \to 0$.

13. A Mathematical Reformulation of NETFD

In this section, we will show a mathematical structure of the formulation of NETFD. Here, for simplicity, we restrict ourselves to the case of the *stationary* quantum stochastic process specified in the previous section.

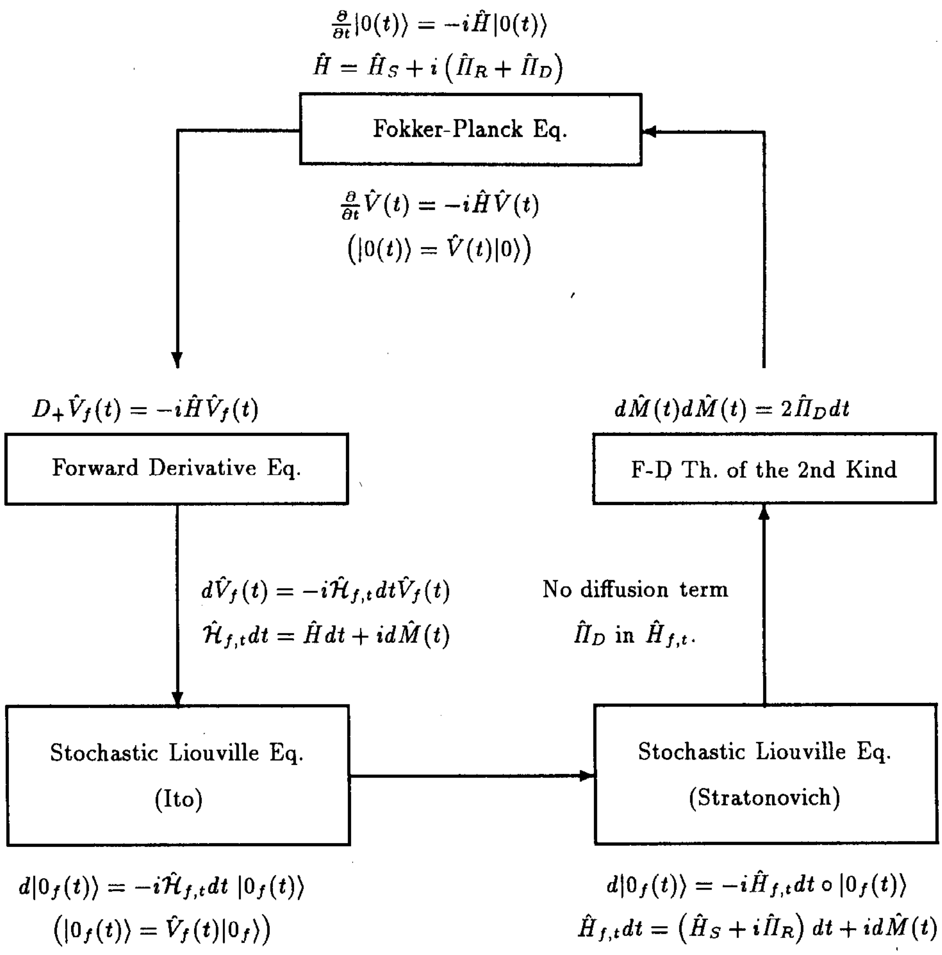

Figure 3: A Mathematical Structure of Quantum Stochastic Equations

13.1. Fokker-Planck Equation

Let us consider a system specified by the Fokker-Planck equation (12.5):

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{13.1}$$

with the infinitesimal time-evolution generator

$$\hat{H} = \hat{H}_S + i\hat{\Pi}, \tag{13.2}$$

where $\hat{H}_S$ is constituted by the system Hamiltonian H_S in the form

$$\hat{H}_S = H_S - \tilde{H}_S. \tag{13.3}$$

We assume that the operator $\hat{H}$ has the *tildian* nature **B1** in section 3.:

$$\left(i\hat{H}\right)^{\sim} = i\hat{H}, \tag{13.4}$$

and has the property **B2**:

$$\langle 1|\hat{H} = 0, \tag{13.5}$$

which guarantees the conservation of probability.

For later convenience, we decompose the generator $\hat{\Pi}$ into two parts as

$$\hat{\Pi} = \hat{\Pi}_R + \hat{\Pi}_D, \tag{13.6}$$

where $\hat{\Pi}_R$ and $\hat{\Pi}_D$ represent respectively a *relaxational* and a *diffusive* time evolution.

The formal solution of (13.1) is given by

$$|0(t)\rangle = \hat{V}(t)|0\rangle, \tag{13.7}$$

with $|0\rangle = |0(0)\rangle$, where $\hat{V}(t)$ is specified by (12.13).

13.2. Forward Derivative and Martingale

Now, let us examine a differential equation

$$D_+\hat{V}_f(t) = -i\hat{H}\hat{V}_f(t), \tag{13.8}$$

given by the Fokker-Planck hat-Hamiltonian $\hat{H}$ and with the forward derivative

$$D_+X(t) = \lim_{\epsilon\to 0} E_{t]}\left(\frac{X(t+\epsilon) - X(t)}{\epsilon}\right). \tag{13.9}$$

The *conditional expectation* $E_{t]}$ is defined, for example, by

$$E_{t]}(X) = e_{t]}Xe_{t]} \otimes 1_{[t}, \tag{13.10}$$

where

$$e_{t]} = 1_{t]} = \int_{-\infty}^{t} d\mu(\tau)|\tau\rangle\langle\tau|, \qquad 1_{[t} = \int_{t}^{\infty} d\mu(\tau)|\tau\rangle\langle\tau|. \tag{13.11}$$

We are assuming that a thermal space for the stochastic operators, which is constituted by a direct product of two Hilbert spaces, i.e. $\mathcal{H} \otimes \mathcal{H}$, can be represented by an orthonormal complete basis $\{|\tau\rangle,\ -\infty < \tau < \infty\}$ satisfying

$$\int_{-\infty}^{\infty} d\mu(\tau)|\tau\rangle\langle\tau| = 1, \qquad \langle\tau|\tau'\rangle d\mu(\tau) = \delta(\tau - \tau')d\mu(\tau). \tag{13.12}$$

with an appropriate measure $d\mu(\tau)$. We see that the *orthogonal projection* $e_{t]}$ satisfies

$$e_{t]}^{\dagger} = e_{t]}, \tag{13.13}$$

$$e_{s]}e_{t]} = e_{t]}e_{s]} = e_{s]}, \qquad \text{for } s \leq t. \tag{13.14}$$

A generalization of Theorem 5.3 in [68] gives the solution of (13.8) in the form

$$\hat{V}_f(t) - \hat{V}_f(s) = -i\int_s^t dt'\hat{H}\hat{V}_f(t') + \hat{M}_0(t) - \hat{M}_0(s), \tag{13.15}$$

where $\hat{M}_0(t)$ is a *martingale* satisfying

$$E_{s]}\left(\hat{M}_0(t)\right) = \hat{M}_0(s). \tag{13.16}$$

The equation (13.15) can be expressed as

$$d\hat{V}_f(t) = -i\hat{H}\hat{V}_f(t)dt + d\hat{M}_0(t), \tag{13.17}$$

with the increment $d\hat{V}_f(t) = \hat{V}_f(t+dt) - \hat{V}_f(t)$ and $d\hat{M}_0(t) = \hat{M}_0(t+dt) - \hat{M}_0(t)$.

13.3. Stochastic Liouville Equation

The equation (13.17) can be written as

$$d\hat{V}_f(t) = -i\hat{\mathcal{H}}_{f,t}dt\hat{V}_f(t), \tag{13.18}$$

with

$$\hat{\mathcal{H}}_{f,t}dt = \hat{H}dt + id\hat{M}(t), \tag{13.19}$$

where we defined the operator $d\hat{M}(t)$ by

$$d\hat{M}(t) = d\hat{M}_0(t)\hat{V}_f^{-1}(t). \tag{13.20}$$

Note that the operator $d\hat{M}(t)$ has a property

$$E_{t]}\left(d\hat{M}(t)\right) = 0, \tag{13.21}$$

which is shown by the property (13.16) of martingale and by the definition (13.20) of $d\hat{M}(t)$.

In order that the generator $\hat{\mathcal{H}}_{f,t}dt$ has a tildian nature and guarantees the conservation of probability, we require that for $d\hat{M}(t)$

$$\left\{d\hat{M}(t)\right\}^{\sim} = d\hat{M}(t). \tag{13.22}$$

We interpret the product $d\hat{M}(t)\hat{V}_f(t)$ in (13.18) as

$$d\hat{M}(t)\hat{V}_f(t) = \left[\hat{M}(t+dt) - \hat{M}(t)\right]\hat{V}_f(t), \tag{13.23}$$

i.e. the time-evolution generator (13.19) is of the Ito type.

Since we will restrict ourselves, in the following in this paper, to the case of *Gaussian* and *white* stationary quantum stochastic process (quantum Wiener process), let us put the following conditions on $d\hat{M}(t)$:

$$d\hat{M}(t) = O(\sqrt{dt}), \tag{13.24}$$

Introducing a stochastic thermal ket-vacuum $|0_f(t)\rangle$ by

$$|0_f(t)\rangle = \hat{V}_f(t)|0_f\rangle, \tag{13.25}$$

with $|0_f\rangle = |0_f(0)\rangle$, and by making use of (13.18), we obtain the stochastic Liouville equation of the Ito type

$$d|0_f(t)\rangle = -i\hat{\mathcal{H}}_{f,t}dt|0_f(t)\rangle. \tag{13.26}$$

Note that, taking the conditional expectation $E_{0]}$ of (13.26), we can get the Fokker-Planck equation (13.1) with the help of the property (13.21) and the interpretation (13.23). In the course of the derivation, we passed the process like

$$E_{0]}\left(|0_f(t)\rangle\right) = E_{0]}(\hat{V}_f(t))|0_f\rangle = \hat{V}(t)|0\rangle = |0(t)\rangle, \tag{13.27}$$

since $|0_f\rangle = |0\rangle$. Equivalently, we see that

$$E_{0]}(\hat{\mathcal{H}}_{f,t}) = \hat{H}. \tag{13.28}$$

13.4. Fluctuation-Dissipation Theorem of the Second Kind

With the help of the connection formula (D.8) between the Ito and the Stratonovich multiplications, we convert (13.18) into the equation of the Stratonovich type as

$$d\hat{V}_f(t) = -i\hat{H}_{f,t}dt \circ \hat{V}_f(t), \tag{13.29}$$

with

$$\hat{H}_{f,t}dt = \hat{\mathcal{H}}_{f,t}dt - \frac{1}{2}id\hat{M}(t)d\hat{M}(t), \tag{13.30}$$

where we used the conditions (13.24).

In order to derive the quantum Langevin equation of the Stratonovich type, we assume that the term of $d\hat{M}(t)d\hat{M}(t)$ cancels the *diffusion term* $\hat{\Pi}_D$ in $\hat{\mathcal{H}}_{f,t}dt$:

$$d\hat{M}(t)d\hat{M}(t) = 2\hat{\Pi}_D dt. \tag{13.31}$$

Then, $\hat{H}_{f,t}dt$ becomes

$$\hat{H}_{f,t}dt = \hat{H}_S + i\hat{\Pi}_K + id\hat{M}(t). \tag{13.32}$$

The relation (13.31) is the content of the fluctuation-dissipation theorem of the second kind.

The stochastic time-evolution equation of the Stratonovich type is thus given by

$$d\hat{V}_f(t) = -i\hat{H}_{f,t}dt \circ \hat{V}_f(t), \tag{13.33}$$

with the infinitesimal time-evolution generator (13.32). Applying $|0_f\rangle$ to (13.33), we finally obtain the stochastic Liouville equation of the Stratonovich type in the form

$$d|0_f(t)\rangle = -i\hat{H}_{f,t}dt \circ |0_f(t)\rangle. \tag{13.34}$$

The Fokker-Planck equation (13.1) can also be derived systematically from the stochastic Liouville equation (13.34) of the Stratonovich type (see Appendix F).

13.5. Langevin Equation

With the help of the algebraic identity

$$dA(t) = d\hat{V}_f^{-1}(t)A\hat{V}_f(t) + \hat{V}_f^{-1}Ad\hat{V}_f(t), \tag{13.35}$$

of the Stratonovich stochastic multiplication with (13.33), and the equation for the inverse of $\hat{V}_f(t)$:

$$d\hat{V}_f^{-1}(t) = i\hat{V}_f^{-1}(t) \circ \hat{H}_{f,t}dt, \tag{13.36}$$

we can derive the Langevin equation of the Stratonovich type for the dynamical quantity

$$A(t) = \hat{V}_f^{-1}(t)A\hat{V}_f(t), \tag{13.37}$$

in the form

$$dA(t) = i[\hat{H}_f(t)dt \,\,^{\circ}_{,}\,\, A(t)], \tag{13.38}$$

where $\hat{H}_f(t)$ and $[X(t) \,^{\circ}_{,}\, Y(t)]$ are defined respectively by (8.41) and (8.42). The equation (13.38) indicates that the quantum Langevin equation of the Stratonovich type is nothing but a Heisenberg equation with the stochastic time-evolution generator $\hat{H}_f(t)dt$.

On the other hand, by means of the algebraic identity

$$dA(t) = d\hat{V}_f^{-1}(t)A\hat{V}_f(t) + \hat{V}_f^{-1}Ad\hat{V}_f(t) + d\hat{V}_f^{-1}Ad\hat{V}_f(t), \tag{13.39}$$

with (13.18), and the equation of the inverse of $\hat{V}_f(t)$ of the Ito type

$$d\hat{V}_f^{-1}(t) = i\hat{V}_f^{-1}(t)\left[\hat{\mathcal{H}}_{f,t}dt - id\hat{M}(t)d\hat{M}(t)\right], \tag{13.40}$$

we obtain the Langevin equation of the Ito type in the form

$$\begin{aligned} dA(t) &= i\left[\hat{H}_f(t),\ A(t)\right] \\ &\quad +\frac{1}{2}d\hat{M}^{(H)}(t)\left[d\hat{M}^{(H)}(t),\ A(t)\right] + \frac{1}{2}\left[A(t),\ d\hat{M}^{(H)}(t)\right]d\hat{M}^{(H)}(t), \end{aligned} \tag{13.41}$$

where $\hat{H}_f(t)$ is defined by (8.41) and $d\hat{M}^{(H)}(t)$ by

$$d\hat{M}^{(H)}(t) = \hat{V}_f^{-1}(t)d\hat{M}(t)\hat{V}_f(t). \tag{13.42}$$

The formula (13.40) can be derived by treating

$$d\left(\hat{V}_f^{-1}(t)\hat{V}_f(t)\right) = 0, \tag{13.43}$$

in terms of the Ito differentiation.

13.6. Determination of $d\hat{M}(t)$

There are at least two options for the determination of the general form of the martingale $d\hat{M}$.

One is derived with the condition

$$\langle 1|d\hat{M}(t) = 0, \tag{13.44}$$

which ensure the conservation of probability within the relevant system. The obtained infinitesimal time-evolution generator $\hat{\mathcal{H}}_{f,t}dt$ has the same structure as the hat-Hamiltonian (8.7) for the Ito stochastic Liouville equation (8.1) in section 8..

The other is derived with the condition

$$d\hat{M}(t)^\dagger = d\hat{M}(t), \tag{13.45}$$

i.e. hermiticy. The infinitesimal time-evolution generator has the same structure as the hat-Hamiltonian studied in section 10.. This case may be intimately related to those investigated by mathematicians [68]-[71]. Its detailed report will be published elsewhere.

14. Relation to Monte Carlo Wave-Function Method

In this section, we will investigate the Fokker-Planck equation (12.5):

$$\frac{\partial}{\partial t}|0(t)\rangle = -i\hat{H}|0(t)\rangle, \tag{14.1}$$

in order to reveal the relation of NETFD to the Monte Carlo wave-function method, i.e. quantum jump simulation [72]-[74].

Let us decompose the hat-Hamiltonian (12.4) as

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \tag{14.2}$$

with

$$\hat{H}_0 = \omega\left(a^\dagger a - \tilde{a}^\dagger \tilde{a}\right) - i\kappa\,(2\bar{n}+1)\left(a^\dagger a + \tilde{a}^\dagger \tilde{a}\right), \tag{14.3}$$

$$\hat{H}_1 = 2i\kappa\left[(\bar{n}+1)\,a\tilde{a} + \bar{n}a^\dagger \tilde{a}^\dagger\right] - 2i\kappa\bar{n}, \tag{14.4}$$

and consider an equation:

$$\frac{\partial}{\partial t}|0_0(t)\rangle' = -i\hat{H}_0|0_0(t)\rangle'. \tag{14.5}$$

Note that $\hat{H}_1$ contains cross terms between tilde and non-tilde operators. We see that $\hat{H}_0$ and $\hat{H}_1$ have the properties

$$\langle 1|\hat{H}_0 = -2i\kappa\,(2\bar{n}+1)\,\langle 1|a^\dagger a, \tag{14.6}$$

$$\langle 1|\hat{H}_1 = 2i\kappa\,(2\bar{n}+1)\,\langle 1|a^\dagger a. \tag{14.7}$$

Introducing the *wave-functions* $|\psi(t)\rangle$ and $|\tilde{\psi}(t)\rangle$ through

$$|0_0(t)\rangle' = |\psi(t)\rangle|\tilde{\psi}(t)\rangle, \tag{14.8}$$

we have from (14.5) Schrödinger equations of the form

$$\frac{\partial}{\partial t}|\psi(t)\rangle = -iH_0|\psi(t)\rangle, \tag{14.9}$$

and its tilde conjugate, where

$$H_0 = \omega a^\dagger a - i\kappa\,(2\bar{n}+1)\,a^\dagger a. \tag{14.10}$$

The Monte Carlo simulations for quantum systems are performed for the Schrödinger equation (14.9) [72]-[74].

The time generation due to the hat-Hamiltonian $\hat{H}_0$ does not preserve the normalization of the ket-vacuum, i.e. the normalized ket-vacuum $|0(t)\rangle$ evolves for the time inclement dt as

$$\begin{aligned}\langle 1|0_0(t+dt)\rangle' &= \langle 1|\left(1 - i\hat{H}_0 dt\right)|0(t)\rangle \\ &= 1 - dp(t),\end{aligned} \tag{14.11}$$

with

$$dp(t) = 2\kappa\,(2\bar{n}+1)\,n(t)dt. \tag{14.12}$$

The recipe of the quantum jump simulation is that, for a time increment dt,

1. When $dp(t) < \varepsilon$ with a given positive constant ε, the normalized ket-vacuum evolves as

$$|0(t)\rangle \longrightarrow |0_0(t+dt)\rangle = \frac{|0_0(t+dt)\rangle'}{1-dp(t)} = \frac{|\psi(t+dt)\rangle}{\sqrt{1-dp(t)}}\frac{|\tilde{\psi}(t+dt)\rangle}{\sqrt{1-dp(t)}}. \tag{14.13}$$

2. In the case $dp(t) > \varepsilon$, a quantum jump comes in:

$$|0_1(t+dt)\rangle = \frac{-i\hat{H}_1 dt|0(t)\rangle}{dp(t)}. \tag{14.14}$$

The time increment dt should be chosen as the condition $dp(t) \ll 1$ being satisfied.

Averaging the processes $|0_0(t)\rangle$ and $|0_1(t)\rangle$ with the respective probability $1 - dp(t)$ and $dp(t)$ (i.e. these ket-vacuums looks like satisfying a certain kind of stochastic Liouville equation):

$$|0(t+dt)\rangle = [1 - dp(t)]\,|0_0(t+dt)\rangle + dp(t)|0_1(t+dt)\rangle, \tag{14.15}$$

we can obtain the Fokker-Planck equation (14.1).

15. Discussions

We showed that, by the success of the formulation of NETFD, we came to be able to investigate dissipative quantum systems systematically upon a unified stand point. The unified formalism for quantum systems covering whole the aspects, *I* to *IV* in Table 1, was realized first by means of the framework of NETFD.

The relation between the Langevin equation and the stochastic Liouville equation is the same as the one between the Heisenberg equation and the Schrödinger equation in quantum mechanics and in quantum field theory. Since they are the stochastic differential equations, there are two types of stochastic multiplication, i.e. the Ito and the Stratonovich types. The Langevin equation (8.39) of the Stratonovich type has the same structure as the Heisenberg equation of motion for analytical quantities. Whereas, the Ito type (8.49) contains an extra term proportional to $dW(t)d\tilde{W}(t)$ due to the difference of stochastic differentiations. Although the stochastic Liouville equations both of the Stratonovich and Ito types, (8.27) and (8.1), have the same form, the latter is more convenient than the former to get the corresponding Fokker-Planck equation (8.24) by taking random average. It is because of the characteristics of the Ito multiplication. The equation of motion for the dynamical variables taken both the random average and the vacuum expectation value can be obtained by two paths, i.e. the one from the Langevin equation directly by taking both the random average and the vacuum expectation, the other from the Fokker-Planck equation by taking the vacuum expectation of the operators corresponding to the dynamical variables. It should be noted that the discovery of the stochastic Liouville equation is the key point for the construction of whole the unified quantum canonical formalism (see Fig. 1).

Note that whole the structure of the formulation is consistent with that of classical one. For example, the Stratonovich stochastic differential equation contains the relaxation generator but does not contain the diffusion generator, whereas the Ito equation does both generators. Note that this consistency is violated for the approach given in section 10. with the *hermitian* interaction hat-Hamiltonian. Note also that this tradition seems to give us the fluctuation-dissipation theorem as shown in section 13.. The relation of the present formulation with those of mathematicians and some physicists [68]-[71],[75]-[79] would be an interesting future problem. For the physical side, the difficulties related in the theories of quantum Langevin equation, claimed by Kubo [80], should be reinvestigated from the unified stand point of NETFD [36]. The first claim is that the representation space of the Langevin equation should be an extended Hilbert space which is constituted by both the one for the relevant system and the one for the random force (an irrelevant system). However, usually the equation of motion for the random force operator is not considered. The second is that the correlations of random force operators for *thermal ensemble* do not satisfy KMS-condition [81,82] in the case of the *white* process for quantum systems. The third claim was how one can obtain the correlation of random force operators for the Langevin equation which is compatible with the master equation derived by the non-conventional treatment of the damping theory [50]-[55],[43],[44], where the effect of non-linearity within a relevant system on its relaxation behavior is taken into account (see [55] for the last claim).

The correspondence of the equation of motion for the ket-vector $\langle\!\langle 1|A(t)$, (8.58) and (8.59), to the Langevin equation in [58] is an attractive future problem in connection with the interpretation of Mori formalism given in section 11.. For spin systems, there is a similar correspondence between the equation of motion for ket-vector and the Langevin equation in [83]. An investigation related to these correspondences may give us a deeper insight for the derivation of the stochastic differential equations from a microscopic point of view. The problem of the representation space within NETFD, e.g. a foundation of the concept of the spontaneous creation of dissipation [13,14], should be studied with the help of the rigged Hilbert space [63]. In this connection, we expect that the mathematical approach in [84]-[86] will provide us with an attractive view point. Upon the unified formulation of NETFD, we may be able to give a further understanding of the quantum jump within the Monte Carlo wave-function method [72]-[74], and also of the quantum-state diffusion method [87]-[89].

With the help of the hat-Hamiltonian for the Fokker-Planck equation, we can construct the Heisenberg equation for coarse grained operators. As was mentioned before, the existence of the Heisenberg equation of motion for coarse grained operators enabled us to construct the canonical formalism of the dissipative quantum fields. It is quite interesting that for somewhat artificial values of μ, ν, i.e. $\mu = 1 + \bar{n}$, $\nu = -\bar{n}$, we can obtain directly the coarse grained equations of motion (4.31) and (4.32) for the stationary case:

$$\frac{d}{dt}a(t) = -i\omega a(t) - \kappa\left[(1+2\bar{n})\,a(t) - 2\bar{n}\tilde{a}^{\dagger}(t)\right], \tag{15.1}$$

$$\frac{d}{dt}a^{\dagger}(t) = i\omega a^{\dagger}(t) + \kappa\left[(1+2\bar{n})\,a^{\dagger}(t) - 2\,(1+\bar{n})\,\tilde{a}(t)\right], \tag{15.2}$$

by taking the random average of the Langevin equations (8.55) and (8.56). For this case, (8.47) tells us that the Stratonovich and the Ito multiplications are identical, and (8.15) gives

$$\langle dW(t)d\tilde{W}(s)\rangle = \langle d\tilde{W}(s)dW(t)\rangle = 0. \tag{15.3}$$

The latter indicates that

$$\bar{n}\left\langle dF(t)dF^{\dagger}(s)\right\rangle = (1+\bar{n})\left\langle dF^{\dagger}(s)dF(t)\right\rangle. \tag{15.4}$$

This is nothing but the KMS-condition [81,82]. The physical meaning of this artificial case is still to be investigated.

The above mentioned future problems are now under investigation, and will be reported elsewhere in the near future.

Let us close this paper by mentioning about those which were not included in the above sections. It was shown that the *divisor* method of the canonical quantum field theory can be generalized to the present dissipative quantum field theory [15,16]. The derivation of the generalized kinetic equation within NETFD were studied [21]. Note that most of the studies by means of NETFD were those in the *kinetic stage*. Thermal processes in the *hydrodynamical stage* has started to be investigated by means of NETFD [23]. There, the concept of non-equilibrium thermodynamics, especially that of the *local equilibrium*, is tried to be interpreted in terms of the concept of quantum field theory. There were several applications of NETFD to optical systems [42]-[47] and spin relaxation [43]. Dynamical rearrangements of vacuum in the thermal space were investigated [64] for the boson transformation and the BCS model. The cases of fermion were not investigated in this paper. It is somewhat straightforward to extend whole the framework to the case of fermion fields.

We expect that the present unified framework of NETFD may open a new field of dissipative quantum field theory which will provide us with a deeper insight of nature from the stand point of *quantum coherence and dissipation* for example.

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A Density Operator Method

Here, we show how we had been dealing with the model within the density operator formalism before NETFD was constructed. The master equation for a damped harmonic oscillator is given by [9]

$$\frac{\partial}{\partial t}\rho_S(t) = -i\left(H_S^\times + i\Pi\right)\rho_S(t), \tag{A.1}$$

with the symbol $H_S^\times X = [H_S,\ X]$, where H_S is the Hamiltonian of the system we are interested in:

$$H_S = \omega a^\dagger a, \qquad \omega = \epsilon - \mu, \tag{A.2}$$

with ϵ and μ being the one-particle energy and the chemical potential, respectively, and where Π is the damping operator:

$$\Pi X = \kappa\left\{[aX,\ a^\dagger] + [a,\ Xa^\dagger]\right\} + 2\kappa\bar{n}[a,\ [X,\ a^\dagger]], \tag{A.3}$$

with $\bar{n}$ being given by (12.3), and

$$\kappa = \Re\mathrm{e}\ g^2 \int_0^\infty dt \sum_{\mathbf{k}} \left\langle [R_{\mathbf{k}}(t),\ R_{\mathbf{k}}^\dagger(0)]\right\rangle_R e^{i\omega t}. \tag{A.4}$$

Here, we have introduced the average, $\langle\cdots\rangle_R = \mathrm{tr}_R \cdots \rho_R$, where the density operator for a reservoir is given by $\rho_R = Z_R^{-1} e^{-\beta H_R}$, $Z_R = \mathrm{tr}_R\, e^{-\beta H_R}$. The coupling constant g represents the strength of the interaction between the damped harmonic oscillator and the reservoir whose temperature is $T = \beta^{-1}$. We see that the one-particle distribution function, defined by $n(t) = \mathrm{tr}\, a^\dagger a \rho_S(t)$, satisfies the Boltzmann equation (12.2).

The above master equation (A.1) can be obtained by projecting out the reservoir by means of the damping theory [9]-[11], starting with the Liouville equation:

$$\frac{\partial}{\partial t}\rho(t) = -iH^\times \rho(t), \tag{A.5}$$

with the model given by the Hamiltonian

$$H = H_S + H_R + H_I, \tag{A.6}$$

where H_I is the Hamiltonian describing the interaction between the system and the reservoir:

$$H_I = g\sum_{\mathbf{k}} \left(aR_{\mathbf{k}}^\dagger + \mathrm{h.c.}\right), \tag{A.7}$$

with $R_{\mathbf{k}}^\dagger$ and $R_{\mathbf{k}}$ being the operators of the reservoir, and H_R is the Hamiltonian of the reservoir the explicit form of which needs not be specified to get the master equation (A.1). The coarse-grained density operator $\rho_S(t)$ is defined by $\rho_S(t) = \mathrm{tr}_R\, \rho(t)$.

Introducing the boson coherent state representation of the anti-normal ordering [59]-[61] through

$$\rho_S(t) = \int \frac{d^2z}{\pi} f_S(t)|z\rangle\langle z|, \tag{A.8}$$

with the boson coherent state $|z\rangle$, defined by $a|z\rangle = z|z\rangle$, we can map the master equation (A.1) into a partial differential equation for the c-number function $f_S(t)$ as [9]

$$\frac{\partial}{\partial t} f_S(t) = \left[-i\omega\left(\partial_* z^* - \mathrm{c.c.}\right) + \kappa\left(\partial_* z^* + \mathrm{c.c.}\right) + 2\kappa\bar{n}\partial_*\partial\right] f_S(t), \tag{A.9}$$

where we have introduced the abbreviation, $\partial = \partial/\partial z$, $\partial_* = \partial/\partial z^*$. This is nothing but a Fokker-Planck equation.

The Fokker-Planck equation (A.9) is transformed into

$$\frac{\partial}{\partial t} F(t) = 2\kappa\left(\xi\partial_\xi + \frac{1}{2} + \bar{n}\partial_\xi\xi\partial_\xi\right) F(t), \tag{A.10}$$

with the help of the relation

$$F(t) = \mathrm{e}^{it\omega(\partial_* z^* - \partial z)} f_S(t), \tag{A.11}$$

where $\xi = |z|^2$, and $\partial_\xi = \partial/\partial\xi$. We can solve (A.10) in the form

$$F(t) = \frac{1}{n(t)}\mathrm{e}^{-\xi/n(t)}, \tag{A.12}$$

with the initial condition $F(0) = f_S(0) = \frac{1}{n}e^{-\xi/n}$. Here, $n(t)$ in (A.12) satisfies the Boltzmann equation (12.2). In deriving the solution (A.12), we have used the Laguerre polynomials

$$L_\ell(\xi) = \frac{1}{\ell!}e^{\xi}(-\partial_\xi \xi \partial_\xi)^\ell e^{-\xi}, \tag{A.13}$$

and the relation

$$\sum_{\ell=0}^{\infty} \frac{1}{\ell!} L_\ell(\xi) x^\ell = \frac{\exp -\xi(x/(1-x))}{1-x}. \tag{A.14}$$

Substituting (A.12) into (A.11), and putting the obtained $f_S(t)$ into (A.8), we have

$$\rho_S(t) = \frac{1}{n(t)} \int \frac{d^2z}{\pi} e^{-|z|^2/n(t)} |z\rangle\langle z|. \tag{A.15}$$

This density operator contains the same information as the thermal ket-vacuum $|0(t)\rangle$ given by (12.6).

It may be worthwhile to note that the relation of the operator algebra for a harmonic oscillator within quantum mechanics to the Hermite polynomials is very much similar to the relation of the operator algebra for a damped harmonic oscillator within NETFD to the Laguerre polynomials.

B The Principle of Correspondence

With the principle of correspondence [90,1,2]:

$$\rho_S(t) \longleftrightarrow |0(t)\rangle, \tag{B.1}$$

$$A_1 \rho_S(t) A_2 \longleftrightarrow A_1 \tilde{A}_2^{\dagger} |0(t)\rangle, \tag{B.2}$$

the master equation (A.1) reduces to the Schrödinger equation (12.5) with the hat-Hamiltonian (12.4). It was noticed first by Crawford [91] that the introduction of two kinds of operators enables us to handle the Liouville equation as the Schrödinger equation.

C Linear Response of Material Systems

Let us consider the linear response to the external field of the system specified by $\hat{H}$.

Since the deformation of the thermal vacuum is given by

$$\delta|0(t)\rangle = e^{-i\hat{H}t}\hat{S}^{(1)}(t,t_0)e^{i\hat{H}t_0}|0(t_0)\rangle, \tag{C.1}$$

with

$$\hat{S}^{(1)}(t,t_0) = -i\int_{t_0}^{t} dt' \hat{H}'(t'), \tag{C.2}$$

the linear response of an observable

$$Q(t) = \sum_{\mathbf{k}} Q_{\mathbf{k}}^{\dagger}(t) + \text{h.c.}, \tag{C.3}$$

is given by

$$\delta\langle Q(t)\rangle = -\sum_{\mathbf{q}}\sum_{\mathbf{k}}\int_0^t dt'\,\Phi_{\mathbf{q},\mathbf{k}}(t,t')\Re\text{e}\,[g_{\mathbf{k}}z_{\mathbf{k}}(t')], \tag{C.4}$$

with

$$\Phi_{\mathbf{q},\mathbf{k}}(t,t') = i\langle 1|[Q_{\mathbf{q}}(t),\ a^\dagger_{\mathbf{k}}(t')]|0\rangle + \text{c.c.}. \tag{C.5}$$

Here, we assumed that the state of the external field is the coherent state defined by

$$b_{\mathbf{k}}|z\rangle = z_{\mathbf{k}}|z\rangle, \tag{C.6}$$

and that the field is given by

$$\Re\text{e}\ g_{\mathbf{k}}z_{\mathbf{k}}(t) = \Re\text{e}\ g^*_{\mathbf{k}}z^*_{\mathbf{k}}(t) = |g_{\mathbf{k}}z_{\mathbf{k}}|\cos(\omega_{\mathbf{k}}t+\phi). \tag{C.7}$$

D Ito and Stratonovich Multiplications

The definitions of the Ito [56] and the Stratonovich [57] multiplications are given, respectively, by

$$X^{(H)}(t)\cdot dY^{(H)}(t) = X^{(H)}(t)\left[Y^{(H)}(t+dt) - Y^{(H)}(t)\right], \tag{D.1}$$

$$dX^{(H)}(t)\cdot Y^{(H)}(t) = \left[X^{(H)}(t+dt) - X^{(H)}(t)\right]Y^{(H)}(t), \tag{D.2}$$

and

$$X^{(H)}(t)\circ dY^{(H)}(t) = \frac{X^{(H)}(t+dt)+X^{(H)}(t)}{2}\left[Y^{(H)}(t+dt) - Y^{(H)}(t)\right], \tag{D.3}$$

$$dX^{(H)}(t)\circ Y^{(H)}(t) = \left[X^{(H)}(t+dt) - X^{(H)}(t)\right]\frac{Y^{(H)}(t+dt)+Y^{(H)}(t)}{2}, \tag{D.4}$$

for arbitrary stochastic operators $X^{(H)}(t)$ and $Y^{(H)}(t)$ in the Heisenberg representation. From (D.1), (D.2) and (D.3), (D.4), we have the formulae which connect the Ito and the Stratonovich products in the differential form

$$X^{(H)}(t)\circ dY^{(H)}(t) = X^{(H)}(t)dY^{(H)}(t) + \frac{1}{2}dX^{(H)}(t)\cdot dY^{(H)}(t), \tag{D.5}$$

$$dX^{(H)}(t)\circ Y^{(H)}(t) = dX^{(H)}(t)\cdot Y^{(H)}(t) + \frac{1}{2}dX^{(H)}(t)\cdot dY^{(H)}(t). \tag{D.6}$$

The connection formulae for the stochastic operators in the Schrödinger representation are given, in the same form as (D.5) and (D.6), by

$$X^{(S)}(t)\circ dY^{(S)}(t) = X^{(S)}(t)dY^{(S)}(t) + \frac{1}{2}dX^{(S)}(t)\cdot dY^{(S)}(t), \tag{D.7}$$

$$dX^{(S)}(t)\circ Y^{(S)}(t) = dX^{(S)}(t)\cdot Y^{(S)}(t) + \frac{1}{2}dX^{(S)}(t)\cdot dY^{(S)}(t), \tag{D.8}$$

where the operators $X^{(S)}(t)$ and $dX^{(S)}(t)$ in the Schrödinger representation are introduced respectively through $X^{(H)}(t) = \hat{V}_f^{-1}(t)X^{(S)}(t)\hat{V}_f(t)$ and $dX^{(H)}(t) = \hat{V}_f^{-1}(t)dX^{(S)}(t)\hat{V}_f(t)$.

E Correlation of Random Force Operators

The random force operators are of the Wiener process whose first and second cumulants are given by real c-numbers:

$$\langle dF(t)\rangle = \left\langle dF^\dagger(t)\right\rangle = 0, \tag{E.1}$$

$$\langle dF(t)dF(t)\rangle = \left\langle dF^\dagger(t)dF^\dagger(t)\right\rangle = 0, \tag{E.2}$$

$$\begin{aligned} \left\langle dF(t)dF^\dagger(t)\right\rangle &= [\text{a real } c\text{-number}], \\ \left\langle dF^\dagger(t)dF(t)\right\rangle &= [\text{a real } c\text{-number}], \end{aligned} \tag{E.3}$$

where $\langle\cdots\rangle = \langle|\cdots|\rangle$ represents the random average referring to the random force operators $dF(t)$.

The random force operators satisfy **Tool 3** in section 3.:

$$\langle|dF^\dagger(t) = \langle|d\tilde{F}(t). \tag{E.4}$$

Applying the connection formula (D.8) to the multiplications, for example, $dW(t)\hat{V}_f(t)$ in the right hand side of the equation (8.32), we have the equation of motion for the time-evolution generator of the Stratonovich type as

$$d\hat{V}_f(t) = -i\hat{H}_{f,t}dt \circ \hat{V}_f(t), \tag{E.5}$$

where $\hat{H}_{f,t}dt$ is the stochastic semi-free hat-Hamiltonian of the Stratonovich type defined by

$$\begin{aligned} \hat{H}_{f,t}dt &= \hat{H}_t dt - i\alpha^{\ddagger}\tilde{\alpha}^{\ddagger}dW(t)d\tilde{W}(t) + i\left[\alpha^{\ddagger}dW(t) + \text{t.c.}\right] \\ &= \omega(t)(a^\dagger a - \tilde{a}^\dagger\tilde{a})dt - i\kappa(t)\left[\alpha^{\ddagger}(\xi a + \eta\tilde{a}^\dagger) + \text{t.c.}\right]dt \\ &\quad + i\left[\alpha^{\ddagger}dW(t) + \text{t.c.}\right]. \end{aligned} \tag{E.6}$$

In deriving the expression (E.6), we demanded that the Stratonovich time-evolution generator should not depend on the diffusion terms, which leads to

$$dW(t)d\tilde{W}(t) = \left\{2\kappa(t)\left[n(t) + \eta\right] + \frac{d}{dt}n(t)\right\}dt. \tag{E.7}$$

This expression is compatible with the assumption that the process is white. Let us put the subscript F on $\Sigma^<(t)$ in the Boltzmann equation (4.14) in order to remember that it is due to the interaction with the random force $dF(t)$:

$$\frac{d}{dt}n(t) = -2\kappa(t)n(t) + i\Sigma_F^<(t). \tag{E.8}$$

Making use of (E.7) and (E.8), we have

$$\begin{aligned} i\Sigma_F^<(t)dt &= -2\kappa(t)\eta dt + dW(t)d\tilde{W}(t) \\ &= -2\kappa(t)\eta dt + \left\langle dF^\dagger(t)dF(t)\right\rangle \\ &\quad + \nu\left\{\left\langle dF(t)dF^\dagger(t)\right\rangle - \left\langle dF^\dagger(t)dF(t)\right\rangle\right\}. \end{aligned} \tag{E.9}$$

where (8.16) has been used within the stochastic convergence, and μ has been erased with the help of (5.26).

It is reasonable to assume that the quantity η may depend on ν, i.e. $\eta = \eta(\nu)$, and that the physical quantities $\kappa(t)$, $i\Sigma_F^{<}(t)$, $\left\langle dF^\dagger(t)dF(t)\right\rangle$, and $\left\langle dF(t)dF^\dagger(t)\right\rangle$ may *not* depend on ν. Then, differentiating (E.9) with respect to ν, we have

$$0 = -2\kappa(t)\frac{\partial\eta}{\partial\nu}dt + \left\langle dF(t)dF^\dagger(t)\right\rangle - \left\langle dF^\dagger(t)dF(t)\right\rangle . \tag{E.10}$$

This leads to

$$\frac{\partial\eta}{\partial\nu} = k(t), \tag{E.11}$$

which is solved as

$$\eta = k(t)\nu + l(t), \tag{E.12}$$

where $k(t)$ and $l(t)$ are real numbers independent of ν. Substituting (E.11) into (E.10), we have

$$\left\langle dF(t)dF^\dagger(t)\right\rangle - \left\langle dF^\dagger(t)dF(t)\right\rangle = 2\kappa(t)k(t)dt. \tag{E.13}$$

By means of (E.12) and (E.13), (E.9) becomes

$$i\Sigma_F^{<}(t)dt = -2\kappa(t)l(t)dt + \left\langle dF^\dagger(t)dF(t)\right\rangle , \tag{E.14}$$

which leads to

$$\begin{aligned}\left\langle dF^\dagger(t)dF(t)\right\rangle &= \left[i\Sigma_F^{<}(t) + 2\kappa(t)l(t)\right] dt \\ &= \left\{2\kappa(t)\left[n(t) + l(t)\right] + \frac{d}{dt}n(t)\right\} dt,\end{aligned} \tag{E.15}$$

where we have used (E.8) at the second equality. The substitution of (E.15) into (E.13) gives us

$$\begin{aligned}\left\langle dF(t)dF^\dagger(t)\right\rangle &= \left\{i\Sigma_F^{<}(t) + 2\kappa(t)\left[k(t) + l(t)\right]\right\} dt \\ &= \left\{2\kappa(t)\left[n(t) + k(t) + l(t)\right] + \frac{d}{dt}n(t)\right\} dt.\end{aligned} \tag{E.16}$$

For the system specified by the Boltzmann equation (12.2), (E.15) and (E.16) reduce, respectively, to

$$\left\langle dF^\dagger(t)dF(t)\right\rangle = 2\kappa\left[\bar{n} + l(t)\right], \tag{E.17}$$

$$\left\langle dF(t)dF^\dagger(t)\right\rangle = 2\kappa\left[\bar{n} + k(t) + l(t)\right]dt. \tag{E.18}$$

Since the Boltzmann equation (12.2) is compatible with the stationary process specified by

$$\left\langle dF^\dagger(t)dF(t)\right\rangle = 2\kappa\bar{n}dt, \tag{E.19}$$

$$\left\langle dF(t)dF^\dagger(t)\right\rangle = 2\kappa(\bar{n} + 1)dt, \tag{E.20}$$

we know that

$$l(t) = 0, \quad k(t) = 1, \tag{E.21}$$

which lead to

$$\eta = \nu \quad (\xi = \mu). \tag{E.22}$$

Substituting (E.21) into (E.15) and (E.16), we obtain (8.18) and (8.19). We also get (8.23) by putting (E.22) into (E.7).

F Another Derivation of the Fokker-Planck Equation

The Fokker-Planck equation (8.24) can be also derived, systematically, from the stochastic Liouville equation (8.27) of the Stratonovich type by means of the method [26,27]:

$$\begin{aligned} \left\langle d\,|0_f(t)\rangle\right\rangle &= d|0(t)\rangle \\ &= -i\left\langle \hat{H}_{f,t}dt \circ |0_f(t)\rangle\right\rangle \\ &= -i\hat{H}dt|0(t)\rangle, \end{aligned} \tag{F.1}$$

with

$$-i\hat{H} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\hat{K}(t_1)dt_1, \tag{F.2}$$

where

$$\hat{K}(t)dt = \sum_{n=1}^{\infty}\hat{K}_n(t)dt, \tag{F.3}$$

with

$$\hat{K}_n(t)dt = (-i)^n\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-2}}\langle\hat{H}_{f,t}dt\circ\hat{H}_{f,t_1}dt_1\circ\cdots\circ\hat{H}_{f,t_{n-1}}dt_{n-1}\rangle_{o.c.}. \tag{F.4}$$

The symbol $\langle\cdots\rangle_{o.c.}$ indicates the *ordered cumulants* [92,10] defined, for example, by

$$\langle X(t)\rangle_{o.c.} = \langle X(t)\rangle, \tag{F.5}$$

$$\langle X(t)X(t_1)\rangle_{o.c.} = \langle X(t)X(t_1)\rangle - \langle X(t)\rangle\langle X(t_1)\rangle, \tag{F.6}$$

$$\begin{aligned} \langle X(t)X(t_1)X(t_2)\rangle_{o.c.} = {} & \langle X(t)X(t_1)X(t_2)\rangle - \langle X(t)X(t_1)\rangle\langle X(t_2)\rangle \\ & -\langle X(t)X(t_2)\rangle\langle X(t_1)\rangle - \langle X(t)\rangle\langle X(t_1)X(t_2)\rangle \\ & +\langle X(t)\rangle\langle X(t_1)\rangle\langle X(t_2)\rangle + \langle X(t)\rangle\langle X(t_2)\rangle\langle X(t_1)\rangle, \end{aligned} \tag{F.7}$$

for any operator $X(t)$.

Using (8.28) with the properties (8.13)–(8.15) for the Wiener process, we obtain the Fokker-Planck generator $\hat{H}$ in (8.24) as

$$\hat{H} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}\left\{\langle\hat{H}_{f,t_1}dt_1\rangle - i\int_0^{t_1}\langle\hat{H}_{f,t_1}dt_1\circ\hat{H}_{f,t_2}dt_2\rangle_{o.c.}\right\}. \tag{F.8}$$

For the mathematical formulation in subsection 13.3., we regard $\langle\cdots\rangle$ in the above formulae as $E_{0]}(\cdots)$.

G Coherent State Representation in NETFD

We introduce a phase-space method for NETFD by means of a generalized coherent state representation [62].

1. The *probability distribution function* $P_f^{(\mu,\nu)}(z,t)$ corresponding to $|0_f(t)\rangle$ is defined by

$$|0_f(t)\rangle = \int_z P_f^{(\mu,\nu)}(z,t)|\Delta^{(\mu,\nu)}(z)\rangle, \tag{G.1}$$

with

$$|\Delta^{(\mu,\nu)}(z)\rangle = \int_\xi \mathrm{e}^{s|\xi|^2/2}\mathrm{e}^{z\xi^*-z^*\xi}|D(\xi)\rangle, \tag{G.2}$$

where $|D(\xi)\rangle$ is specified by

$$\tilde{\alpha}^{\dagger}|D(z)\rangle = -z|D(z)\rangle, \quad \alpha|D(z)\rangle = -\partial_*|D(z)\rangle, \tag{G.3}$$

and

$$\langle 1|D(z)\rangle = \pi\delta^{(2)}(z), \qquad \delta^{(2)}(z) = \delta(\Re\mathrm{e}(z))\delta(\Im\mathrm{m}(z)). \tag{G.4}$$

Here, we introduced abbreviations $\int_z = \int d^2z/\pi$, and $\partial = \partial/\partial z$, $\partial_* = \partial/\partial z^*$. The parameter $s = \nu - \mu$ specifies the ordering of operators, e.g. $s = 1$ for normal ordering, $s = 0$ for anti-normal ordering and $s = 1/2$ for Weyl ordering. Equation (G.1) shows the correspondence between thermal space and phase-space as

$$\alpha|0_f(t)\rangle \longleftrightarrow zP_f^{(\mu,\nu)}(z,t), \quad \tilde{\alpha}^{\dagger}|0_f(t)\rangle \longleftrightarrow -\partial_* P_f^{(\mu,\nu)}(z,t). \tag{G.5}$$

Note that the tilde invariance, $|0_f(t)\rangle^\sim = |0_f(t)\rangle$, reads

$$P_f^{(\mu,\nu)}(z,t)^* = P_f^{(\mu,\nu)}(z,t), \tag{G.6}$$

and that α and $\alpha^{\dagger}$ are canonical operators satisfying the canonical commutation relation

$$[\alpha,\ \alpha^{\dagger}] = 1. \tag{G.7}$$

2. The phase-space quantity $G^{(\mu,\nu)}(z_1, z_1^*, z_2, z_2^*)$ for the operator $G(a, a^\dagger, \tilde{a}^\dagger, \tilde{a})$ in the thermal space is defined through

$$G(a, a^\dagger, \tilde{a}^\dagger, \tilde{a}) = \int_{z_1}\int_{z_2} G^{(\mu,\nu)}(z_1, z_1^*, z_2, z_2^*)\Delta^{(\mu,\nu)}(z_1)\tilde{\Delta}^{(\mu,\nu)}(z_2), \tag{G.8}$$

with

$$\Delta^{(\mu,\nu)}(z) = \int_\xi \mathrm{e}^{s|\xi|^2/2}\mathrm{e}^{z\xi^*-z^*\xi}D(\xi), \quad D(\xi) = \mathrm{e}^{\xi a^\dagger - \xi^* a}. \tag{G.9}$$

Then, for the state

$$G(a, a^\dagger, \tilde{a}^\dagger, \tilde{a})|0_f(t)\rangle = \int_z F^{(\mu,\nu)}(z, z^*, t)|\Delta^{(\mu,\nu)}(z)\rangle, \tag{G.10}$$

we obtain

$$\begin{aligned} F^{(\mu,\nu)}(z,z^*,t) &= e^{\nu\partial^1\partial^2_* - \mu\partial^1_*\partial^2} \\ &\quad \times G^{(\mu,\nu)}(z_1 + \nu\partial_*, z_1^* - \mu\partial, z_2 \\ &\quad - \mu\partial_*, z_2^* + \nu\partial) P_f^{(\mu,\nu)}(z,t)\Big|_{\substack{z_1 = z_2 = z \\ z_1^* = z_2^* = z^*}} \end{aligned} \tag{G.11}$$

3. The expectation value of the observable operator

$$G(a,a^\dagger) = \int_z F^{(\mu,\nu)}(z,z^*)\Delta^{(\mu,\nu)}(z), \tag{G.12}$$

is given by

$$\langle 1|G(a,a^\dagger)|0_f(t)\rangle = \int_z F^{(\nu,\mu)}(z,z^*)P_f^{(\mu,\nu)}(z,t). \tag{G.13}$$

4. As for the random force operators $dW(t)$ and $d\tilde{W}(t)$, we cast the mapping correspondence between thermal space and phase-space as

$$dW(t) \longleftrightarrow dW(t), \quad d\tilde{W}(t) \longleftrightarrow dW^*(t). \tag{G.14}$$

The stochastic process for these random forces in phase-space are specified by (8.13)–(8.15) with the replacement of the operators according to the correspondence (G.14). Namely,

$$\Big\langle dW(t)\Big\rangle = \Big\langle dW^*(t)\Big\rangle = 0, \tag{G.15}$$

$$\Big\langle dW(t)dW(s)\Big\rangle = \Big\langle dW^*(t)dW^*(s)\Big\rangle = 0, \tag{G.16}$$

$$\Big\langle dW(t)dW^*(s)\Big\rangle = \Big[i\Sigma^<(t) + 2\nu\kappa(t)\Big]\delta(t-s)dt\,ds. \tag{G.17}$$

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КАНОНІЧНИЙ ФОРМАЛІЗМ ДИСИПАТИВНИХ КВАНТОВИХ СИСТЕМ. НЕРІВНОВАЖНА ТЕРМОПОЛЬОВА ДИНАМІКА

Т. Арімітсу

Представлено канонічний формалізм квантових систем в далекому від рівноваги стані – нерівноважну термопольову динаміку (НТПД), що об'єднує різні точки зору й охоплює всі аспекти нерівноважної статистичної механіки, такі як рівняння Больцмана, Фокера-Планка, Ланжевена, а також стохастичне рівняння Ліувіля.

Показано як отримується напів-вільний генератор часової еволюції квантового рівняння Фокера-Планка в нестаціонарному випадку з дотриманням кількох основних вимог, що накладаються деякими фундаментальними властивостями рівняння

Ліувіля. На основі цього генератора показано як будується канонічна теорія дисипативних квантових систем. Оператори знищення й породження представлені через часовозалежне перетворення Боголюбова.

Показано, що в рамках НТПД є два шляхи – дві можливості введення зовнішнього поля. Один спосіб – через ермітовий “hat”-гамільтоніан, інший – через неермітовий “hat”-гамільтоніан. З старим “hat”-гамільтоніаном представлено $\hat{S}$-матрицю та метод твірного функціоналу з метою отримати співвідношення між методом НТПД та одним із шляхів замкненого часу Швінгера.

З останньою неермітовою взаємодією “hat”-гамільтоніана отримано загальний вираз стохастичного напів-вільного генератора часової еволюції для нестаціонарного гаусіана білого квантового стохастичного процесу. Також отримано в загальному кореляцію оператора випадкової сили. На основі цього генератора показано як може бути сконструйована об’єднана теорія квантових стохастичних диференціальних рівнянь. Об’єднаним чином досліджено стохастичні рівняння системи Ліувіля та Ланжевена як в формі Іто, так і в формі Стратоновича.

Вся теорія НТПД також представлена в формі c-чисел, що дозволяє в рамках НТПД розглядати й когерентні стани.

Система стохастичних диференціальних рівнянь також сконструйована на “hat”-гамільтоніані з ермітовою взаємодією. Подана інтерпретація формули Морі в рамках теорії НТПД. Подано математичне формулювання НТПД, де стохастичний генератор часової еволюції записано через мартингали. В рамках НТПД переглянуто метод хвильових функцій Монте-Карло (це моделювання квантового переходу).