UDK 532,533,537.7; PACS 05.70Fh

PARTITION FUNCTION STEP-BY-STEP INTEGRATION WITHOUT APPROXIMATIONS. ONE-DIMENSIONAL FERROMAGNET WITH LONG-RANGE POTENTIAL

YU. V. KOZITSKY

Lviv Academy of Commerce
68 I. Franko St., UA-290011 Lviv, Ukraine

Received December 12, 1994

One-dimensional Ising-like ferromagnetic model with a pair spin-spin interaction decaying as $(\text{dist})^{-1-\lambda}$ is considered. For this model the step-by-step integration procedure in the spirit of I.R.Yukhnovskii [1] is developed. Its main feature is the absence of any approximations.

The concept of self-similarity [1] appearing at the critical point is one of the most productive idea in modern theories of critical phenomena. In the approach of I.R.Yukhnovskii [1] based on the collective variables method this idea was embodied in the scheme of step-by-step integration in the space of collective variables corresponding to the waves of a spin density in the three-dimensional Ising model. The significant feature of this scheme is the application of a number of approximations. They result in a substitution of the initial symmetry of considered model (the translational symmetry) by a new one. This is a symmetry of self-similarity. Mathematically it can be described as a substitution of the initial translationally invariant model by its hierarchical version [3,4]. The latter possesses all properties of known Dyson's hierarchical model [5], and that's why admits more advanced nathematically strict consideration. The other results of mentioned approximations consist in the fact that due to them a very considerable description on the physical level of strictness was made [1]. Unfortunately the mathematical evaluation of errors appearing in this scheme has not been performed and seems to be unpossible. Therefore having in mind the successes gained on the physical level from the one hand, and the vague situation with the direct control of errors, from the other one, we are going to return to the very beginning of the I.R.Yukhnovskii's scheme and elaborate a mathematically strict version of step-by-step integration procedure. The first attempt in this direction is presented below.

We start from the formal Hamiltonian of considered model given by the formula

$$H = -\frac{1}{2} \sum_{s,s' \in Z} J(s,s')\sigma_s \sigma_{s'}, \tag{1}$$

where Z is a set of integer numbers, $\{\sigma_s \mid s \in Z\}$ are one - dimentional random variables (spins) whose initial (free) probability distributions are identical and defined by the measure $d\chi(\sigma_s)$. The latter is supposed to belong to the Simon-Griffiths class [6]. The potential $J(s, s')$ is chosen as

$$J(s,s') = J(s',s) = \mid s - s' + 1 \mid^{-1-\lambda}, s \neq s' \tag{2}$$

and $J(s, s) = 0$. For given nonnegative integer m let

$$\Xi_m = \{-2^m, -2^m+1, ..., -1, 0, 1, ..., 2^m - 1\} \tag{3}$$

be a specially chosen subset of Z. We also introduce

$$\Lambda_s^{(n)} = \{2^n s, 2^n s + 1, ..., 2^n s + 2^n - 1\} \tag{4}$$

with $s \in Z$ and nonnegative integer n. For introduced sets we have

$$\Xi_m = \bigcup_{s \in \Xi_{m-n}} \Lambda_s^{(n)}; \quad 0 \leq n \leq m; \tag{5}$$

$$\Lambda_s^{(n)} = \bigcup_{s_1, \in \Lambda_s^{(n-k)}} \Lambda_{s_1}^{(k)}; \quad 0 \leq k \leq n. \tag{6}$$

Every Ξ_m we will consider as an additive group, therefore for every couple of elements of Ξ_m $s + s'$ means a group addition. For such a couple let

$$\rho_m(s, s') = \min\{\mid s - s' \mid; 2^{m+1} - \mid s - s' \mid\}; \tag{7}$$

therewith $\rho_m = 0, 1, ..., 2^m$. In order to define the Hamiltonian on the group Ξ_m we introduce

$$\begin{aligned} &J_m(s, s') = J_m(\rho_m(s, s')); \\ &J_m(0) = 0; \quad J_m(\rho) = (1 + \rho)^{-1-\lambda}, \quad \rho = 1, 2, ..., 2^m - 1; \\ &J_m(2^m) = 2^{-m(1+\lambda)}; \end{aligned} \tag{8}$$

and than

$$H_m = -\frac{1}{2} \sum_{s, s' \in \Xi_m} J_m(s, s') \sigma_s \sigma_{s'}. \tag{9}$$

The essence of our approach is placed in the following representation of H_m:

$$H_m = -\frac{1}{2} \sum_{k=0}^{m} \sum_{s, s' \in \Xi_{m-k}} I^{(k)}(m \mid s, s') \sigma(\Lambda_s^{(k)}) \sigma(\Lambda_{s'}^{(k)}) \tag{10}$$

where $\sigma(\Lambda) = \sum_{s \in \Lambda} \sigma_s$ and for $s' \geq s$, $\alpha = 0, 1$; $\alpha' = 0, 1$;

$$I^{(k)}(m \mid 2s + \alpha, 2s' + \alpha') =$$

$$
= \begin{cases} b^{(k)}(m \mid 2s' + \alpha' - 2s - \alpha) - b^{(k)}(m \mid 2s' - 2s + 1) + \\ \quad + a^{(k)}(m \mid 2s' - 2s + 1); \qquad \text{for } s' - s < 2^{m-k-1} \\ b^{(k)}(m \mid 2^{m-k} - \mid \alpha - \alpha' \mid) - b^{k)}(m \mid 2^{m-k}) + \\ \quad + a^{(k)}(m|2^{m-k}); \qquad \text{for } s' - s = 2^{m-k-1} \\ b^{(k)}(m|2^{m+1-k} - 2s' - \alpha' + 2s + \alpha) - \\ \quad - b^{(k)}(m|^{m+1-k} - 2s' + 2s + 1) + \\ \quad + a^{(k)}(m|2^{m+1-k} - 2s' + 2s + 1) \qquad \text{for } s' - s > 2^{m-k-1} \end{cases} \tag{11}
$$

and

$$
\begin{aligned} I^{(k)}(m|s,s) &= 0; \\ I^{(k)}(m|s',s) &= I^{(k)}(m|s,s'); \\ I^{(k)}(m|2s,2s+1) &= b^{(k)}(1). \end{aligned} \tag{12}
$$

Here we have used the following notations

$$
b^{(k)}(m|\rho) = J_m(2^k\rho + 2^k - 1) - \sum_{l=0}^{k-1} a^{(l)}(m|2^{k-l}\rho + 2^{k-l} - 1); \tag{13}
$$

for ρ=1,2,...,2^m-1, and

$$
b^{(k)}(m|2^{m-k}) = J_m(2^m) - \sum_{l=0}^{k-1} a^{(l)}(m|2^{m-l}). \tag{14}
$$

In the expressions (11)-(14) the functions $a^{(l)}(m|\rho)$ can be chosen arbitrarily. We may use this opportunity to let the functions $b^{(k)}(m|\rho)$ and $I^{(k)}(m|\rho)$ possess the scaling properties of the function $J_m(\rho)$. We put for ρ=1,2,...,2^{m-k}-1

$$
a^{(k)}(m|\rho) = \delta^k(1-\delta)[2^k(\rho+1)]^{-1-\lambda}; \tag{15}
$$

and

$$
a^{(k)}(m|2^{m-k}) = \delta^k(1-\delta)2^{-m(1+\lambda)}; \tag{16}
$$

where δ is certain positive parameter to be chosen. Here we remark that the choice δ=1 corresponds to the vanishing of all $a^{(l)}$. In what follows we obtain

$$
b^{(k)}(m|\rho) = \delta^k \cdot 2^{-k(1+\lambda)}(\rho+1)^{-1-\lambda}; \tag{17}
$$

for ρ=1,2,...,2^{m-k}-1 and

$$
b^{(k)}(m|2^{m-k}) = \delta^k 2^{-m(1+\lambda)}. \tag{18}
$$

Hereafter the expressions (11)-(14) yield

$$
I^{(k)}(m|s,s') = 2^{-\Theta k} I(m-k|s,s') \tag{19}
$$

with

$$\Theta = 1 + \lambda - \log \delta \tag{20}$$

and $s, s' \in \Xi_{m-k}$. In order to describe the function $I(n|s, s')$ we introduce the following notation:

$$I_{\alpha\alpha'}(n|s, s') = I(n|2s + \alpha, \quad 2s' + \alpha'), \tag{21}$$

where $s, s' \in \Xi_{n-1}$, and α, α'=0,1. Thereby we have for $0 < \rho_{n-1}(s, s') < < 2^{n-1}$:

$$I_{\alpha\alpha}(n|s, s') = (2\rho_{n-1}(s, s') + 1)^{-1-\lambda} - \delta(2\rho_{n-1}(s, s') + 2)^{-1-\lambda}; \tag{22}$$

and for $\rho_{n-1}(s, s') = 2^{n-1}$:

$$I_{\alpha\alpha}(n|s, s') = (1 - \delta)2^{-n(1+\lambda)}; \tag{23}$$

for $0 < s' - s < 2^{n-1}$:

$$I_{01}(n|s, s') = (1 - \delta)(2\rho_{n-1}(s, s') + 2)^{-1-\lambda}; \tag{24}$$

for $s' - s = 2^{n-1}$:

$$I_{01}(n|s, s') = (1 - \delta)2^{-n(1+\lambda)}; \tag{25}$$

and for $s' - s > 2^{n-1}$:

$$I_{01}(n|s, s') = (2\rho_{n-1}(s, s'))^{-1-\lambda} - \delta(2\rho_{n-1}(s, s') + 2)^{-1-\lambda}. \tag{26}$$

For the case $s' < s$ we put

$$I_{01}(n|s, s') = I_{10}(n|s', s) \tag{27}$$

and for $0 < s' - s < 2^{n-1}$:

$$I_{10}(n|s, s') = (2\rho_{n-1}(s, s'))^{-1-\lambda} - \delta(2\rho_{n-1}(s, s') + 2)^{-1-\lambda}; \tag{28}$$

for $s' - s = 2^{n-1}$:

$$I_{10}(n|s, s') = I_{01}(n|s, s') \tag{29}$$

and for $s' - s > 2^{n-1}$:

$$I_{10}(n|s, s') = (1 - \delta)(2\rho_{n-1}(s, s') + 2)^{-1-\lambda}. \tag{30}$$

Similarly we put

$$I_{10}(n|s, s') = I_{01}(n|s', s) \tag{31}$$

for the case $s' < s$. It should be remarked that the functions $I_{\alpha\alpha'}(n|s, s')$ are translationally invariant on Ξ_{n-1}, that is

$$I_{\alpha\alpha'}(n|s, s') = I_{\alpha\alpha'}(n|s + t, s' + t) \tag{32}$$

for all $t \in \Xi_{n-1}$, where $s + t$ means the group addition.

Now let us consider the following function:

$$G_k(z) = \int_{R^{\Xi_{n+m}}} \exp\left\{ \sum_{s\in\Xi_{n+m-k}} z_s \cdot 2^{-\frac{1}{2}k\Theta}\sigma(\Lambda_s^{(k)}) + \right.$$

$$\left. + \frac{\beta}{2}\sum_{l=0}^{k} \sum_{s,s'\in\Xi_{n+m-l}}^{k} I^{(l)}(n+m|s,s')\sigma(\Lambda_s^{(l)})\sigma(\Lambda_{s'}^{(l)}) \right\} \prod_{s\in\Xi_{n+m}} d\chi(\sigma_s), \quad (33)$$

where $\beta > 0$ is the inverse temperature and $z = \{z_s | s \in \Xi_{n+m-k}\}$. In what follows the function

$$F_k(z) = G_k(z)/G_k(0) \quad (34)$$

is a generating function of a normed total spin of block $\Lambda^{(k)}$:

$$2^{-k\Theta/2}\sigma(\Lambda^{(k)}) = \frac{1}{|\Lambda^{(k)}|^{\Theta/2}} \sum_{s\in\Lambda^{(k)}} \sigma_s,$$

whereas the function $G_k(z)$ corresponds to the partially integrated partition function appearing in the framework of I.Yukhnovskii method mentioned above. Indeed, in (33) the potential of the model described by the Hamiltonian (9), (10) is taken into account only partially because of the fact that the corresponding sum is restricted by the value of k instead of $n+m$ as in (10). The block stucture of the term describing the spin-spin interaction in (33) yields the sequence of $G_k(z)$ to be done recursively. In order to do this we remark that these functions can be continued on $C^{\Xi_{n+m-k}}$ as entire ones. It is caused by the corresponding properties of the measure $d\chi(\sigma)$ (see [4] for further details). Moreover, for every function we can define the action of the operator

$$\exp\left\{ \frac{1}{2} \sum_{s,s'\in\Xi_{n+m-k}} A(s,s')D_s D_{s'} \right\} G_k(z), \quad (35)$$

where $D_s = \partial/\partial z_s$ and $A(s,s') \geq 0$ are arbitrary. The result of this action is that the function (35) possesses all analytical properties of $G_k(z)$. Having this in mind we obtain:

$$G_k(z) = \exp\left\{ \frac{\beta}{2} \sum_{s,s'\in\Xi_{n+m-k}} 2^{k\Theta} I^{(k)}(n+m|s,s')D_s D_{s'} \right\} \times$$

$$\times \int_{R^{\Xi_{n+m}}} \exp\left\{ \sum_{s\in\Xi_{n+m-k}} z_s \cdot 2^{-\frac{1}{2}\Theta k}\sigma(\Lambda_s^{(k)}) + \right. \quad (36)$$

$$\left. + \frac{\beta}{2}\sum_{l=0}^{k-1} \sum_{s,s'\in\Xi_{n+m-l}} I^{(l)}(n+m|s,s')\sigma(\Lambda_s^{(l)})\sigma(\Lambda_{s'}^{(l)}) \right\} \times \prod_{s\in\Xi_{n+m}} d\chi(\sigma_s).$$

The property (6) yields

$$\sigma(\Lambda_s^{(k)}) = \sum_{s_1\in\Lambda_s^{(1)}} \sigma(\Lambda_{s_1}^{(k-1)}). \quad (37)$$

For $s_1 \in \Xi_{n+m-k+1}$ we put

$$\bar{z}_{s_1} = z_s \quad if \quad s_1 \in \Lambda_s^{(1)} \tag{38}$$

and having in mind the correspondence (37) we obtain:

$$\sum_{s \in \Xi_{n+m-k}} z_s 2^{-\frac{1}{2}k\Theta} \sigma(\Lambda_s^{(k)}) = \sum_{s_1 \in \Xi_{n+m-k+1}} \bar{z}_{s_1} 2^{-\frac{1}{2}\Theta} \cdot 2^{-\frac{1}{2}(k-1)\Theta} \sigma(\Lambda_{s_1}^{(k-1)}). \tag{39}$$

The latter yields in (36):

$$G_k(z) = \exp\left\{\frac{\beta}{2} \sum_{s,s' \in \Xi_{n+m-k}} 2^{k\Theta} I^{(k)}(n+m|s,s') D_s D_{s'}\right\} \times$$

$$\times \int_{R^{\Xi_{n+m}}} \exp\left\{\sum_{s \in \Xi_{n+m-k+1}} \bar{z}_s 2^{-\frac{1}{2}\Theta} \cdot 2^{\frac{-1}{2}(k-1)\Theta} \sigma(\Lambda_s^{k-1}) + \right.$$

$$\left. + \frac{\beta}{2} \sum_{l=0}^{k-1} \sum_{s,s' \in \Xi_{n+m-l}} I^{(l)}(n+m|s,s') \sigma(\Lambda_s^{(l)}) \sigma(\Lambda_{s'}^{(l)})\right\} \prod_{s \in \Xi_{n+m}} d\chi(\sigma_s) =$$

$$= \exp\left\{\frac{\beta}{2} \sum_{s,s' \in \Xi_{n+m-k}} I(n+m-k|s,s') D_s D_{s'}\right\} G_{k-1}\left(\bar{z} 2^{-\frac{1}{2}\Theta}\right); \tag{40}$$

with

$$G_0(z) = \int_{R^{\Xi_{n+m}}} \exp\left\{\sum_{s \in \Xi_{n+m}} z_s \sigma_s + \frac{\beta}{2} \sum_{s,s' \in \Xi_{n+m}} I^{(0)}(n+m|s,s') \sigma_s \sigma_{s'}\right\} \times$$

$$\times \prod_{s \in \Xi_{n+m}} d\chi(\sigma_s). \tag{41}$$

Here the scaling property (19) and the definition (33) have been taken into account. By putting in (40) $k = n$ we obtain the recursion for the sequence $G_n(z)$ we have mentioned above:

$$G_n(z) = \exp\left\{\frac{\beta}{2} \sum_{s,s' \in \Xi_m} I(m|s,s') D_s D_{s'}\right\} G_{n-1}\left(\bar{z} 2^{-\frac{1}{2}\Theta}\right), \tag{42}$$

where the variables $\bar{z}$ and z are connected by the relation (38). This relation corresponds to the "transition on a new lattice": the fundamental element of I.Yukhnovsky's scheme [1] (see also [3,4]). In order to avoid the problem of infinite volume divergencies we may substitute $G_n(z)$ by the properly normed function $F_n(z)$ given by (34). Thus we obtain

$$F_n(z) = \frac{1}{Z_n^{(m)}} \exp\left\{\frac{\beta}{2} \sum_{s,s' \in \Xi_m} I(m|s,s') D_s D_{s'}\right\} F_{n-1}\left(\bar{z} 2^{-\frac{1}{2}\Theta}\right),$$

$$Z_n^{(m)} = \exp\left\{\frac{\beta}{2} \sum_{s,s' \in \Xi_m} I(m|s,s') D_s D_{s'}\right\} F_{n-1}\left(\bar{z} 2^{-\frac{1}{2}\Theta}\right)\Big|_{z=0}. \tag{43}$$

The functions F_{n-1} and F_n in (43) depend on different number of variables: the former possesses them in the number $|\Xi_{m+1}| = 2|\Xi_m|$. From the other hand, in order to proceed enlarging n in (43) we must enlarge m. Proceeding this enlarging we may achieve the thermodynamic limit ($m \to \infty$) as well as the limit of the sequence $\{F_n(z)\}$. The latter must be a function of an infinite number of variables that require the appropriate mathematical methods to be applied [7]. This is supposed to be done in our next paper.

The research described in this publication was made possible in part by Grant N UCN 000 from the International Science Foundation.

References

[1] I.R.Yukhnovskii. Phase transitions of second order. Collective variables method. Singapour, World Scientific, 1987.

[2] Ya.G.Sinai Theory of phase transitions: Rigorous results. London: Pergamon Press, 1982.

[3] Yu.V.Kozitsky, I.R.Yukhnovskii. Generalized hierarchical model of a scalar ferromagnet in collective variables method. // Teor. Mat. Fiz., 1982, vol. 51, p. 268-277 (in Russian).

[4] Yu.V.Kozitsky. Hierarchical model of a vector ferromagnet. Self-similar block-spin distributions and the Lee-Yang theorem. Reports on Math. Phys., 1988, vol. 26, No 3, p. 429-445.

[5] F.J.Dyson. Existence of a phase transition in one-dimensional Ising ferromagnets. Commun. Math. Phys., 1969, vol. 12, p. 91-107.

[6] B.Simon, R.Griffiths. The $(\phi^4)_2$ field theory as a classical Ising model. Commun. Math. Phys., 1973, vol. 33, p. 145-164.

[7] Yu.M.Berezansky, Yu.G.Kondratiev. Spectral methods in infinite dimensional analysis. Naukova Dumka, Kiev, 1988 (in Russian, English translation to appear in Kluwer Academic Publishers).

ІНТЕГРУВАННЯ СТАТИСТИЧНОЇ СУМИ БЕЗ НАБЛИЖЕНЬ. ОДНОВИМІРНИЙ ФЕРОМАГНЕТИК З ДАЛЕКОСЯЖНИМ ПОТЕНЦІАЛОМ ВЗАЄМОДІЇ

Ю.В. Козицький

Розглядається одновимірна модель феромагнетика ізінгівського типу з парною взаємодією спін-спін, що спадає за законом $(\mathrm{dist})^{-1-\lambda}$. Для цієї моделі розвинуто процедуру покрочного інтегрування в дусі І.Р.Юхновського. Її основна особливість полягає у тому, що вона проводиться без жодних наближень.