УДК 536.75; PACS 04.25.Nx

# FIELD ASPECTS OF THE METHOD OF COLLECTIVE VARIABLES FOR THE SYSTEM OF CHARGED PARTICLES

L.F.Blazhyjevskii

*Ivan Franko Lviv State University*
*Chair of Theoretical Physics*
*12 Drahomanov St., 290005 Lviv–05, Ukraine*

Received June 9, 1995

The connection between the method of collective variables and the field theory methods is analyzed using as an example the relativistic system of charged particles. It is shown that partition sum can be written in the form of integral by the paths of particles in the configuration space, classical action of which is not additive, after the elimination of field variables. The correction to the energy of heat radiation is calculated.

## Introduction

The method of collective variables is one of the original mathematical methods in modern statistical physics. It's basis was created more than 40 years ago mainly by Bohm, Pines, Bogolubov, Zubarev, while considering the problems of plasma theory and superfluidity. The subsequent development of the method and it's application to the different problems of equilibrium statistical theory were carried out by Yukhnovskii. In early sixties he proposed "The method of displacements and collective variables", where classical and quantum schemes of collective approach were put together and which was successfully applied to the studies of quantum plasma and non-ideal bose-gas. At the same time Dr. Yukhnovskii and his followers developed the traditional approach of the method of collective variables (MCV), basing on the use of transition function. This approach appeared to be widely used in the classical system of particles with complex electrostatic interaction, in the theory of metals and liquids and in the theory of phase transitions. The results of the studies mentioned above and bibliography can be found in the monographs [1–3], in the review [4] and some papers included into journal [5].

The Method of Collective Variables is rested on the idea of introducing along with individual variables some additional (collective) variables, which describe the collective motion of the particles. Therefore, the introducing of collective coordinates can be considered as the turn from the theory of direct particles interactions to some kind of fields theory, which is formulated in terms of path integrations. The field aspect of collective integrations method becomes particularly clear if this method is applied to the equilibrium system of charged particles in weak relativistic (post-Newton) approximation. This approach was proposed in joint publication [6] of Yukhnovskii and the author. Here the possibility of generalization of this approach to the relativistic case is considered.

## 1. Path integral for statistical density matrix

Consider the system of N spinless particles, having mass m and charge e. In post-Newton approximation this system is described by Darvin Hamiltonian

$$L = L_0 + L_1, \quad L_0 = -\sum_j mc^2 \left(1 - \frac{v_j^2}{c^2}\right)^{1/2},$$
$$L_1 = \frac{1}{2}\sum_{j\neq l}\sum_{\mathbf{k}} \frac{4\pi e^2}{Vk^2} \mathrm{e}^{i\mathbf{k}\mathbf{r}_{jl}} \left\{1 - \frac{1}{c^2}\left[(\mathbf{v}_j\mathbf{v}_l) - \frac{1}{k^2}(\mathbf{k}\mathbf{v}_j)(\mathbf{k}\mathbf{v}_l)\right]\right\}, \tag{1.1}$$

here $\mathbf{v}$, $c$ - speed of the particles and light, correspondingly, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, $V$ – system volume. It is convenient to preserve the exact relativistic expression for $L_0$.

The traditional approach of statistic mechanics is based on Hamiltonian function use. However, the transition to the hamiltonian variables is not obvious since $L$ complicatedly depends on particles velocities. Particularly, it can be shown that finding the velocities as functions of momentums demands, in case of long-range interactions, the consideration of all $1/c$ power terms in the formal series. It is due to the fact that all these terms are of the same order of magnitude if one considers the thermodynamical limit (the existence of the latter is necessary to take into account in statistical mechanics.) The formulation of statistical mechanics in terms of functional integrals has been developed in author's papers [7,8]; in this approach all initial relations of the theory are expressed in terms of Lagrange function variables only. The formulation stems from certain modification of Feynman method of integration over variables. For example, the statistical density matrix

$$K(\mathbf{r}, \mathbf{r}'; \beta) = \mathrm{e}^{-\beta\hat{H}} \prod_j \delta(\mathbf{r}_j - \mathbf{r}_j'),$$

($H$ – Hamilton operator, $\beta^{-1}$ – temperature) can be written in the form of path integral by the paths in configuration space, particularly:

$$K(\mathbf{r}, \mathbf{r}'; \beta) = \mathrm{reg} C^{3N} \int D^{3N}u(\tau) \mathrm{e}^W \prod_j \delta\left(\mathbf{r}_j - \mathbf{r}_j' - \hbar\int_0^\beta d\tau \mathbf{u}_j(\tau)\right), \tag{1.2}$$

$$W = \int_0^\beta d\tau L\Big(i\mathbf{u}(\tau), \mathbf{r}(\tau)\Big), \tag{1.3}$$

$$Du(\tau) = \prod_{0\le\tau\le\beta} du(\tau), \quad -\infty \le u(\tau) \le +\infty,$$

$$C^{-1} = \int Dz(\tau)\exp\left(-\frac{1}{2}\int_0^\beta d\tau z^2(\tau)\right).$$

The functional $L(i\mathbf{u}(\tau), \mathbf{r}(\tau))$ can be derived from classical Lagrange function of the system, if in the latter one substitutes the velocities $\mathbf{v}_j$ by $i\mathbf{u}_j(\tau)$

$(i^2 = -1)$, and coordinates $\mathbf{r}_j$ by expressions

$$\mathbf{r}_j(\tau) = \mathbf{r}_j - \hbar \int_{\tau}^{\beta} d\tau' \mathbf{u}_j(\tau') \quad or \quad \mathbf{r}_j(\tau) = \mathbf{r}'_j + \hbar \int_{0}^{\tau} d\tau' \mathbf{u}_j(\tau'). \tag{1.4}$$

$W$ has the meaning of classical action being dependent on Lagrange function variables for the theory with imaginary time $t = \hbar\beta/i$ (Euclidian theory). The "reg" symbol means that the result of path integral calculation has to be regularized. This regularization means emission of the terms, which are proportional to $\left[\delta(0)\right]^n (n = 1, 2, ..., \delta(0) = \delta(\tau)_{|\tau=0}$, $\delta(\tau)$- Dirac delta-function. In the approximation of path integral by finite number of usual integrals the role of $\delta(0)$ plays $1/\Delta\tau$. We won't use symbol "reg" in the future, however, we'll keep in mind the necessity, if required, of regularization mentioned above.

The relations for matrix $K$ listed above are general. Obviously, the functional $W$ in case of post-Newton system of charged particles is determined by formulas (1.3'), (1.1). It can be easily written in the form

$$W = W_0 + W_s + W_1, \tag{1.5}$$

$$W_0 = -\int_0^{\beta} d\tau \sum_j mc^2\left(1 + u_j^2(\tau)/c^2\right)^{1/2},$$

$$W_s = \frac{1}{2}\int_0^{\beta} d\tau \sum_j \sum_{\mathbf{k}} e^2 \nu_{\mathbf{k}}^2 \mathbf{f}_{\mathbf{k}}^2(\tau),$$

$$\frac{1}{2}W_1 = \int_0^{\beta} d\tau \sum_{\mathbf{k}} \nu_{\mathbf{k}}^2 \left(\mathbf{X}_{\mathbf{k}}(\tau)\mathbf{X}_{-\mathbf{k}}(\tau)\right) \equiv$$

$$-\frac{1}{2}\int_0^{\beta} d\tau \sum_{\mathbf{k}} \nu_{\mathbf{k}}^2 |\mathbf{X}_{\mathbf{k}}(\tau)|^2.$$

Here we use the following notations:

$$\mathbf{X}_{\mathbf{k}}(\tau) = \sum_j e e^{-i\mathbf{k}\mathbf{r}_j(\tau)} \mathbf{f}_{\mathbf{k}}(\tau), \quad (\mathbf{X}_{-\mathbf{k}} = -\mathbf{X}_{\mathbf{k}}^*),$$

$$\mathbf{f}_{\mathbf{k}}(\tau) = \frac{\mathbf{k}}{|\mathbf{k}|} + \frac{1}{c}\left[\frac{\mathbf{k}}{|\mathbf{k}|}\mathbf{u}_j(\tau)\right], \quad \nu_{\mathbf{k}}^2 = \frac{4\pi}{Vk^2}. \tag{1.6}$$

Let's point out that term $W_S$ in (1.5) is introduced in order to compensate self-action terms in $W_1$ which are absent in initial expression (1.1).

To finish this part we write the formula for partition sum, derived by integration of diagonal elements of matrix $K$:

$$Z_N = \int d^{3N}rC^{3N} \int D^{3N}u(\tau) e^{W_0 + W_S + W_1} \prod_j \delta\left(\hbar \int_0^{\beta} d\tau \mathbf{u}_j(\tau)\right). \tag{1.7}$$

In non-relativistic case ($c \to \infty$) Lagrange function (1.1) is equal to the difference between kinetics energy of the particles and the energy of Coulumb interaction. One can show that in this case the relation (1.1) has the form of usual Feynman integral by the paths.

## 2. The introduction of field variables

We introduce in (1.7) the additional integration over the field variables. It can be done by expressing $\exp W_1$ in terms of Gauss path integral, particularly

$$e^{W_1} \equiv \exp\frac{1}{2}\int_0^\beta d\tau \sum_{\mathbf{k}} \nu_{\mathbf{k}}^2 \left(\mathbf{X}_{\mathbf{k}}(\tau)\mathbf{X}_{-\mathbf{k}}(\tau)\right) = \tag{2.1}$$

$$B\int D\mathbf{R}_{\mathbf{k}}(\tau)\exp\int_0^\beta d\tau \sum_{\mathbf{k}} \left\{\frac{1}{2}\Big(\mathbf{R}_{\mathbf{k}}(\tau)\mathbf{R}_{-\mathbf{k}}(\tau)\Big) - i\nu_{\mathbf{k}}\Big(\mathbf{R}_{\mathbf{k}}(\tau)\mathbf{X}_{-\mathbf{k}}(\tau)\Big)\right\}.$$

The constant $B$ is derived using the condition which demands the integral over $\mathbf{R}_{\mathbf{k}}$ to be equal to 1 while $\mathbf{X}_{\mathbf{k}} = 0$. Let's point out that variable $\mathbf{R}_{\mathbf{k}}(\tau)$ is complex. Since $\mathbf{X}_{\mathbf{k}}^* = -\mathbf{X}_{-\mathbf{k}}$ and so $\mathbf{R}_{\mathbf{k}}^* = -\mathbf{R}_{-\mathbf{k}}^*$, therefore $\mathbf{R}_{\pm\mathbf{k}} = \pm\mathbf{R}_k^c + i\mathbf{R}_{\mathbf{k}}^s$. Then

$$\frac{1}{2}\sum_{\mathbf{k}}(\mathbf{R}_{\mathbf{k}}\mathbf{R}_{-\mathbf{k}}) = -\frac{1}{2}\sum_{\mathbf{k}}|\mathbf{R}_{\mathbf{k}}|^2 = -\frac{1}{2}|\mathbf{R}_0|^2 - \sum_{\mathbf{k}>0}\left\{(\mathbf{R}_{\mathbf{k}}^c)^2 + (\mathbf{R}_{\mathbf{k}}^s)^2\right\},$$

and the right-hand part in (2.1) is the shortened form of the following expression

$$B\int\prod_\tau\left\{d\mathbf{R}_0(\tau)\prod_{\mathbf{k}>0} d\mathbf{R}_{\mathbf{k}}^c(\tau)d\mathbf{R}_{\mathbf{k}}^s(\tau)\right\}\exp\int_0^\beta d\tau\left\{-\frac{1}{2}\mathbf{R}_0^2 - i\nu_0(\mathbf{R}_0\mathbf{X}_0)\right\}\times$$

$$\times\exp\int_0^\beta d\tau\sum_{\mathbf{k}>0}\left\{-(\mathbf{R}_{\mathbf{k}}^c)^2 - (\mathbf{R}_{\mathbf{k}}^s)^2 - 2i\nu_{\mathbf{k}}(\mathbf{R}_{\mathbf{k}}^c\mathbf{X}_{\mathbf{k}}^c) - 2i\nu_{\mathbf{k}}(\mathbf{R}_{\mathbf{k}}^s\mathbf{X}_{\mathbf{k}}^s)\right\}.$$

Substituting (2.1) into (1.7), we obtain

$$Z_N = \int d^{3N}rC^{3N}\int D^{3N}u(\tau)\times \tag{2.2}$$

$$B\int D\mathbf{R}_{\mathbf{k}}(\tau)e^{W[\mathbf{R},\mathbf{u},\mathbf{r}]}\prod_j\delta\left(\hbar\int_0^\beta d\tau\,\mathbf{u}_j(\tau)\right),$$

$$W[\mathbf{R},\mathbf{u},\mathbf{r}] = W_p + \tilde{W}_f + W_{int}, \quad W_p = W_0 + W_s,$$

$$\tilde{W}_f = \frac{1}{2}\int_0^\beta d\tau\sum_{\mathbf{k}}\Big(\mathbf{R}_{\mathbf{k}}(\tau)\mathbf{R}_{-\mathbf{k}}(\tau)\Big),$$

$$W_{int} = -\int_0^\beta d\tau\sum_{\mathbf{k}} i\nu_{\mathbf{k}}\Big(\mathbf{R}_{\mathbf{k}}(\tau)\mathbf{X}_{-\mathbf{k}}(\tau)\Big),$$

here $W_0$, $W_s$, $\mathbf{X_k}$ are derived from formulas (1.5), (1.6).

The relation (2.2) can be derived using collective variables method as well, if $\mathbf{X_k}$

$$e^{W_1} = \int dZ_k(\tau) J[\mathbf{Z_k}; \mathbf{u}, \mathbf{r}] \exp \frac{1}{2} \int_0^\beta d\tau \sum_{\mathbf{k}} i\nu_{\mathbf{k}}^2 \left(\mathbf{Z_k}(\tau)\mathbf{Z_{-k}}(\tau)\right),$$

here transition function $J$ is chosen by infinite product of $\delta$-functions [6]:

$$J = \prod_\tau \prod_{\mathbf{k}} \delta \left( \mathbf{Z_k}(\tau) - e \sum_j e^{-i\mathbf{k}\mathbf{r}_j(\tau)} \left\{ \frac{\mathbf{k}}{|\mathbf{k}|} + \frac{1}{c} \left[ \frac{\mathbf{k}}{|\mathbf{k}|} \mathbf{u}_j(\tau) \right] \right\} \right).$$

Now, using the integral representation of $\delta$ function and calculating Gauss path integral by $\mathbf{Z_k}(\tau)$, one can obtain formula (2.1) for $\exp W_1$.

Let's write $Z_N$ in the form, convenient for further calculations. Particularly, we change the integration order over $\mathbf{r}_j$,$\mathbf{u}_j(\tau)$ $\mathbf{R_k}(\tau)$ and take into account that $W_p$ and $W_{int}$ are unar quantities. Then one can easily get the expression for $Z_N$ in the following form

$$Z_N = B \int D\mathbf{R}_k(\tau) e^{\tilde{W}_f} \left[Z_1(\mathbf{R})\right]^N . \tag{2.3}$$

Here $Z_1(\mathbf{R})$ is one-particle partition sum

$$Z_1(\mathbf{R}) = \int d^3 r C^3 \int D^3 u(\tau) e^{W(1)} \delta \left( \hbar \int_0^\beta d\tau \mathbf{u}(\tau) \right), \tag{2.4}$$

and $W(1)$ is defined by the expression which is summed over the particles in $W_p + W_{int}$.

The integral (2.3) has to be regularized also. It is due to the change of integration order while moving from formula (2.2) to (2.3).

## 3. Relativistic system of charged particles

Let's consider the relations from previous chapter in more detail. As it can be seen from (2.2), the functional $W[\mathbf{R}, \mathbf{u}, \mathbf{r}]$ has the form of action for the system of particles and some fictitious field $\mathbf{R_k}(\tau)$. The structure of $W_{int}$ allows to identify the field variables. Indeed, the part of action which describes the interaction of charged particles and electromagnetic field has the following form in Euclidian theory:

$$\frac{1}{V} \int_0^\beta d\tau \sum_j \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}_j} \left\{ i\frac{e}{c} \left( \mathbf{u}_j(\tau) \mathbf{A_k}(\tau) \right) - e\varphi_{\mathbf{k}}(\tau) \right\}, \tag{3.1}$$

here $\varphi_{\mathbf{k}}(\tau)$ $\mathbf{A_k}(\tau)$ – Fourier representation for scalar and vector potentials of electromagnetic field. Considering notations (1.6) we can easily conclude that $W_{int}$ in (2.2) is equal to expression in (3.1) if

$$-i\frac{\sqrt{4\pi V}}{k^2}(\mathbf{k}\mathbf{R_k}) = \varphi_{\mathbf{k}}, \quad -\frac{\sqrt{4\pi V}}{k^2}[\mathbf{k}\mathbf{R_k}] = \mathbf{A_k}. \tag{3.2}$$

That is, the integration variable $\mathbf{R_k}(\tau)$ can be related to the electromagnetic field potentials in the following way:

$$\mathbf{R_k} = \frac{1}{\sqrt{4\pi V}}\Big\{[\mathbf{kA_k}] + i\mathbf{k}\varphi_\mathbf{k}\Big\} \tag{3.3}$$

Substituting (3.3) to the formula for $\tilde{W}_f$ we obtain

$$\tilde{W}_f = \int\limits_0^\beta d\tau \sum_\mathbf{k} \frac{k^2}{8\pi V}\left\{|\varphi_\mathbf{k}|^2 - |\mathbf{A_k}|^2\right\} = \int\limits_0^\beta d\tau \sum_\mathbf{k} \frac{1}{8\pi V}\left\{|\mathbf{E_k}|^2 - |\mathbf{H_k}|^2\right\}, \tag{3.4}$$

here $\mathbf{E_k}$, $\mathbf{H_k}$ are Fourie representations for electrical and magnetic field. Therefore, the expression $W[\mathbf{R},\mathbf{u},\mathbf{r}]$ which is integrated in (2.2) can be said to be the action integral for charged particles and field in post-Newton approximation.

The structure of functional $W[\mathbf{R},\mathbf{u},\mathbf{r}]$ enables to make generalization of post-Newton relations onto relativistic case. Indeed, the expressions for $W_0$, $W_{int}$ have the same form in the relativistic theory of charged particles. We can expect that in order to obtain a relativistic formula for partition sum one can just substitute the "static action" $\tilde{W}_f$ in $W[\mathbf{R},\mathbf{u},\mathbf{r}]$ by exact relation for free electromagnetic field action integral, that is, to replace the fictitious field by the real one. To take into account field's degrees of freedom one should complement the expression in brackets (3.4) by the term $-|\dot{\mathbf{A}}_\mathbf{k}|^2/(\hbar ck)^2$, here the point means the derivation over $\tau$. Considering relations (3.2) we write action integral in the following form:

$$W_f = \frac{1}{2}\int\limits_0^\beta d\tau \sum_\mathbf{k}\left\{(\mathbf{R_k R_{-k}}) + \frac{1}{(\hbar ck)^2}\left[(\dot{\mathbf{R}}_\mathbf{k}\dot{\mathbf{R}}_\mathbf{-k}) - (\mathbf{k}\dot{\mathbf{R}}_\mathbf{k})(\mathbf{k}\dot{\mathbf{R}}_\mathbf{-k})/k^2\right]\right\}. \tag{3.5}$$

This permits to describe the partition sum of relativistic system "charged particles + field" by the following formula:

$$\Xi = \int d^{3N}rC^{3N}\int D^{3N}u(\tau)\bar{B}\int D\mathbf{R_k}(\tau)\prod_j \delta\left(\hbar\int\limits_0^\beta d\tau \mathbf{u}_j(\tau)\right)\times$$
$$\exp(W_f + W_0 + W_s + W_{int}), \tag{3.6}$$

here $W_f$ is defined from (3.5), $W_0$, $W_s$, $W_{int}$ are the same as in (2.2), constant $\bar{B}$ is derived in order to find the partition sum of free electromagnetic field in case of particles absence. It is obvious that formula (3.6) can be rewritten in the form

$$\Xi = \bar{B}\int D\mathbf{R_k}(\tau)e^{W_f}\left[Z_1(\mathbf{R})\right]^N \tag{3.7}$$

with the same $Z_1(\mathbf{R})$ as in (2.3). Let's point out that integral over field variables in (3.6) is the regular Feynman integral by paths.

The relations for $\Xi$ listed above are complied with the results obtained in the fields theory. It becomes clear if we change the integration over $\mathbf{R}_k(\tau)$

in (3.6) to the integration over $\mathbf{A_k}(\tau)$, $\varphi_\mathbf{k}(\tau)$. Then the expression $W_0 + W_f + W_{int}$ would have the usual form of action integral for charged particles and field. Substitution of three integrals by four leads to appearance of extra multiplier $\prod_\tau \prod_\mathbf{k} \delta(\mathbf{kA_k})$ in integral, as it should be expected since electromagnetic field is gauged. (It is well-known [9] that quantization of gauged fields is done by path integration of expressions $\Phi \exp \frac{i}{\hbar}[\text{action}]$, here the functional $\Phi$ sets the type of gauging.) Therefore, the relation (3.6) differs from the similar formula in fields theory by self-action terms only - they are partially eliminated in post Newton approximation, but are present if field approach is applied. The reason for it is that the fields theory has "naked" masses and charges and it's results should be renormalized. Particularly, in weak relativistic approximation, using Lagrange formulation of the theory, this renormalization actually leads to the elimination of self-action and substitution of "naked" masses and charges by physical ones. This is why, formula (3.6) should be considered as the result of fields theory, where renormalization is partially done.

## 4. Path integral for partition sum of the particles having non-additive action

The path integral (3.6) is Gauss-type as to variables $\mathbf{R_k}(\tau)$. This makes feasible to write $\Xi$ in the form of product of free field partition sum and partition sum of particles with direct interaction. We consider free field case first ($N = 0$). It is convenient to change temporarily integration variables in (3.7), according to $\mathbf{R}_{\mathbf{k},n}$

$$\mathbf{R_k}(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \mathbf{R}_{\mathbf{k},n} e^{i\omega_n \tau}, \quad \omega_n = \frac{2\pi}{\beta} n.$$

Then,

$$\Xi(N=0) = \bar{B} I \int \prod_\mathbf{k} \prod_n d\mathbf{R}_{\mathbf{k},n} \exp \Phi, \tag{4.1}$$

$$\Phi = \frac{1}{2\beta} \sum_\mathbf{k} \left\{ \sum_{n \neq 0} \left[ \delta_{\mu\nu} + l^\perp_{\mu\nu} \left( \frac{\omega_n}{\hbar c k} \right)^2 \right] R^\mu_{k,n} R^\nu_{-k,-n} + \left( \mathbf{R}_{\mathbf{k},0} \mathbf{R}_{-\mathbf{k},0} \right) \right\},$$

here $\mu, \nu = \{x, y, z\}$, $I$ - transformation Jacobi factor, $\delta_{\mu\nu}$- Cronecker factor,

$$l^\perp_{\mu\nu} = \delta_{\mu\nu} - k_\mu k_\nu / k^2 \equiv \delta_{\mu\nu} - l^{\parallel}_{\mu\nu}.$$

Reducing $\Phi$ to canonical form and returning back to initial variables $\mathbf{R}_k(\tau)$, we found

$$\Xi(N=0) = \left\{ \prod_\mathbf{k} \prod_{n \neq 0} \det \left| \delta_{\mu\nu} + l^\perp_{\mu\nu} \left( \frac{\omega_n}{\hbar c k} \right)^2 \right| \right\}^{-1/2} \times$$
$$\bar{B} \int D\mathbf{R_k}(\tau) \exp \frac{1}{2} \int_0^\beta d\tau \sum_\mathbf{k} (\mathbf{R_k R_{-k}}). \tag{4.2}$$

The calculation of product over $n$ is not difficult. It can be shown that if $\bar{B}$ match the condition

$$\Big(\prod_{\mathbf{k}}\prod_{n\neq 0} 4\pi^2 n^2\Big)^{-1}\bar{B}\int D\mathbf{R}_{\mathbf{k}}(\tau)\exp\frac{1}{2}\int_0^\beta d\tau\sum_{\mathbf{k}}(\mathbf{R}_{\mathbf{k}}\mathbf{R}_{-\mathbf{k}})=1,$$

one can deduce from (4.2) the partition sum of harmonic oscillators:

$$\Xi(N=0)=\exp\left\{-2\sum_{\mathbf{k}}\ln(1-e^{-\beta\hbar ck})-\beta\sum_{\mathbf{k}}\hbar ck\right\}$$
$$\equiv Z_f\exp(-\beta\sum_{\mathbf{k}}\hbar ck), \tag{4.3}$$

here $Z_f$ is partition sum of free field.

The integration over $\mathbf{R}_{\mathbf{k}}(\tau)$ in case of $N\neq 0$ is done in the same way. The integral over field variables in (3.6) differs from (4.1) by presence of the additional term in exponent, which is linear by $\mathbf{R}_{\mathbf{k}n}$. It can be eliminated by shift transformation. After calculations are made, we get $\Xi = Z_f\tilde{Z}_N$, here $\tilde{Z}_N$ is defined by formula (1.7) in which $W_1$ is substituted by

$$S_1=-\frac{1}{2}\int_0^\beta d\tau\int_0^\beta d\tau'\sum_{j,l}\sum_{\mathbf{k}}e^2\nu_{\mathbf{k}}^2\exp\left\{i(\mathbf{k}\mathbf{r}_j(\tau))-(\mathbf{k}\mathbf{r}_l(\tau'))\right\}\times$$
$$\left[\delta(\tau-\tau')+\frac{1}{c^2}l^{\perp}_{\mu\nu}u^{\mu}_j(\tau)u^{\nu}_l(\tau')D_k(\tau-\tau')\right]; \tag{4.4}$$

$$D_k(\tau-\tau')=\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}e^{i\omega_n(\tau-\tau')}\left[1+\left(\frac{\omega_n}{\hbar ck}\right)^2\right]^{-1}=$$
$$\frac{1}{2}\hbar ck\ \mathrm{ch}\left[\hbar ck\left(\frac{\beta}{2}-|\tau-\tau'|\right)\right]\mathrm{sh}\left[\hbar ck\frac{\beta}{2}\right]. \tag{4.5}$$

Derived expression $\tilde{Z}_N$ can be considered as partition sum of relativistic particles having direct (non-field) interaction. The expression (4.5) is temperature photon Green function. In post Newton approximation $D_k(\tau-\tau')=\delta(\tau-\tau')$. Then $S_1=W_1$ and $\tilde{Z}_N$ complies with (1.7).

It is worthwhile mentioning that the same result for $\tilde{Z}_N$ can be derived in a different way. Let's eliminate from the very beginning the field variables in action integral for charged particles and field. It can be easily shown that field equations $\delta(W_f+W_{int})=0$ in terms of $\mathbf{R}_{\mathbf{k}}(\tau)$ and their solutions have the following form:

$$\frac{1}{(\hbar ck)^2}l^{\perp}_{\mu\nu}\ddot{R}^{\nu}_{\mathbf{k}}-R^{\mu}_{\mathbf{k}}+i\nu_{\mathbf{k}}X^{\mu}_{\mathbf{k}}(\tau)=0,$$

$$R^{\mu}_{\mathbf{k},n}=i\nu_{\mathbf{k}}\left\{l^{\parallel}_{\mu\nu}+l^{\perp}_{\mu\nu}\left[1+\left(\frac{\omega_n}{\hbar ck}\right)^2\right]^{-1}\right\}X^{\nu}_{\mathbf{k},n}.$$

Using the last relation in expressions for action we find $W_f + W_{int} = S_1$. This approach of field degrees of freedom elimination is typical for Wheleer-Feynman electrodynamics. Therefore the functional $W_0 + S_1 = S$ can be considered as the alternative to Wheeler-Feynman theory with imaginary time. The presence of two integrations over $\tau$ in (4.4) means that action is not additive. For such systems Hamilton and Lagrange functions do not exist in traditional sense. Assuming that formula (1.2) is correct in case of non-additive action as well, one can easily find that the above relation for $\tilde{Z}_N$ is derived by simple substitution the functional $W$ in (1,2) by $S$.

## 5. Classical relativistic gas of charged particles

As it follows from (3.7), in order to find $\Xi$ we should calculate one-particle partition sum $Z_1(\mathbf{R})$ determined by relation (2.4). We restrict ourselves to consideration of charged particles classical gas model. In this case the expression for one - particle action gets simplified since, accordingly to (1.4), the functional dependence of coordinates on velocities disappears when $\hbar = 0$. Then, we use cumulant expansion for $Z_1(\mathbf{R})$. Considering terms having first order of charge magnitude in cumulant powers we obtain

$$Z_1(\mathbf{R}) = Z_1^0 \exp\left\{ \frac{1}{2}\int_0^\beta d\tau \sum_{\mathbf{k}} e^2 \nu_{\mathbf{k}}^2 \langle\langle \mathbf{f}_{\mathbf{k}}^2(\tau)\rangle\rangle + \right.$$
$$\left. \frac{1}{2}\int_0^\beta d\tau \int_0^\beta d\tau' \sum_{\mathbf{k}} e^2 \nu_{\mathbf{k}}^2 R_{\mathbf{k}}^{\mu}(\tau) R_{-\mathbf{k}}^{\nu}(\tau') \langle\langle f_{\mathbf{k}}^{\mu}(\tau) f_{\mathbf{k}}^{\nu}(\tau')\rangle\rangle \right\}, \tag{5.1}$$

here $Z_1^0$ - partition sum of one particle and

$$\langle\langle(\ldots)\rangle\rangle = \frac{1}{Z_1^0}\int d^3 r C^3 \int D^3 u(\tau)(\ldots) \mathrm{e}^{W_0(1)} \delta\left(\hbar \int_0^\beta d\tau \mathbf{u}(\tau)\right). \tag{5.2}$$

Path integrals (5.2) are calculated using the following relations [8]:

$$\mathrm{reg} C^3 \int D^3 u(\tau)\,[\ldots \mathbf{u}(\tau_0)\ldots]\, \mathrm{e}^{W_0(1)} \delta\left(\hbar \int_0^\beta d\tau \mathbf{u}(\tau)\right) =$$
$$\int \frac{d^3 p}{(2\pi b)^3} e^{-\beta \mathcal{E}(p)} \mathrm{reg}\left[\ldots \frac{1}{i}\left(\frac{\delta}{\delta \varphi(\tau_0)} + \frac{\delta}{\delta \mathbf{p}} \mathcal{E}(\mathbf{p} - \varphi(\tau_0))\right)\ldots\right]_{|\varphi=0},$$
$$\mathcal{E}(\mathbf{p}) = (m^2 c^4 + c^2 \mathbf{p}^2)^{1/2}.$$

It can be shown then that

$$\langle\langle f_{\mathbf{k}}^{\mu}(\tau) f_{\mathbf{k}}^{\nu}(\tau)\rangle\rangle = \pi_{\mu\nu} = l_{\mu\nu}^{\parallel}\pi^{\parallel} + l_{\mu\nu}^{\perp}\pi^{\perp},$$
$$\langle\langle f_{\mathbf{k}}^{\mu}(\tau) f_{\mathbf{k}}^{\nu}(\tau')\rangle\rangle = \pi_{\mu\nu} - l_{\mu\nu}^{\perp}\beta\pi^{\perp}\delta(\tau - \tau'),$$
$$\pi^{\parallel} = 1, \quad \pi^{\perp} = -\left\langle \frac{\mathbf{p}^2 - (\mathbf{kp})^2/k^2}{2\mathcal{E}^2(p)/c^2} \right\rangle_0 = -\left\langle \frac{c^2 p^2}{3\mathcal{E}^2(p)} \right\rangle_0 \equiv -\frac{1}{\delta}, \tag{5.3}$$

here $\langle\ldots\rangle_0$ means averaging over Maxwell distribution. Using these results in (3.7) we find

$$\Xi = (Z_1^0)^N \exp\left\{\frac{1}{2}\sum_{\mathbf{k}}\frac{\mathcal{H}^2}{k^2}\pi_{\mu\mu}\right\}\bar{B}\int D\mathbf{R}_{\mathbf{k}}(\tau)\exp\Phi[R], \tag{5.4}$$

$$\Phi[R] = \frac{1}{2}\sum_{\mathbf{k}}\left\{\int_0^\beta d\tau\left(\gamma_{\mu\nu}R_{\mathbf{k}}^{\mu}(\tau)R_{-\mathbf{k}}^{\nu}(\tau) + l_{\mu\nu}^{\perp}\dot{R}_{\mathbf{k}}^{\mu}(\dot{\tau})\dot{R}_{-\mathbf{k}}^{\mu}(\tau)/(\hbar ck)^2\right) + \int_0^\beta d\tau\int_0^\beta d\tau'\frac{\mathcal{H}^2}{\beta k^2}\pi_{\mu\nu}R_{\mathbf{k}}^{\mu}(\tau)R_{-\mathbf{k}}^{\nu}(\tau)\right\}, \tag{5.5}$$

$$\gamma_{\mu\nu} = \delta_{\mu\nu} + l_{\mu\nu}^{\perp}\kappa^2/k^2\delta \equiv l_{\mu\nu}^{\parallel}\gamma^{\parallel} + l_{\mu\nu}^{\perp}\gamma^{\perp}, \quad \mathcal{H}^2 = 4\pi Ne^2\beta/V. \tag{5.6}$$

As in case of free field the integration over $\mathbf{R}_{\mathbf{k}}(\tau)$ leads to changes in variables, which, in turn, brings $\Phi$ to canonical form. We reduce to this form the first term in (5.5). As the result, the tensors $l_{\mu\nu}^{\perp}$, $\pi_{\mu\nu}$ are replaced by $(l^{\perp}\gamma^{-1})_{\mu\nu}$, $(\pi\gamma^{-1})_{\mu\nu}$, here $\gamma^{-1}$ - matrix inverse to $\gamma$. It can be easily shown that $(l^{\perp}\gamma^{-1})_{\mu\nu} = l_{\mu\nu}^{\perp}/\gamma^{\perp}$, $(\pi\gamma^{-1})_{\mu\nu} = l_{\mu\nu}^{\parallel}\pi^{\parallel}/\gamma^{\perp} + l_{\mu\nu}^{\perp}\pi^{\perp}/\gamma^{\perp}$. Jacobian of transformation should be put 1 due to the relation $\prod_{\tau} a(\tau) = \exp\{\delta(0)\int d\tau \ln a(\tau)\}$ and regularization mentioned above. Later, changing the variables into $\mathbf{R}_{\mathbf{k},n}$, we get integral (5.4) in the form which differs from (4.1) by the relation for $\Phi$ function only. Now it is defined by relation

$$\Phi = \frac{1}{2\beta}\sum_{\mathbf{k}}\left\{\sum_{n\neq 0}\left[\delta_{\mu\nu} + l_{\mu\nu}^{\perp}\left(\frac{\omega_n}{\hbar ck}\right)^2\frac{1}{\gamma^{\perp}}\right]R_{\mathbf{k},n}^{\mu}R_{-\mathbf{k},-n}^{\nu} + \left[\delta_{\mu\nu} + \frac{\mathcal{H}^2}{k^2}(\pi\gamma^{-1})_{\mu\nu}\right]R_{\mathbf{k},0}^{\mu}R_{-\mathbf{k},-0}^{\nu}\right\}. \tag{5.7}$$

We point out the renormalization of photon energy spectra - we have $\hbar ck$ instead of $\hbar ck(\gamma^{\perp})^{1/2} = \tilde{\omega}$ in (4.1). Repeating the calculations from previous section we found $\Xi$. Let's write the result in the following form:

$$\ln\left[\Xi/(Z_1^0)^N\right] = \sum_{\mathbf{k}}\left\{-\beta\hbar\tilde{\omega} - 2\ln(1-\mathrm{e}^{-\beta\hbar\tilde{\omega}}) + \frac{\mathcal{H}^2}{2k^2}(\pi^{\parallel} + 2\pi^{\perp}) - \frac{1}{2}\ln\left(1 + \frac{\mathcal{H}^2}{k^2}\pi^{\parallel}/\gamma^{\parallel}\right) - \ln\left(1 + \frac{\mathcal{H}^2}{k^2}\pi^{\perp}/\gamma^{\perp}\right)\right\}.$$

Free energy is derived from this relation. Considering values $\pi$, $\gamma$ in (5.3), (5.6) we obtain

$$F = F_{0N} + F^{\parallel} + F^{\perp} + \hat{F}_f + F_V, \tag{5.8}$$

$$F_{0N} = -\frac{1}{\beta}\ln Z_1^0, \quad F^{\|} = -\frac{1}{2\beta}\sum_{\mathbf{k}}\left\{\frac{\mathcal{H}^2}{k^2} - \ln\left(1+\frac{\mathcal{H}^2}{k^2}\right)\right\},$$

$$F^{\perp} = \frac{1}{\beta}\sum_{\mathbf{k}}\left\{\frac{\mathcal{H}^2}{k^2\delta} - \ln\left(1+\frac{\mathcal{H}^2}{k^2\delta}\right)\right\},$$

$$\tilde{F}_f = \frac{2}{\beta}\sum_{\mathbf{k}}\ln\left(1-\exp\left[-\beta\hbar c\sqrt{k^2+\mathcal{H}^2/\delta}\right]\right),$$

$$F_V = \sum_{\mathbf{k}}\hbar c\sqrt{k^2+\mathcal{H}^2/\delta}.$$

Here $F_{0N}$ is free energy of relativistic non-interacting particles. $F^{\|}$ : $F^{\perp}$ set the corrections (adjustments) due to Coulomb and magnetic interaction and have the same form as in the weak relativistic theory. After calculation of the sums we find

$$F^{\|} + F^{\perp} = -\frac{1}{3}Ne^2\mathcal{H}(1-2/\delta^{3/2}). \tag{5.9}$$

In the post Newton approximation $\delta = \beta mc^2$ and (5.8) coincides with Debay correction for the weak relativistic electron gas [7]. $\tilde{F}_f$ is "field" part in the expression for free energy. It can be written in the following form:

$$\tilde{F}_f = -\frac{V}{3\pi^2\beta^4(\hbar c)^3}\int\limits_a^\infty dy\frac{(y^2-a^2)^{3/2}}{e^y-1}, \quad a^2 = (\beta\hbar\omega_0)^2\left[\frac{\beta mc^2}{\delta}\right],$$

here $\omega_0$ - frequency of plasma oscillations. If value of $a$ is small

$$\tilde{F}_f \simeq -\frac{V\pi^2}{45\beta^4(\hbar c)^3} + N\frac{\pi e^2}{3\beta\hbar c\delta} \equiv F_f + F'. \tag{5.10}$$

The first term here is the free energy of the field. The second one can be associated with the energy of charged particles radiation. Indeed, in non-relativistic approximation the energy of charged particle radiation is defined by the formula $\mathcal{E}' = \frac{2e^2}{3c^3}\ddot{r}^2\Delta t$. Under the heat motion $\ddot{r}^2\Delta t \sim V^2/\Delta t = 3/\beta m\Delta t$. Taking into account that in Euclidian theory $\Delta t \sim \beta\hbar$ we found $\mathcal{E}' = 2e^2/\hbar\beta^2mc^3$. This result qualitatively complies with (5.10). If $\delta = \beta mc^2$ $\frac{\partial}{\partial\beta}(\beta F'/N) = -\pi e^2/3\hbar\beta^2mc^3$. The last term in (5.8) is responsible for the energy of zero oscillations. In post-Newton approximation $F_V$ does not depend on the temperature. Therefore, $F_v$ can be associated with the energy of field in vacuum state.

## References


[1] Yukhnovskii I.R., Holovko M.F. Statistical theory of classical equilibrium systems. Kiev, Naukova Dumka, 1980, 372 p. (in Russian).

[2] Yukhnovskii I.R.,Gurskii Z.A. Quntum and statistical Theory of disordered systems. Kiev, Naukova Dumka, 1991, 288 p. (in Russian).

[3] Yukhnovskii I.R. Phase transitions of second order. Collective variables method. Kiev, Naukova Dumka, 1985, 224 p. (in Russian); Singapore, World Scientific, 1987, (English translation).
[4] Yukhnovskii I.R. Displacement and collective variables method. // Ukr. Fiz. Jorn., 1967, vol., 12, No 10, p. 1677-1694 (in Russian).
[5] The problems of modern statistical physics. Kiev, Naukova Dumka, 1985, 288 p, (in Russian).
[6] Yukhnovskii I.R., Blazhyjevskii L.F. Collective variables and functional representation for density matrix of weakly relativistic systems. / Preprint of Ukrainan Ac. Sci., Kiev, 1985, Inst. for. Theor. Phys., ITP-85-84R, 32 p. (in Russian).
[7] Blazhyjevskii L.F. About one application of path integrals in statistical thermodynamics. // Ukr. Jorn., 1979, vol. 24, No 11. p. 1737-1745 (in Russian).
[8] Blazhyjevskii L.F. Path integrals in configurational space in weakly nonequilibrium theory of many particles. // Teor. Mat. Fiz., 1986, vol. 66, No 3, p. 409-421 (in Russian).
[9] Popov V.N. Path integrals in quantum field theory and statistical physics. Moscow, Atomizzdat, 1976, 256 p. (in Russian).

# ПОЛЬОВІ АСПЕКТИ МЕТОДУ КОЛЕКТИВНИХ ЗМІННИХ ДЛЯ СИСТЕМ ЗАРЯДЖЕНИХ ЧАСТИНОК

Л.Ф.Блажиєвський

На прикладі релятивістської системи заряджених частинок у стані статистичної рівноваги аналізується зв'язок методу колективних змінних з польовою теорією. Показано, що після виключення польових змінних статистичну суму системи можна записати у вигляді інтеграла за траєкторіями конфігураційного простору частинок, класична дія для яких має неадитивний характер. Обчислена поправка до енергії теплового випромінювання.