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CALCULATION OF THE DENSITY MATRIX OF LIQUID ^{4}He

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A system of N atoms of liquid ^{4}He is under consideration. The calculation of the one-particle density matrix based on the pair-correlation function method is presented. Number of terms is factorized and summized with extracting the small terms. The results obtained have the form of the two terms sum where the first term have the structure similar to the density matrix of ideal bose-gas. The equations for calculating the temperature of bose-condensation are presented.

1. Introduction

In this paper we present the result of a calculation of the single-particle reduced density matrix of superfluid ^{4}He. Our method of calculation is to apply pair-correlation function method, developed in [1,2].

In section III and IV the different ways to obtain the equation for the temperature of phase transition in ^{4}He based on density matrix are outlined.

We try to obtain the results rested on the measured values only. It can be fulfilled by representing the results into the forms, where the liquid-structure function $S(\mathbf{k})$ is the single parameter. Then we can take $S(\mathbf{k})$ for example from the experimental studies of liquid helium.

2. Basic assumption

We consider a system of N atoms of ^{4}He, enclosed in the volume V at the temperature T. By definition the s-particle reduced density matrix of such system is:

$$F_s(\mathbf{r_1},\ldots,\mathbf{r_s}|\,\mathbf{r}'_1,\ldots,\mathbf{r}'_s)=\frac{V^s}{Z_N}\int \mathrm{d}\mathbf{r}_{s+1}\ldots\int \mathrm{d}\mathbf{r}_N R_N(\mathbf{r_1},\ldots,\mathbf{r_N}|\,\mathbf{r}'_1,\ldots,\mathbf{r}'_N),$$

where the general density matrix:

$$R_N(\mathbf{r_1},\ldots,\mathbf{r_N}|\,\mathbf{r}'_1,\ldots,\mathbf{r}'_N)=\sum_n \Psi_n^*(\mathbf{r}'_1,\ldots,\mathbf{r}'_N)\; e^{-\beta\hat{H}_N}\Psi_n(\mathbf{r_1},\ldots,\mathbf{r_N}); \quad (2.1)$$

$$Z_N=\int \mathrm{d}\mathbf{r}_1\ldots\int \mathrm{d}\mathbf{r}_N R_N(\mathbf{r_1},\ldots,\mathbf{r_N}|\,\mathbf{r_1},\ldots,\mathbf{r_N})$$

— partition function of the system described by the Hamiltonian operator $\hat{H}_N$. For the s-particle reduced density matrix s=1,2,3,..., and in the bulk limit $s/N \to 0$; with the condition $N \to \infty$, $V \to \infty$, $N/V = \rho \to$ const.

The momentum distribution of atoms is given by the single-particle reduced density matrix $F_1(\mathbf{r}|\mathbf{r}')$:

$$N_{\mathbf{q}} = \frac{N}{V} \int e^{-i\mathbf{q}\mathbf{R}} F_1(R) \mathrm{d}\mathbf{R}, \qquad \mathbf{R} = \mathbf{r}_1 - \mathbf{r}'_1,$$

where

$$F_1(\mathbf{r}|\mathbf{r}') = F_1(|\mathbf{r} - \mathbf{r}'|).$$

We start from the expression for the general density matrix obtained in [3]:

$$R_N(x|x') = P_N(x|x')\, R^0_N(x|x'),$$

$$R^0_N(x|x') = \frac{1}{N} \left[\frac{m*}{2\pi\beta\hbar^2}\right]^{\frac{3}{2}N} \sum_Q \exp\left\{-\frac{m*}{2\beta\hbar^2} \sum_{j=1}^{N} \left(\mathbf{r}_j - \mathbf{r}'_{Q\,j}\right)^2\right\}. \qquad (2.2)$$

where the sum goes over all permutations of the coordinates $\mathbf{r}_1, \ldots \mathbf{r}_N$.

$$P_N(x|x') = e^{-\beta E_0} \Psi_0(x') \Psi_0(x) V^N,$$

where

$$\Psi_0(x) \equiv \Psi_0(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \frac{1}{\sqrt{V^N}} \exp\left\{\frac{1}{2} \sum_{\mathbf{q}\neq 0} a_2(q) \rho_{\mathbf{q}} \rho_{-\mathbf{q}}\right\}$$

— the wave function of ground state of the system,

$$E_0 = \frac{N(N-1)}{2V}\, \nu_0 - \frac{1}{4} \sum_{\mathbf{q}\neq 0} (\alpha_q - 1)^2 \frac{\hbar^2 q^2}{2m},$$

$$\alpha_q = \sqrt{1 + \frac{2N}{V} \nu_q \Big/ \frac{\hbar^2 q^2}{2m}}, \qquad a_2(q) = -\frac{\alpha_q - 1}{2}.$$

The density matrix R^0_N corresponds to the formation of the bose-condensate fraction. It also should be noted, that the mass of particles was transformed ($m \to m^*$), according to the equation:

$$\frac{m}{m^*} = 1 - \frac{1}{3N} \sum_{\mathbf{q}\neq 0} \frac{(S_{\mathbf{q}} - 1)^2}{S_{\mathbf{q}} + 1},$$

it is clear that $m^* > m$, that means the decrease of bose-condensation temperature. This equation must be solved numerically. Calculation of this type have been performed in [4-6]. For the effective mass of liquid ^{4}He at the density $\rho = 0.02185 \mathring{A}^{-3}$ yet $m^*_0 = 1.7m$ [17,18].

The factor $P_N(x|x')$ in (1.2) takes into account the hard-core repulsion of atoms due to the decrease of the free volume per particle that results in the increase of the phase transition temperature.

3. One-particle redused density matrix

Single-particle density matrix can be obtained by integration of general density matrix:

$$\frac{1}{V}F_1(\mathbf{r}_1|\mathbf{r}'_1) = \frac{1}{Z_N}\int \mathrm{d}\mathbf{r}_2 \ldots \mathrm{d}\mathbf{r}_\mathbf{N} R^0_N(\mathbf{r}_1,\ldots,\mathbf{r}_N|\mathbf{r}'_1,\mathbf{r}_2\ldots,\mathbf{r}_N)\times$$
$$\exp\left\{-\beta E_0 + \frac{1}{2}\sum_{\mathbf{k}\neq 0} a_2(k)\left[\rho_\mathbf{k}\rho_{-\mathbf{k}} + \rho'_\mathbf{k}\rho'_{-\mathbf{k}}\right]\right\}. \tag{3.1}$$

Note that the structure of density matrix R^0_N, mentioned in (2.2) is similar to the structure of N range determinant, in which all terms will be taken with "+":

$$R^0_N(\mathbf{r}_1,\ldots,\mathbf{r}_N|\mathbf{r}'_1,\mathbf{r}_2\ldots,\mathbf{r}_N) = \frac{1}{N}\begin{vmatrix} K^0_{1\,1'} & K^0_{1\,2} & \ldots & \ldots & K^0_{1\,N} \\ K^0_{2\,1'} & 1 & \ldots & \ldots & K^0_{2\,N} \\ K^0_{3\,1'} & K^0_{3\,2} & 1 & \ldots & K^0_{3\,N} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ K^0_{N\,1'} & K^0_{N\,2} & \ldots & \ldots & 1 \end{vmatrix}_+ . \tag{3.2}$$

Here a matrix element is

$$K^0_{i\,j} = \left(\frac{m^*}{2\pi\beta\hbar^2}\right)^{\frac{3}{2}} e^{-\frac{m^*}{2\beta\hbar^2}\sum\limits_{j=1}^{N}(\mathbf{r}_i-\mathbf{r}_j)^2}. \tag{3.3}$$

Let us expand the determinant (3.2) and extract the terms with indexes $1'$ and 1 in each term. Next we shall construct the number of elements containing all permutations of first s elements. Extracting the first row of the determinant, next, the first column, we can separate the multiplier depending on $\mathbf{r}_1$ and $\mathbf{r}'_1$ in each term. Afterwards we shall continue this process with the minors of rank $N-2$, $N-3$, etc.

The set of terms received has the structure of product of first s elements of the determinant into the new determinant rank $N-s$. All new determinants are composed according to the following rules: the determinant rank $N-2$ does not contain first and j-th rows, and first (to index $1'$) and j-th columns, etc. Arranging the sum over the indexes of the elements with simultaneous addition of all permutations of elements of mute indexes we can symmetrize all terms of the sum, beginning from the third one.

The density matrix (3.2) can be rewriten in the form of number of terms containing the density matrices of the system of $N-s$ particles, where $s = 1, 2, \ldots, N-1$. Let us note these matrices are R^0_{N-s}. It should be also noted that all obtained density matrices do not depend on $\mathbf{r}_1$ and $\mathbf{r}'_1$.

Thus we can write:

$$R^0_N(\mathbf{r}_1,\ldots,\mathbf{r}_N|\mathbf{r}'_1,\mathbf{r}_2\ldots,\mathbf{r}'_N) = \frac{1}{N}K^0_{11'}R^0_{N-1} + \frac{1}{N(N-1)}\times$$
$$\sum_{2\leqslant j\leqslant N} K^0_{1j}K_{j1'}q^0R^0_{N-2} + \frac{1}{N(N-1)(N-2)}\sum_{2\leqslant j\neq i\leqslant N}\Big[K^0_{1j}K^0_{ij}K^0_{j1'} +$$
$$+ K^0_{1j}K^0_{ji}K^0_{i1'}\Big]R^0_{N-3} + \frac{1}{N(N-1)(N-2)(N-3)}\times$$

$$\sum_{2\leqslant i<j<l\leqslant N}\Big[K^0_{1j}K^0_{ij}K^0_{jl}K^0_{l1'}+K^0_{1j}K^0_{ji}K^0_{il}K^0_{l1'}+K^0_{1i}K^0_{il}K^0_{lj}K^0_{j1'}+$$

$$K^0_{1j}K^0_{jl}K^0_{li}K^0_{i1'}++K^0_{1l}K^0_{lj}K^0_{ji}K^0_{i1'}+K^0_{1l}K^0_{li}K^0_{ij}K^0_{j1'}\Big]\ R^0_{N-4}+... \quad (3.4)$$

It is helpful now to introduce the collective variables $\rho_{\mathbf{k}}$ and $\xi_{\mathbf{k}}$ according to the relations (see [1,2]):

$$\begin{aligned}\rho^{N-s}_{\mathbf{k}} &= \frac{1}{\sqrt{N-s}}\sum_{j=s+1}^{N}e^{-i\mathbf{k}\mathbf{r}_j};\\ \xi_{\mathbf{k}} &= \frac{1}{\sqrt{N}}\sum_{j=1}^{s}e^{-i\mathbf{k}\mathbf{r}_j},\end{aligned} \quad (3.5)$$

that is

$$\rho=\sqrt{\frac{N-s}{N}}\ \rho^{N-s}_{\mathbf{k}}+\xi_{\mathbf{k}}.$$

Substituting (3.5) into (3.1) we are able to reduce the density matrix to the sum of the terms including the density matrices R^0_{N-s}, according to (3.4). Thus we can separate the integration over $(N-s)$ variables (on which R^0_{N-s} depend) and next step is an integration over $\mathbf{r}_1,\ldots,\mathbf{r}_s$.

Introduction of the collective variables permit us to split the density matrix into parts, each one being dependent on the $\mathbf{r}_s+1,\ldots,\mathbf{r}_N$ or on the first s coordinates. Only after that we have a possibility to separate these parts completely.

Let us introduce the notation follows. We can define the function dependent only on the first s coordinates by:

$$g_s(1,2,\ldots,s|1',2,\ldots,s)=\exp\left\{\frac{1}{2}\sum_{\mathbf{k}\neq 0}a_2(k)\ (\xi_{\mathbf{k}}\xi_{-\mathbf{k}}+\xi'_{\mathbf{k}}\xi'_{-\mathbf{k}})\right\}\times$$

$$\times\left\langle\exp\left\{\sum_{\mathbf{k}\neq 0}a_2(k)\sqrt{\frac{N-s}{N}}\rho^{N-s}_{\mathbf{k}}\ (\xi_{-\mathbf{k}}+\xi'_{-\mathbf{k}})\right\}\right\rangle_{N-s}$$

where $\langle\ldots\rangle_{N-s}$ denote:

$$\langle\ldots\rangle_{N-s}=\frac{1}{Q_{N-s}}\int \mathrm{d}\,r_{s+1}\ldots\int \mathrm{d}\,r_N\,(\ldots)\,R^0_{N-s}(s+1,\ldots,N)\times$$

$$\times\exp\left\{\sum_k a_2(k)\frac{N-s}{N}\rho^{N-s}_k\rho^{N-s}_{-k}\right\},$$

Q_{N-s} is the part of density matrix depending on the rest $(N-s)$ coordinates, which are introduced by the relations

$$\begin{aligned}Q_{N-s} &= \int \mathrm{d}\mathbf{r}_{s+1}\ldots\int \mathrm{d}\mathbf{r}_N R^0_{N-s}(s+1,\ldots,N)\times\\ &\exp\left\{2a_0+\sum_{\mathbf{k}\neq 0}a_2(k)\frac{N-s}{N}\rho^{N-s}_{\mathbf{k}}\rho^{N-s}_{-\mathbf{k}}\right\}.\end{aligned} \quad (3.6)$$

It should be also noted that Q_{N-s} is quite similar to the partition function of the system of $(N-s)$ particles, but differs in by some constants. It follows immediately from (3.6) . Q_{N-s} goes to the partition function only in the bulk limit, thus we will keep our definition.

After all transformations we have to rewrite the one-particle density matrix in the form:

$$F_1(\mathbf{r}_1|\mathbf{r}_1') = \frac{V}{N}\frac{Q_{N-1}}{Q_N}K^0_{11'}g_1(1|1') + \frac{V}{N}\frac{Q_{N-2}}{Q_N}\int \mathrm{d}\mathbf{r}_2 K^0_{12}K^0_{21'} \times$$
$$g_2(1,2|1',2) + \frac{V}{N}\frac{Q_{N-3}}{Q_N}\int \mathrm{d}\mathbf{r}_2 \int \mathrm{d}\mathbf{r}_3 K^0_{12}K^0_{23}K^0_{31'}g_3(1,2,3|1',2,3) +$$
$$\frac{V}{N}\frac{Q_{N-4}}{Q_N}\int \mathrm{d}\mathbf{r}_2 \int \mathrm{d}\mathbf{r}_3 \int \mathrm{d}\mathbf{r}_4 K^0_{12}K^0_{23}K^0_{34}K^0_{41'} \times$$
$$g_4(1,2,3,4|1',2,3,4) + \dots \ . \tag{3.7}$$

Next let us evaluate g_s and Q_{N-s}. For this goal we are to find g_s first. We suppress the dependence on coordinates and use the abbreviated notations:

$$g_s = g_s(1,2,\dots,s|1',2,\dots,s).$$

g_s may be considered as exponent of irreducible "averages". Due to the translational invariancy of ρ_k^{N-s} the first irreduceble average is equal to zero. First term tends to zero because $\langle \rho_k^{N-s} \rangle = 0$.

It follows also that $\langle \rho_{k_1}^{N-s}\rho_{k_2}^{N-s}\rangle_{N-s} \neq 0$ only for $k_1 = -k_2$.

Finally one can write g_s by:

$$g_s = \exp\left\{\frac{1}{2}\sum_{\mathbf{k}\neq 0}\left[a_2(k)\left(\xi_{\mathbf{k}}\xi_{-\mathbf{k}} + \xi'_{\mathbf{k}}\xi'_{-\mathbf{k}}\right) + a_2^2(k)\frac{N-s}{N}S^{N-s}(\mathbf{k})\left|\xi_{\mathbf{k}} + \xi'_{\mathbf{k}}\right|^2\right]\right\}, \tag{3.8}$$

where we consider only two first terms. The liquid-structure function $S^{N-s}(\mathbf{k})$ for $N-s$ particles is

$$S^{N-s}(\mathbf{k}) = \langle \rho_{\mathbf{k}}^{N-s}\rho_{-\mathbf{k}}^{N-s}\rangle_{N-s}. \tag{3.9}$$

In the same approximation for Q_{N-s} we have:

$$Q_{N-s} = Z^0_{N-s}\exp\left\{-\frac{1}{2}\sum_{\mathbf{k}\neq 0}\ln\left[1 - 2a_2(k)\frac{N-s}{N}S_0^{N-s}(\mathbf{k})\right]\right\} \tag{3.10}$$
$$Z^0_{N-s} = \int \mathrm{d}\mathbf{r}_{s+1}\dots\int \mathrm{d}\mathbf{r}_N\, R^0_{N-s}(s+1,\dots,N),$$

where $S_0^{N-s}(\mathbf{k})$—the liquid-structure function of ideal Bose-gas.

As was mentioned above Q_{N-s} is not exactly the partition function of the system. It has an additional exponent as it follows immediately from the right-hand side of the equation (3.10).

Next it is possible to expand Q_{N-s}/Q_N in powers of the s/N

$$\lim_{s/N\to 0} Q_{N-s}/Q_N = \left(z^0 \exp\left\{-\frac{1}{2N}\sum_{\mathbf{k}\neq 0}\frac{2a_2(k)\rho\frac{dS_0(\mathbf{k})}{d\rho}}{1-2a_2(k)S_0(\mathbf{k})}\right\}\right)^S \times \exp\left\{-\frac{1}{2N}S\sum_{\mathbf{k}\neq 0}\frac{2a_2(k)S_0(\mathbf{k})}{1-2a_2(k)S_0(k)}\right\}, \quad (3.11)$$

where $z^0 = Z^0_{N-1}/Z^0_N$ and it can be defined by $z^0 = e^{\beta\mu_0}$, μ_0—the chemical potential.

Let us turn back to (3.9). The liquid-structure function $S^{N-s}(\mathbf{k})$ may be obtained in the form of derivative of Q_{N-s} so that the final result is:

$$S^{N-s}(\mathbf{k}) = \frac{S_0^{N-s}(\mathbf{k})}{1-2a_2(k)\frac{N-s}{N}S_0^{N-s}(\mathbf{k})}.$$

We are still faced with the task of collecting $F_1(\mathbf{r}|\mathbf{r}')$. For this goal we have to decompose g_s.

For the decomposition of the g_s we shall factorize it using the collective variables $\xi_{\mathbf{k}}$ for each s. So equation(3.6) can be modified to read

$$g_s = F_s^0 \exp\left\{-\frac{1}{2N}s\sum_{\mathbf{k}\neq 0}\frac{2a_2(k)(S_0(\mathbf{k})-1)}{1-2a_2(k)S_0(\mathbf{k})}\right\}, \quad (3.12)$$

where

$$\begin{aligned} F_s^0 &= F_1^0(1|1')\prod_{j=2}^{s}\sqrt{F_2^0(1,j)F_2^0(1',j)}\prod_{2\leqslant i<j\leqslant s}F_2^0(i,j), \\ F_1^0(1|1') &= \exp\left\{\frac{1}{N}\sum_{\mathbf{k}\neq 0}\frac{a_2^2(k)S_0(\mathbf{k})}{1-2a_2(k)S_0(\mathbf{k})}\left(e^{i\mathbf{k}(\mathbf{r}_1-\mathbf{r}'_1)}-1\right)\right\}, \\ F_2^0(i,j\,|\,i,j) &\equiv F_2^0(i,j) = e^{-\tilde{\Phi}(|r_i-r_j|)}. \end{aligned}$$

Exponent in (3.12) may be added to z^0 in each term of density matrix. Let us label this part z.

Thus, adopting the mentioned factorization, we arrive at the expression:

$$\begin{aligned} F_1(\mathbf{r}_1|\mathbf{r}'_1) = \frac{V}{N}F_1^0(1|1')\Big\{&zK^0_{11'} + z^2\int \mathrm{d}\mathbf{r}_2 K^0_{12}\sqrt{F_2^0(1,2)}K^0_{21'}\sqrt{F_2^0(1',2)} + \\ &z^3\int \mathrm{d}\mathbf{r}_2\int \mathrm{d}\mathbf{r}_3 K^0_{12}\sqrt{F_2^0(1,2)}K^0_{23}F_2^0(2,3)K^0_{31'}\sqrt{F_2^0(1',3)}\times \\ &\sqrt{F_2^0(1,3)F_2^0(1',2)} + z^4\int \mathrm{d}\mathbf{r}_2\int \mathrm{d}\mathbf{r}_3\int \mathrm{d}\mathbf{r}_4 K^0_{12}\sqrt{F_2^0(1,2)}\times \\ &K^0_{23}F_2^0(2,3)K^0_{34}F_2^0(3,4)K^0_{41'}\sqrt{F_2^0(1',4)}\sqrt{F_2^0(1',4)F_2^0(1,3)}\times \\ &\sqrt{F_2^0(1,4)F_2^0(1',2)F_2^0(1',3)}\,F_2^0(2,4) + \ldots\Big\}. \end{aligned} \quad (3.13)$$

4. Summation of the density matrix

Let us transform (3.13) in the form which allows to collect $F_1(\mathbf{r}_1|\mathbf{r}'_1)$. In this case, we shall separate the leading terms in the one-particle density matrix. Let us introduce

$$\begin{aligned} h_3 &= \sqrt{F_2^0(1,3)F_2^0(1',2)} - 1, \\ h_4 &= \sqrt{F_2^0(1,3)F_2^0(1,4)F_2^0(1',2)F_2^0(1',3)}F_2^0(2,4) - 1, \\ &\vdots \end{aligned}$$

Thus

$$\begin{aligned} F_1(r_1|r'_1) &= F_1^0(1|1')\frac{1}{N}\sum_{\mathbf{q}} e^{i\mathbf{qR}}\left\{\frac{z^2(K_q^*)^2}{1-zK_q} + zK_q^0\right\} + \\ &\quad F_1^0(1|1')\frac{V}{N}\sum_{n\geqslant 3} z^n D_n, \end{aligned} \tag{4.1}$$

$$D_n = \int \mathrm{d}\mathbf{r}_2 \ldots \int \mathrm{d}\mathbf{r}_N K_{12}^0 \sqrt{F_2^0(1,2)} K_{1'n}^0 \sqrt{F_2^0(1',n)} \prod_{2\leqslant i<j\leqslant n} \left[K_{ij}^0 F_2^0(i,j) h_n\right].$$

$$\begin{aligned} K_q^* &= \int K^*(\mathbf{R}) e^{-i\mathbf{qR}} \mathrm{d}\mathbf{R}, \qquad K^*(\mathbf{R}) = K^0(\mathbf{R})\sqrt{F_2^0(R)}; \\ K_q &= \int K(\mathbf{R}) e^{-i\mathbf{qR}} \mathrm{d}\mathbf{R}, \qquad K(\mathbf{R}) = K^0(\mathbf{R}) F_2^0(R); \\ K_q^0 &= \int K^0(\mathbf{R}) e^{-i\mathbf{qR}} \mathrm{d}\mathbf{R}, \\ \mathbf{R} &= \mathbf{r} - \mathbf{r}'; \end{aligned}$$

Or, under the assumption $D_n = 0$:

$$F_1(r_1|r'_1) = F_1^0(1|1')\frac{1}{N}\sum_{q} e^{i\mathbf{qR}}\left\{\frac{z^2(K_q^*)^2}{1-ZK_q} + ZK_q^0\right\}. \tag{4.2}$$

Note, that equation (4.2) for ideal Bose-gas passes into

$$F_1^{\mu}(r_1|r'_1) = \frac{1}{N}\sum_{\mathbf{q}} e^{i\mathbf{qR}}\frac{zK_q^0}{1-zK_q^0}.$$

From the normalization condition the equation (4.2) we obtain for z_n

$$1 = \frac{1}{N}\sum_{\mathbf{q}}\left\{\frac{z^2(K_q^*)^2}{1-zK_q} + zK_q^0\right\},$$

that yields $1 - z_c K_0 = 0$ in the λ-point. Therefore for z_c we have $z_c = 1/K_0$, where $K_0 = K_q(q=0)$.

The equation for the critical temperature after substituting z_c yet:

$$1 = \frac{1}{N}\sum_{\mathbf{q}}\left\{\frac{(K_q^*/K_0)^2}{1-K_q/K_0}+\frac{K_q^0}{K_0}\right\}. \tag{4.3}$$

Now we turn to our assumption for D_n. Let us consider a few first terms

$$\begin{aligned}
D_3 &= \int \mathrm{d}\mathbf{r}_2 \int \mathrm{d}\mathbf{r}_3 K_{12}^0 \sqrt{F_2^0(1,2)} K_{23}^0 F_2^0(2,3) K_{31'}^0 \sqrt{F_2^0(1',3)} h_3 \\
D_4 &= \int \mathrm{d}\mathbf{r}_2 \int \mathrm{d}\mathbf{r}_3 \int \mathrm{d}\mathbf{r}_3 K_{12}^0 \sqrt{F_2^0(1,2)} K_{23}^0 F_2^0(2,3) K_{34}^0 F_2^0(3,4) K_{41'}^0 \times \\
&\quad \sqrt{F_2^0(1',4)} h_4 \\
&\quad \vdots
\end{aligned}$$

Unfortunately, these terms are difficult to be estimated. The terms involving h_i are expected to be small in interesting temperature range. We remain the numerical calculations of this problem for the future.

5. Different form of equation for density matrix

An alternative form of equation (4.1) can be obtained by rewriting equation (3.13) in the different form. We shall write the coefficients

$$\begin{aligned}
h_2^* &= \sqrt{F_2^0(1,2)F_2^0(1',2)} - F_2^0(1,2)F_2^0(1',2); \\
h_3^* &= \sqrt{F_2^0(1,2)F_2^0(1,3)F_2^0(1',2)F_2^0(1',3)}F_2^0(2,3) - \\
&\quad F_2^0(1,2)F_2^0(2,3)F_2^0(1',2); \\
h_4^* &= \sqrt{F_2^0(1,2)F_2^0(1,3)F_2^0(1,4)F_2^0(1',2)F_2^0(1',3)F_2^0(1',4)} \times \\
&\quad F_2^0(2,3)F_2^0(3,4)F_2^0(2,4) - F_2^0(1,2)F_2^0(2,3)F_2^0(3,4)F_2^0(1',4) \\
&\quad \vdots
\end{aligned}$$

In this case for F_1 we obtain:

$$\begin{aligned}
F_1(r_1|r_1') &= F_1^0(1|1')\frac{1}{N}\sum_{\mathbf{q}} e^{i\mathbf{q}\mathbf{R}}\left\{\frac{z^2(K_q)^2}{1-zK_q}+zK_q^0\right\} + \\
&\quad F_1^0(1|1')\frac{V}{N}\sum_{n\geqslant 3} z^n D_n^*,
\end{aligned} \tag{5.1}$$

where

$$D_2^* = \int \mathrm{d}\mathbf{r}_2 K_{12}^0 K_{1'2}^0 h_2^*, \quad D_3^* = \int \mathrm{d}\mathbf{r}_2 \int \mathrm{d}\mathbf{r}_3 K_{12}^0 K_{1'2}^0 h_2^*,$$

and

$$D_n^* = \int \mathrm{d}\mathbf{r}_2 \ldots \int \mathrm{d}\mathbf{r}_N K_{12}^0 K_{1'n}^0 \prod_{2\leqslant i<j\leqslant n} K_{ij}^0 h_n^*.$$

K_q and K_q^0 are defined by relations in (4.1).

The equation for the λ-point temperature yet

$$1 = \frac{1}{N} \sum_{\mathbf{q}} \left\{ \frac{(K_q/K_0)^2}{1 - K_q/K_0} + \frac{K_q^0}{K_0} \right\}. \tag{5.2}$$

where D_n^* are neglected.

Note, that different ways to collect the equation for density matrix exist, as we have shown above. Thus we obtain some similar equations for finding T_λ. It is clear that it requires more detailed numerical calculations of D_n and D_n^* which were neglected without sufficient proves. A final remark must be made concerning the use of formal parameters. We need only the liquid-structure factor of system to solve equation (4.3) and equation (5.2). So it is possible to make use of data from the experimental works of a number of papers [7-11] or from the theoretical calculations.

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РОЗРАХУНОК МАТРИЦІ ГУСТИНИ РІДКОГО ^{4}He

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В статті розглянута система N атомів рідкого ^4He. З повної матриці густини системи, при допомозі методу парних кореляційних функцій, отримана одночастинкова матриця густини у вигляді суми елементів, які підлягають факторизації. Після виділення малих членів та згортання ряду був отриманий вираз для одночастинкової матриці густини у вигляді суми двох доданків, перший з яких по структурі подібний до матриці густини ідеального бозе-ґазу. Наведені формули для розрахунку температури бозе-конденсації.