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STAR EXPONENTS IN POLYMER THEORY: RENORMALIZATION GROUP RESULTS IN THREE DIMENSIONS

C. VON FERBER[1], YU.V. HOLOVATCH[2]

[1] *Fachbereich Physik der Universität Essen*
D-45117 Essen, Germany
[2] *Institute for Condensed Matter Physics*
of the Ukrainian National Academy of Sciences
1 Svientsitskii St., UA-290011 Lviv, Ukraine

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We study networks of self avoiding polymer chains of arbitrary but fixed topology and calculate critical exponents governing their scaling properties (star exponents). Calculations are performed in the frames of the fixed-dimension field theoretical approach. Renormalization group functions in the Callan-Symanzik scheme are obtained in three-loop approximation and are analysed directly in three dimensions. Perturbation theory expansions are resummed with the use of Padé-Borel transformation. The results obtained are in a good agreement with Monte-Carlo and ε-expansion data.

1. Introduction

Long polymer chains immersed in a good solvent can be described in frames of field-theoretical renormalization group approach widely used in the theory of critical phenomena. In the early 70-ies P.G. de Gennes was one of the first to realize that a model of long polymer chains in terms of self-avoiding walks (SAW) in the limit of an infinite number of SAW steps $N \to \infty$ can be mapped onto a magnetic n-component spin model with $O(n)$-symmetry in the limit $n \to 0$ [1,2]. In particular, the critical exponents ν and γ describing the behavior of the correlation length (ξ) and magnetic susceptibility (χ) of the $O(n)$-model in the vicinity of critical temperature T_c,

$$\xi \sim \tau^{-\nu}, \tag{1.1}$$

$$\chi \sim \tau^{-\gamma}, \quad \tau = \frac{T - T_c}{T_c}, \tag{1.2}$$

in polymer theory correspond to size and configuration number exponents ν and γ describing the average square end-to-end distance R^2

$$R^2 \sim N^{2\nu}, \quad N \gg 1 \tag{1.3}$$

of a single chain of N monomers and the number $\mathcal{Z}_N$ of possible ways of realization of a SAW of N steps on a given lattice:

$$\mathcal{Z}_N \sim \mu^N N^{\gamma-1}, \quad N \gg 1, \tag{1.4}$$

where μ is the non-universal connectivity constant.

Natural interest in generalization of the theory of "long chains" to the case when these chains have a possibility to be tied together and thus to form "stars" and networks of arbitrary topology lead to the notion of star exponents which govern their scaling properties [3-7]. Zero-mass renormalization group scheme as well as direct renormalization by minimal subtraction with successive ε-expansion were used to analyse critical behavior of polymer networks and stars. The approach we are going to use here foots on the massive field theory renormalization scheme (Callan-Symanzik equations) and it enables one to perform calculations directly in the fixed space dimension d of interest (e.g. $d = 3$) avoiding application of the ε-expansion.

In this article we present results for star exponents obtained directly in three dimensions in three-loop approximation. The paper is organized as follows. In Sec.2 we give some definitions and review recent results obtained in the theory of polymer networks and stars; Sec.3 is devoted to the description of the massive field theory renormalization scheme we use here, expressions for the renormalization group functions in three-loop approximation are given there as well; in Sec.4 we perform the resummation of the series obtained and give the results for star exponents.

2. Star polymers

Generalizing the above concept, it is quite natural to describe the topology of a polymer network in a good solvent as that of a set of long chains tied together at their endpoints. It was shown [4-6] that the scaling properties of such a network are determined by its "star-like" vertices, connecting the extremities of the chains. Let us consider first a single star polymer (Fig.1) with F legs of N monomers each. Then it can be shown that for long chains $N \to \infty$ its number of self-avoiding configurations will scale according to [4]:

$$\mathcal{Z}_{N,F} \sim \mu^{FN} N^{\gamma_F - 1}, \quad N \gg 1. \tag{2.1}$$

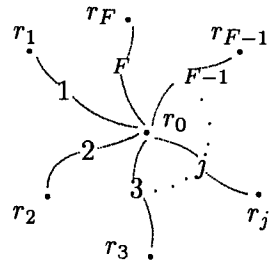

Figure 1. Star polymer.

Formula (2.1) can be considered as the generalization of (1.4), μ still being the non-universal connectivity constant and values γ_F give us the first example of star exponents. In the case $F = 1, 2$ one still has the single polymer chain: $\gamma_1 = \gamma_2 = \gamma$ and for $F \geq 3$ γ_F form a set of independent critical exponents. For the two-dimensional case the values of γ_F are known exactly [3]:

$$\gamma_F = [68 + 9F(3 - F)]/64. \tag{2.2}$$

The exponents γ_F are not the only possible exponents appearing in polymer physics which can not be expressed in terms of "usual" SAW exponents γ and ν (1.3), (1.4). For example, one can consider two polymer stars having F and F' legs, correspondingly (Fig.2). Due to the hard-core repulsion the probability of approaching the cores of these two polymers at distance r vanishes as

$$P(r) \sim r^{\theta_{F,F'}}, \quad r \to 0. \tag{2.3}$$

Here $\theta_{F,F'}$ form a set of universal contact exponents.

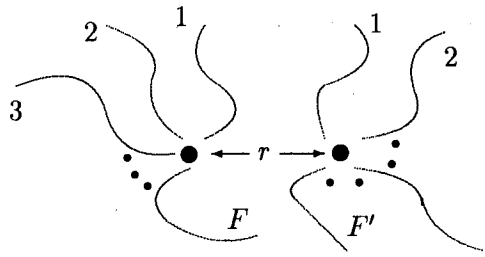

Figure 2. Two star polymers of functionalities F and F' at a distance r.

One can continue introducing other sets of exponents caused by the star-like nature of considered chains and describing their different properties. Let us note however that only one of those sets is independent and the rest can be expressed in terms of it. As it was pointed out in [4] polymer networks can be described in terms of a single infinite series of independent universal critical exponents γ_F. All geometrical exponents can be expressed in terms of γ_F. For the two-star contact exponents $\theta_{F,F'}$ the corresponding expression reads [4]:

$$\nu\theta_{F,F'} - 1 = \gamma_F + \gamma_{F'} - \gamma_{F+F'}. \tag{2.4}$$

Let us now consider a general network G of arbitrary but fixed topology, made of F chains of equal length N and tied together in vertices (see Fig.3). One has for the asymptotic number of self-avoiding configurations $\mathcal{Z}_G$:

$$\mathcal{Z}_G \sim \mu^{FN} N^{\gamma_G - 1}, \quad N \gg 1, \tag{2.5}$$

where the expression for the critical exponent γ_G reads:

$$\gamma_G - 1 + \nu d = \sum_{F \geq 1} n_F \Big[(\gamma_F - 1 + \nu d) - \frac{F}{2}(\gamma_2 - 1 + \nu d) \Big], \tag{2.6}$$

here n_F is the number of vertices with F legs.

Thus knowing one set of exponents for single star polymers one can obtain geometrical exponents characterizing any polymer network [3,7]. So in what follows below we will consider mainly the properties of a single polymer star immersed in a good solvent with arbitrary but fixed number of arms (see Figure 1).

We are using the continuous chain model introduced by Edwards [10]. Considering an ensemble of F disconnected branches and puting correspondence to each branch a a path $\mathbf{r}_a(s)$, parametrized by $0 \leq s \leq S_a, a =$

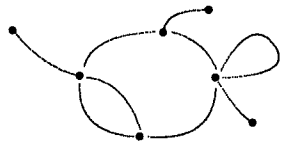

Figure 3. A polymer network G. It is characterized by the numbers n_F of F-leg vertices. Here $n_1 = 3$, $n_3 = 2$, $n_4 = 1$, $n_5 = 1$.

$1, 2, ..., F$ (S_a being the Gaussian surface of ath branch) one can describe such a system by the following Hamiltonian:

$$\frac{1}{k_B T} H_F = \frac{1}{2} \sum_{a=1}^{F} \int_0^{S_a} ds \left(\frac{d\mathbf{r}(s)}{ds}\right)^2 + \frac{u_0}{2} \sum_{a,b=1}^{F} \int_0^{S_a} ds \int_0^{S_b} ds' \delta^d(\mathbf{r}_a(s) - \mathbf{r}_b(s')). \tag{2.7}$$

The partition function is obtained as a functional integral over all possible configurations of polymer system divided by its volume Ω thus dividing out identical configurations just translated in space:

$$\mathcal{Z}\{S_a\} = \frac{1}{\Omega} \int D[\mathbf{r}_1, ..., \mathbf{r}_F] \exp[-\frac{H_F}{k_B T}], \tag{2.8}$$

here the symbol $D[\mathbf{r}_1, ..., \mathbf{r}_f]$ includes normalization such that $\mathcal{Z}\{S_a\} = 1$ for $u_0 = 0$. To make the exponential of δ-functions and the functional integral well defined in bare theory a cutoff s_0 has to be introduced such that all simultaneous integrals of any variables s and s' are cut off by $|s - s'| > s_0$. With this we can relate the Gaussian surface S of a path to the notion of steps N of a self avoiding walk by

$$N = S/s_0.$$

Now the partition function for the polymer star reads:

$$\mathcal{Z}_F\{S_a\} \equiv \mathcal{Z}_F\{S_1, ..., S_F\} = \frac{1}{\Omega} \int D[\mathbf{r}_1, ..., \mathbf{r}_F] \exp[-\frac{H_F}{k_B T}] \times \prod_{a=2}^{F} \delta^d(\mathbf{r}^a(0) - \mathbf{r}^1(0)), \tag{2.9}$$

here the product of δ-functions

$$\prod_{a=2}^{F} \delta^d(\mathbf{r}^a(0) - \mathbf{r}^1(0))$$

ensures the "star-like" configuration of a set of F chains, i.e. "enforces" them to have a common endpoint in $\mathbf{r}^1(0)$ (c.f. fig.1).

Besides the partition functions one can define correlation functions of an F-arm star. For the core-endpoint correlation function one has [3,7]:

$$\hat{P}_F(\mathbf{r}_0;\mathbf{r}_1,...,\mathbf{r}_F;S_1,...,S_F) = \langle \prod_{a=1}^{F} \delta^d(\mathbf{r}^a(0)-\mathbf{r}_0)\delta^d(\mathbf{r}^a(S_a)-\mathbf{r}_a)\rangle, \quad (2.10)$$

here $\mathbf{r}_0$ is the coordinate of the core, $\mathbf{r}_1,...,\mathbf{r}_F$ being coordinates of chain endpoints, $S_1,...,S_F$ being their Gaussian surfaces. The averaging in (2.10) means:

$$\langle ... \rangle = \frac{\frac{1}{\Omega}\int D[\mathbf{r}_1,...,\mathbf{r}_F]\exp[-\frac{H_F}{k_BT}](...)}{\mathcal{Z}_F\{S_a\}}.$$

For the Green functions one has:

$$(2\pi)^d\delta^d[\mathbf{p}_0+\sum_{a=1}^{F}\mathbf{p}_a]G_F(\mathbf{p}_0;\mathbf{p}_1,...,\mathbf{p}_F;S_1,...,S_F) =$$

$$\mathcal{Z}_F(S_1,...,S_F)\int\prod_{a=0}^{F}[d^d\mathbf{r}_a\exp[i\mathbf{p}_a\mathbf{r}_a]\hat{P}_F(\mathbf{r}_0;\mathbf{r}_1,...,\mathbf{r}_F;S_1,...,S_F). \quad (2.11)$$

The mapping to field theory is performed by a Laplace transformation from the Gaussian surfaces S_a to conjugated chemical potential variables ("mass variables") μ_a:

$$\tilde{\mathcal{Z}}_F\{\mu_a\} = \int\prod_{b=1}^{F} dS_b\exp[-\mu_bS_b]\mathcal{Z}_F\{S_a\}. \quad (2.12)$$

As a result an ensemble of F polymer chains is described by a field theory with the Lagrangian $\mathcal{L}\{\phi_a,\mu_a\}$ involving a separate field $\vec{\phi}_a \equiv \vec{\phi}_a(\mathbf{r})$ for each chain:

$$\mathcal{L}\{\phi_b,\mu_b\} = \frac{1}{2}\sum_{a=1}^{F}\int d^d\mathbf{r}(\mu_a|\vec{\phi}_a|^2+\frac{1}{2}|\nabla\vec{\phi}_a|^2)+\frac{u_0}{8}\sum_{a,a'=1}^{F}\int d^d\mathbf{r}|\vec{\phi}_a|^2|\vec{\phi}_{a'}|^2, \quad (2.13)$$

here $\vec{\phi}_a$ is an n-component vector field $\vec{\phi}_a = (\phi^1,...,\phi^n)$, $|\vec{\phi}_a|^2 = \sum_{\alpha=1}^{n}(\phi_a^\alpha)^2$. However, the perturbation expansion of field theory (2.13) at arbitrary value of n results in particular in some diagrams, which do not appear in polymer theory . As it is well known [1,2] such diagrams (involving closed loops of propagator lines connected to the remainder of the diagram by interaction lines only) can be suppressed by taking the limit $n = 0$.

Returning now to the partition function of a single polymer star $\mathcal{Z}_F\{S_b\}$ and using the Laplace transformation (2.12) we can represent its Laplace transform $\tilde{\mathcal{Z}}_F\{\mu_b\}$ in the following form:

$$\tilde{\mathcal{Z}}_F\{\mu_b\} = \frac{1}{\Omega}\int d^d\mathbf{r}_0\prod_{a=1}^{F}d^d\mathbf{r}_a\int D[\phi_a(r)]\times$$
$$\prod_{a=1}^{F}\phi_a(r_0)\phi_a(r_a)\exp[-\mathcal{L}\{\phi_b,\mu_b\}], \quad (2.14)$$

at $n = 0$. In field theory the star vertex is related to the local composite operator $\prod_{a=1}^{F} \phi_a$ of F distinct zero component fields. Formally this is the $n = 0$ limit of an operator known in n-component field theory [12]:

$$\sum_{\alpha_1,...,\alpha_F=1}^{n} N^{\alpha_1,...,\alpha_F} \phi^{\alpha_1}(r)....\phi^{\alpha_F}(r), \tag{2.15}$$

where $N^{\alpha_1,...,\alpha_F}$ is a zero trace symmetric $SO(n)$ tensor:

$$\sum_{\alpha=1}^{n} N^{\alpha,\alpha,\alpha_3,...,\alpha_F} = 0.$$

Now it can be checked diagrammatically that Green functions involving an operator with $N^{\alpha_1,...,\alpha_F}$ symmetry coupled to F external fields $\phi_a^{\alpha}(r_a)$ in the limit $n = 0$ coincide with Green functions (2.11) of the F-arm star polymer. Thus in order to find star exponents in the field-theoretical renormalization group program one may consider the problem of renormalization of composite operators (2.15). This will be the subject of the subsequent section.

3. Renormalization group functions in three loop approximation

As was shown in the previous section considering the behavior of the star-like vertices in polymer theory one faces the problem of renormalization of a field theory containing two couplings one being of $O(n)$ symmetry and described by tensor $S_{\alpha_1,...,\alpha_4}$, the other being of traceless $SO(n)$ symmetry $(N^{\alpha_1,...,\alpha_F})$:

$$\mathcal{L}\{\phi_b, \mu_b\} = \frac{1}{2} \int d^d\mathbf{r}[(\mu_0^2|\vec{\phi}|^2 + |\nabla\vec{\phi}|^2) + \frac{u_0}{4!} S_{\alpha_1,...,\alpha_4} \phi^{\alpha_1}....\phi^{\alpha_4} +$$
$$\frac{v_0}{4!} N^{\alpha_1,...,\alpha_F} \phi^{\alpha_1}....\phi^{\alpha_F}]. \tag{3.1}$$

First of all let us point out that the problem of composite field renormalization in the case of traceless symmetry we are going to tackle here was considered in the frames of ε-expansion in [12] in the field-theory context, in polymer context ε-expansion results up to ε^3-order were analysed in [6,7]. We will return to these results later. The distinct feature of our study is that we are going to apply here equations of the massive field theory. As is well known in order to analyze the expressions appearing in such an approach one can either apply ε-expansion (and thus re-derive the results of [6,7,12]), or as it was proposed by Parisi [13] consider renormalization group functions directly in the dimension of space of interest (here we are interested in the three dimensional case). So performing calculations in the spirit of [13] enables us to proceed directly in three-dimensions.

The critical properties of the field theory (3.1) can be extracted from the coefficients $\beta_u, \beta_v, \gamma_\phi, \gamma_{\phi^2}$ of the Callan-Symanzik equation for the renormalized N-point vertex function Γ_R^N (see e.g. [14-16]). As far as we are interested in the renormalization group functions and their derivatives over the coupling constants at the Heisenberg fixed point ($u^* \neq 0, v^* = 0$) we need

to calculate vertex functions of the symmetric interaction $S_{\alpha_1,\ldots,\alpha_4}\phi^{\alpha_1}\ldots\phi^{\alpha_4}$ Γ^2, Γ^4 together with F-point vertex function Γ^F with only one $N^{\alpha_1,\ldots,\alpha_F}\phi^{\alpha_1}$ $\ldots\phi^{\alpha_F}$ insertion (the other terms will either contain some trace of tensor $N^{\alpha_1,\ldots,\alpha_F}$ or will be proportional to some power of v and will disappear in the Heisenberg fixed point). The graphs for Γ^F are obtained from the usual four-point graphs by replacing each four-point vertex in turn by $vN^{\alpha_1,\ldots,\alpha_F}$. In three loop approximation which we are going to consider here there appear two more graphs in Γ^F which can not be produced in this manner (they are shown in fig.4).

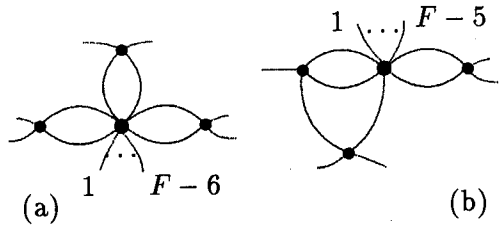

Figure 4. Additional graphs appearing in the function Γ^F in three-loop approximation.

Finally the expressions for β- and γ-functions read [1]:

$$\beta_u = -(4-d)u\left[1 - u + \beta_u^{2LA}u^2 + \beta_u^{3LA}u^3 + \ldots\right], \tag{3.2}$$

$$\beta_v = -(4-d)v[\frac{\delta_F}{(4-d)} - \frac{F(F-1)}{n+8}(u + \beta_v^{2LA}u^2 + \beta_v^{3LA}u^3 + \ldots], \tag{3.3}$$

$$\gamma_\phi = -(4-d)\frac{2(n+2)}{(n+8)^2}u^2\left[2i_2 + (4i_2 - 3i_8)u + \ldots\right], \tag{3.4}$$

$$\tilde{\gamma}_{\phi^2} = (4-d)\frac{n+2}{n+8}u\left[1 + \gamma^{2LA}u + \gamma^{3LA}u^2 + \ldots\right], \tag{3.5}$$

here δ_F is the engineering dimension of the coupling v:

$$\delta_F = F + d - \frac{Fd}{2}$$

and the expressions for two-loop ($\beta_u^{(2LA)}$, $\beta_v^{(2LA)}$, $\gamma^{(2LA)}$) and three-loop ($\beta_u^{(3LA)}$, $\beta_v^{(3LA)}$, $\gamma^{(3LA)}$) contributions read:

$$\beta_u^{(2LA)} = \frac{8}{(n+8)^2}[(5n+22)(i_1 - \frac{1}{2}) + i_2(n+2)], \tag{3.6}$$

[1] We changed the scale for the renormalized couplings $u = u'(n+8)D/6$, $v = v'D/6$ and beta functions $\beta_u(u) = 6\beta_{u'}(u')/[(n+8)D]$, $\beta_v(u) = 6\beta_{v'}(u')/D$, ($D$ being the one-loop integral: $D = \frac{1}{(2\pi)^d}\int \frac{d\vec{k}}{(k^2+1)^2}$, u', v' being the renormalized couplings corresponding to the bare couplings u_0, v_0), to define a convenient numerical scale in which the first two coefficients of $\beta_u(u)$ are -1 and 1.

$$\beta_v^{(3LA)} = -(2i_1(n+4F-2)+2i_2(n+2)/(F-1)+ (-n-4F+2))/(n+8), \quad (3.7)$$

$$\gamma^{(2LA)} = \frac{6}{(n+8)}(1-2i_1), \quad (3.8)$$

$$\beta_u^{(3LA)} = \frac{1}{(n+8)^2}\sum_{j=0}^{8} b_u^j i_j, \quad (3.9)$$

$$\beta_v^{(3LA)} = \frac{1}{(n+8)^2}\sum_{j=0}^{8} b_v^j i_j, \quad (3.10)$$

$$\gamma^{(3LA)} = \frac{1}{(n+8)^2}[10(n+8)-(44n+280)i_1 + +(8-3d)(n+2)i_2+12(n+2)i_3+ 24(n+8)i_4+6(n+8)i_5+18(n+2)i_6]. \quad (3.11)$$

In (3.9), (3.10) $i_0 \equiv 1$. And for the coefficients b_u^j,b_v^j one has:

$$\begin{aligned}
b_u^0 &= -8(4n^2+61n+178),\\
b_u^1 &= 4(31n^2+430n+1240),\\
b_u^2 &= (3dn^2+30dn+48d+8n^2+80n+128),\\
b_u^3 &= -12(n^2+10n+16),\\
b_u^4 &= -48(n^2+20n+60),\\
b_u^5 &= -24(2n^2+21n+58),\\
b_u^6 &= -6(3n^2+22n+56),\\
b_u^7 &= -24(5n+22),\\
b_u^8 &= -12(n^2+10n+16); \quad (3.12)\\
b_v^0 &= (n^2+8nF+6n+20F^2-28F+56),\\
b_v^1 &= -4(n^2+7nF+5n+18F^2-28F+54),\\
b_v^2 &= -(3dnF-3dn+6dF-6d+4n^2- 8nF+48n-16F+80)/(F-1),\\
b_v^3 &= 12(n+2),\\
b_v^4 &= 12(nF+2n+4F^2-10F+20),\\
b_v^5 &= 3(n^2+4nF-2n+4F^2+12F-24),\\
b_v^6 &= 6(n+2F^2-10F+18),\\
b_v^7 &= 4(nF-2n+14F-28),\\
b_v^8 &= 3(n^2+10n+16)/(F-1); \quad (3.13)
\end{aligned}$$

here $i_1 - i_8$ are the integrals originating from the corresponding two- and three-loop Feynman graphs. Their numerical values at $d = 3$ are as follows [17,18]:

$$\begin{aligned}
&i_1 = 2/3; \quad i_2 = -2/27; \quad i_3 = -.0376820725;\\
&i_4 = .3835760966; \quad i_5 = .5194312413; \quad i_6 = 1/2;\\
&i_7 = .1739006107; \quad i_8 = -.0946514319. \quad (3.14)
\end{aligned}$$

Correspondence between $i_1 - i_8$ and the appropriate Feynman graphs is given in [18]. For the additional graphs appearing in Γ^F the corresponding normalized numerical values are: 1 (fig.4a) and i_1 (fig.4b). In the case $d = 3$ expression for β_u and gamma-functions coincide with three-loop parts of appropriate expressions obtained in [19] and at arbitrary value of d they were given in [20].

Note that expressions (3.2) - (3.5) are written for the arbitrary number of field components n and thus contain more information that is necessary for the polymer case $n = 0$. This case was considered by Wallace and Zia [12] in three loop ε expansion motivated by the question of relevance of the v-coupling in the renormalization group sense. These authors introduced a series of critical exponents α_F: the value of the critical exponent α_F is connected with the anomalous dimension x'_F of v at the $O(n)$-symmetrical fixed point:

$$(2-\eta)\alpha_F = x'_F - \frac{F}{2}\eta,$$
$$x'_F = \frac{\partial \beta_v(u,v)}{\partial v} - \delta_F, \quad \text{at} \quad u = u^*, \quad v = v^* = 0. \tag{3.15}$$

The relation to the star exponents is given in the limit $n = 0$ by

$$\gamma_F - 1 = -\nu(2-\eta)\alpha_F + [\nu(2-\eta) - 1]F. \tag{3.16}$$

Thus the expressions (3.2)-(3.5) may be used to study the stability of the $O(n)$-symmetric fixed point to the perturbation introduced by the traceless coupling. In frames of the ε-expansion the following expression for α_F was obtained in [12] in ε^3-approximation:

$$\begin{aligned}
\alpha_F &= \frac{\varepsilon F(F-1)}{2(n+8)}[1 + \frac{\varepsilon}{2(n+8)}(2 - 4F - n + \frac{2(9n+42)}{n+8}) + \\
&\frac{\varepsilon^2}{2(n+8)^4}\{4(n+8)[(F-2)(n+14)(n+8) - \\
&6(5n+22)]\zeta(3) - \frac{1}{2}n^4 - 18n^3 - 86n^2 - 168n + 592 + \\
&2(F-4)(n+8)[8k(n+8) + n^2 - 24n - 136]\}],
\end{aligned} \tag{3.17}$$

where $\zeta(3) \simeq 1.202$ is the Riemann ζ function.

4. Star exponents in three dimensions

We now proceed with the calculation of the star exponents. Combining (3.2)-(3.5) with (3.16) one obtains the following expression for the function $\gamma_F(u)$ ($\gamma_F = \gamma_F(u^*)$):

$$\gamma_F = 1 + (4-d)F[\gamma_F^{1LA}u + \gamma_F^{2LA}u^2 + \gamma_F^{3LA}u^3], \tag{4.1}$$
$$\gamma_F^{1LA} = \frac{n - F + 3}{2(n+8)}, \tag{4.2}$$
$$\gamma_F^{2LA} = \frac{-1}{(n+8)^2}[(-nF + 7n - 4F^2 + 6F + 10)i_1 +$$

$$(dn^2 - dnF + 5dn - 2dF + 6d - 4n^2 + 6nF - 34n + 8F^2 - 4F - 44)/4], \quad (4.3)$$

$$\gamma_F^{3LA} = \frac{1}{(n+8)^3}\sum_{j=0}^{8}\gamma_F^j i_j, \quad (4.4)$$

$$\begin{aligned}
\gamma_F^0 &= \frac{-1}{8}(-d^2n^3 + d^2n^2F - \\
&\quad 7d^2n^2 + 4d^2nF - 16d^2n + 4d^2F - 12d^2 + \\
&\quad 8dn^3 - 10dn^2F + 82d * n^2 - 8dnF^2 - \\
&\quad 36dnF + 236dn - 16dF^2 - 32dF + 208d - 16n^3 + \\
&\quad 28n^2F - 260n^2 + 64nF^2 + 72nF - 1112n + \\
&\quad 80F^3 - 128F^2 + 400F - 1504), \\
\gamma_F^1 &= \frac{-1}{2}(dn^2F - 13dn^2 + \\
&\quad 4dnF^2 + 2dnF - 54dn + 8dF^2 - 56d - 8n^2F + 100n^2 - \\
&\quad 44nF^2 + 604n - 72F^3 + 152F^2 - 328F + 1000), \\
\gamma_F^2 &= \frac{-d}{2}(n^2 - nF + 5n - 2F + 6), \\
\gamma_F^3 &= -6(-n^2 + nF - 5n + 2F - 6), \\
\gamma_F^4 &= -6(-2n^2 + nF^2 + nF - 22n + 4F^3 - \\
&\quad -14F^2 + 30F - 52), \\
\gamma_F^5 &= \frac{-3}{2}(n^2F - 3n^2 + 4nF^2 - 6nF - \\
&\quad 18n + 4F^3 + 8F^2 - 36F - 8), \\
\gamma_F^6 &= -3(-3n^2 + nF - 13n + 2F^3 - \\
&\quad 12F^2 + 28F - 30), \\
\gamma_F^7 &= -2(nF^2 - 3nF + 2n + 14F^2 - 42F + 28). \quad (4.5)
\end{aligned}$$

Expression (4.2) as well as (3.2)-(3.5) is known to be asymptotic and the appropriate resummation procedure is to be applied in order to extract from them reliable information. We choose here the simple Padè-Borel resummation technique. The procedure is as follows: first one solves the equation for the fixed point:

$$\beta_u(u^*) = 0, \quad (4.6)$$

and then the obtained value of Heisenberg fixed point $u^* \neq 0, v^* = 0$ is substituted into the corresponing series for the critical exponent (say, $\gamma_F(u)$) and one more resummation of the series for the critical exponent is performed:

$$\gamma_F = \gamma_F(u^*) \quad (4.7)$$

and in this way the numerical value of critical exponent is obtained.

So in fact one has to deal subsequently with several (in the example given above: with two) asymptotic series. In the ε-expansion technique there exists a natural expansion parameter (ε) allowing one to pass from one series to the other in a self-consistent way, i.e. to be in the frames of the

same (chosen from the very beginning) approximation in the powers of ε. In the expansions given above the expansion parameter is the number of loops (=certain power of the coupling constants). And once one has solved the equation like (4.6) the expansion parameter "disappears" and the result for the fixed point is just a number [2]. So here some arbitrariness appears. One can either follow the scheme of dealing with "numbers", i.e. subsequently solving the equations:

$$\beta_u^{res}(u^*) = 0, \tag{4.8}$$

$$\gamma_F = \gamma_F^{res}(u^*), \tag{4.9}$$

(here $\beta^{res}, \gamma_F^{res}$ mean the expressions for the corresponding renormalization group functions resummed in some way) and substituting the numerical value of the fixed point u^* obtained from (4.8) into (4.9). It is quite clear that in this case this numerical value (obtained, say, in l-loop approximation) will contribute in the same way in all orders of the perturbation theory in the expression for γ_F. The alternative to such a scheme of calculations is the so-called pseudo-ε-expansion [3], where an auxiliary parameter (let it be t) is introduced to keep track of the orders of perturbation theory:

$$\beta_u(u,t) = -tu + \sum_{j=2}^{\infty} \beta_u^{(j-1)LA} u^j, \tag{4.10}$$

β_u^{jLA} being j-loop contributions. Of course the equality holds:

$$\beta_u(u, t=1) = \beta_u(u).$$

Now one can obtain the value of the fixed point u^* as a series in t [4] :

$$u^*(t) = \sum_j u^{*,(j)} t^j. \tag{4.11}$$

This series can be substituted into (4.7) resulting in:

$$\gamma_F(t) = \sum_j \tilde{\gamma}_F^j t^j, \tag{4.12}$$

and the final formula for critical exponent reads:

$$\gamma_F = \gamma_F^{res}(u^*, t=1), \tag{4.13}$$

where γ_F^{res} means the resummed (with respect to t) series (4.12). Both schemes explained above are used in analysis of the asymptotic series arising in the fixed-dimensional renormalization group framework. Here we will follow the second one. Performing the expansion (4.11) for the expression for the β-function (3.2) and substituting the series for the Heisenberg fixed

[2] We are speaking here only about the analysis of the series obtained in a certain (l-loop) approximation. Of course one should start from the same approximation in the number of loops for all renormalization group functions.

[3] First introduced by B.Nickel: see Ref.19 in [21].

[4] In fact powers of t in a certain term of perturbation theory correspond to the number of loops in the loop integrals and this enables us to separate contributions from different orders of the perturbation theory while substituting results of (4.6) into (4.7).

Table 1. Values of the star exponent γ_F obtained in three-dimensional theory (columns 2,3) in comparison with the results of ε-expansion (columns 4,5,6) and Monte-Carlo simulations (column 7). See the text for a full description.

F	$d=3$		ε-expansion [6,7]			MC [22,23]
3	1.06	1.05	1.05	1.05	1.07	1.09
4	0.86	0.86	0.84	0.83	0.85	0.88
5	0.61	0.61	0.53	0.52	0.55	0.57
6	0.32	0.32	0.14	0.18		0.16
7	-0.02	-0.01	-0.33	-0.20		
8	-0.40	-0.36	-0.88	-0.60		(-0.99, -0.30)
9	-0.80	-0.72	-1.51	-1.01		

point $u^* \neq 0$, $v^* = 0$ into (4.1) one finally obtains the following expansion of γ_F in powers of the "pseudo-ε" parameter t:

$$\gamma_F(t) = \sum_{j=0}^{3} \tilde{\gamma}_F^j t^j. \tag{4.14}$$

The coefficients $\tilde{\gamma}_F^j$ for $j = 0 - 3$ read:

$$\tilde{\gamma}_F^0 = 1, \tag{4.15}$$

$$\tilde{\gamma}_F^1 = (4-d)F\gamma_F^{(1LA)}, \tag{4.16}$$

$$\tilde{\gamma}_F^2 = (4-d)F\gamma_F^{2LA} + \frac{8(4-d)F\gamma_F^{1LA}}{(n+8)^2} \times ((5n+22)(i_1 - \frac{1}{2}) + (n+2)i_2), \tag{4.17}$$

$$\begin{aligned}\tilde{\gamma}_F^3 = {} & (4-d)F\gamma_F^{3LA} + \frac{16(4-d)F\gamma_F^{2LA}}{(n+8)^2}[(5n+22)(i_1 - \frac{1}{2}) + \\ & (n+2)i_2] + (4-d)F\gamma_F^{1LA}[\frac{128}{(n+8)^4}((5n+22)(i_1 - \frac{1}{2}) + \\ & +(n+2)i_2)^2 + \frac{1}{(n+8)^3}(-32n^2 - 488n - 1424 + 4(31n^2 + \\ & 430n + 1240)i_1 + (3d+8)(n+2)(n+8)i_2 - \\ & 12(n+2)(n+8)i_3 - 48(n^2 + 20n + 60)i_4 - \\ & 24(2n^2 + 21n + 58)i_5 - 6(3n^2 + 22n + 56)i_6 - \\ & 24(5n+22)i_7 - 12(n+2)(n+8)i_8)].\end{aligned} \tag{4.18}$$

and the coefficients γ_F^{1LA}, γ_F^{2LA}, γ_F^{3LA} are given in (4.2)-(4.4). Eq. (4.14) together with (3.2)-(3.5), (4.1) contains one of the main results of this paper.

Now one can apply the Padè-Borel resummation in t to the series (4.14) and then at $t = 1$ one obtains the values for the star exponent γ_F given for different F in the second and third columns of Table 1. The second column contains the value of γ_F obtained directly from resummation of the series (4.14); while in the third column we give γ_F obtained on the base of the resummed series for the exponent x'_F (namely such a way of calculation of γ_F was chosen in [6,7] in the frames of the ε-expansion method). The next columns give the results obtained by the ε^3-expansion based on: simple Padè approximation (the 4th column) and Padè-Borel analysis neglecting or exploiting exact results for $d = 2$ (the 5th and 6th columns respectively) [6,7]. The last column contains Monte-Carlo data [22,23]. For low number of arms $F \leq 5$ the results of the different approaches agree reasonably well and are also close to the values obtained by MC simulation. We have used two different renormalization schemes as well as different procedures for the resummation of the resulting asymptotic series. Table 1 gives thus a test for the stability of the results under changes of the calculational scheme. Obviously for higher number $F > 5$ of arms coincedence of the results is no longer good. The main reason for this is that calculating the exponents combinatorial factors lead to an expansion in $F\varepsilon$ for the ε expansion and of Fg when directly expanding in a renormalized coupling g. For such large values of the expansion parameter even resummation of the series fails. For large numbers of arms other approaches to the theory of polymer stars like a self consistent field approximation might be more useful. We conclude that the Parisi method of massive renormalization in fixed dimension as it is widely applied in the theory of critical phenomena used together with an appropriate resummation scheme is a powerful tool also for the calculation of exponents in polymer theory in the present case leading to a good test of previous results and methods.

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References

[1] P. G. De Gennes. Exponents for the excluded volume problem as derived by the Wilson method. // Phys. Lett. A, 1972, vol. 38, No 5, p. 339-340.

[2] P. G. De Gennes. Scaling concepts in polymer physics. New York, Ithaca, Cornell University Press, 1979.

[3] B. Duplantier. Polymer network of fixed topology: renormalization, exact critical exponent γ in two dimensions, and $d = 4 - \varepsilon$. // Phys. Rev. Lett., 1986, vol. 57, No 8, p. 941-944; B. Duplantier, H. Saleur. Exact surface and wedge exponents for polymers in two dimensions. // Phys. Rev. Lett., 1986, vol. 57, No 25, p. 3179-3182.

[4] B. Duplantier. Statistical mechanics of polymer networks of any topology. // J.Stat.Phys., 1989, vol. 54, No 3/4, p. 581-680.

[5] B. Duplantier. Exact contact critical exponents of a self avoiding polymer chain in two dimensions. // Phys. Rev. B, 1987, vol. 35, p. 5290-5293.

[6] B. Duplantier, C. von Ferber, U. Lehr, L. Schäfer. Renormalization of polymer networks and stars. in: "Renormalization Group'91, 2nd. Int. Conf., 3-6 Sept.

1991, Dubna, USSR", Ed. by D.V.Shirkov V.B.Priezzhev, Singapore, World Scientific, 1992, p. 473-595.
[7] L. Schäfer, C. von Ferber, U. Lehr, B. Duplantier. Renormalization of polymer networks and stars. // Nucl. Phys. B, 1992, vol. 374, No 3, p. 473-495.
[8] J. des Cloizeaux. Short range correlation between elements of a long polymer in a good solvent. // J. Phys. (Paris), 1980, vol. 41, No 3, p. 223-238.
[9] J. des Cloizeaux. Lagrangian Theory for a self avoiding random chain. // Phys. Rev., 1974, vol. 10, p. 1665-1669.
[10] S. F. Edwards. Proc. Phys. Soc., 1965, vol. 85, p. 613; Proc. Phys. Soc., 1966, vol. 88, p. 265.
[11] J. des Cloizeaux, G. Jannink. Les polymères en solution: Leur modélization et leur structure. Les Ulis, Les éditions de physique, 1987.
[12] D. J. Wallace, R. K. P. Zia. Harmonic perturbations of generalized Heisenberg spin systems. // J. Phys. C., 1975, vol. 8, No 6, p. 839-843.
[13] G. Parisi. Field-theoretic approach to second-order phase transitions in two- and three-dimensional systems. // J. Stat. Phys., 1980, vol. 23, No 1, p. 49 - 82.
[14] D. Amit. Field theory, the renormalization group, and critical phenomena. New York, Mc Graw-Hill int. Book Co., 1978.
[15] E. Brezin, J. C. Le Guillou, J. Zinn-Justin. Field theoretical approach to critical phenomena. In: "Phase transitions and critical phenomena", ed. by C.Domb and M.S.Green., New York, Academic Press, 1976, vol. 6, p. 125 -247.
[16] J. Zinn-Justin. Euclidean field theory and critical phenomena. New York, Oxford University Press, 1989.
[17] B. G. Nickel, D. I. Meiron, G. A. Baker, Jr. Compilation of 2-pt. and 4-pt. graphs for continuous spin models. Univ. of Guelf Report, 1977.
[18] Yu. Holovatch, T. Krokhmal's'kii. Compilation of two-point and four-point graphs in field theory in noninteger dimensions. // J. Math. Phys., 1994, vol. 35, No 8, p.3866-3880.
[19] G. A. Baker, Jr., B. G. Nickel, D. I. Meiron. Critical indices from perturbation analysis of the Callan-Symanzik equation. // Phys. Rev. B, 1978, vol. 17, No 3, p.1365-1374.
[20] Yu. Holovatch. Critical exponents of Ising-like systems in general dimensions. // Theor. Math. Phys. (Moscow), 1993, vol. 96, No 3, p.482-495.
[21] J. C. Le Guillou, J. Zinn-Justin. Critical exponents from field theory. // Phys. Rev. B, 1980, vol. 21, No 9, p. 3976-3998.
[22] J. Batoulis, K. Kremer. Thermodynamic properties of star polymers: good solvents. // Macromolecules, 1989, vol. 22, p. 4277-4285.
[23] A. J. Barret, D. L. Tremain. Lattice walk models of star polymers with many arms. // Macromolecules, 1987, vol. 20, p. 1687-1692.

ЗІРКОВІ ПОКАЗНИКИ ІЗ ТЕОРІЇ ПОЛІМЕРІВ: РЕНОРМГРУПОВІ РЕЗУЛЬТАТИ У ТРИВИМІРНОМУ ПРОСТОРІ

К. фон Фербер, Ю.В. Головач

Досліджуються сітки довільної, але фіксованої топології, утворені полімерними ланцюгами і обчислюються критичні показники, що характеризують їх масштабні властивості (зіркові

показники). Обчислення проводяться у рамках теоретико-польового підходу при фіксованій вимірності простору. Ренормгрупові функції в схемі Каллана-Симандзіка отримано у трипетлевому наближенні і проаналізовано безпосередньо у трьох вимірах. Ряди теорії збурень пересумовано за допомогою перетворення Паде-Борела. Отримані результати добре узгоджуються з даними Монте-Карло та ϵ-розкладу.