CONDENSED MATTER PHYSICS

$SU(N) \to SU(2)$ symmetry breaking in quantum antiferromagnets

A.K. Kolezhuk[1,2], T.L. Zavertanyi[1]

[1] Institute of High Technologies, Taras Shevchenko National University of Kyiv, Ministry of Education and Science, 03022 Kyiv, Ukraine

[2] Institute of Magnetism, National Academy of Sciences of Ukraine and Ministry of Education and Science of Ukraine, 36–b Vernadskii Av., 03142 Kyiv, Ukraine

We study a $SU(2)$-symmetric spin-3/2 system on a bipartite lattice close to the antiferromagnetic $SU(4)$-symmetric point, which can be described by the CP^3 model with a perturbation breaking the symmetry from $SU(4)$ down to $SU(2)$ and favoring the Néel ordering. We show that the effective theory of the perturbed model is not the usual $O(3)$ nonlinear sigma model (NLSM), but rather the $O(3) \times O(2)$ NLSM. We show that in the presence of perturbation, the topological charge q of the CP^3 field is connected to the $O(3)$-NLSM type topological charge of the spin texture Q (defined in a usual way via the unit Néel vector) by the relation $q = 3Q$, thus under the influence of the perturbation unit-charge skyrmions of CP^3 model bind into triplets. We also show that in the general spin-S case, symmetry breaking from $SU(2S + 1)$ to $SU(2)$ results in the general relation $2SQ_{O(3)} = q_{CP^{2S}}$ between CP^{2S} and $O(3)$ charges, so one can expect $2S$-multiplet binding of skyrmions.

Key words: frustrated magnets, skyrmions, cold gases in optical lattices

1. Introduction and model

Numerous studies in the past two decades have firmly positioned ultracold gases as an extremely versatile "toolbox" capable of simulating a wide range of problems originating in condensed matter physics and field theory. Particularly, multicomponent ultracold gases in optical lattices[1–3] allow one to model spin systems with strong non-Heisenberg exchange interactions[4–9], normally inaccessible in solid state magnets. Presence of controllable strong higher-order (biquadratic, biqubic, etc.) exchange interactions allows one to explore models with enhanced $SU(N)$ symmetry with $N > 2$, which have been a subject of extensive theoretical studies [10–16]. Realization of $SU(N)$ antiferromagnets with N up to 10 was suggested[17, 18] and realized in experiments [19]. Spin systems with strong higher-order exchange interactions may exhibit phases with unconventional (multipole) order and can be considered as a special type of frustrated magnets.

It has been shown [20, 21] that in spin-1 systems close to the antiferromagnetic $SU(3)$ point, a perturbation that breaks this symmetry down to $SU(2)$ can lead to an interesting effect: unit-charge topological excitations (skyrmions, hedgehogs) of the effective CP^2 theory describing the $SU(3)$-symmetric model bind into doublets that correspond to unit-charge topological excitations of the effective $O(3)$ nonlinear sigma model (NLSM) theory describing the $SU(2)$-symmetric model.

In the present work, we study spin-3/2 systems close to the antiferromagnetic $SU(4)$ point, and show that a similar effect of binding topological excitations into triplets exists when the symmetry gets broken down to $SU(2)$. We further show that this result can be generalized: for a system with underlying spin S close to the antiferromagnetic $SU(2S + 1)$ point, a perturbation that brings the symmetry down to $SU(2)$ can lead to the formation of $2S$-multiplets of topological excitations.

We start with a system of spin-3/2 fermions on a bipartite optical lattice in d spatial dimensions ($d = 1, 2$) that can be described by the following Hamiltonian in the s-wave scattering approximation [6]:

$$\widehat{H} = -t \sum_{\sigma=\pm 1/2, \pm 3/2} \sum_{\langle ij \rangle} \left(c_{\sigma,i}^{\dagger} c_{\sigma,j} + \text{h.c.} \right) + \sum_{i} \sum_{F=0,2} U_F \sum_{m=-F}^{F} P_{Fm,i}^{\dagger} P_{Fm,i}, \tag{1.1}$$

where $c_{\sigma,i}$ are the spin-3/2 fermionic operators at the lattice site i, t is the effective hopping amplitude between two neighboring sites (which is for simplicity assumed to be the same for all spatial directions),

$$P_{Fm,i} = \sum_{\sigma\sigma'} \left\langle Fm \left| \frac{3}{2}\sigma, \frac{3}{2}\sigma' \right\rangle c_{\sigma,i} c_{\sigma',i} \right.$$

are the operators describing an on-site pair with the total spin F, and interaction strengths U_0, U_2 are proportional to the scattering lengths in the $F = 0$ and $F = 2$ channels, respectively.

At quarter filling (one particle per site), and in the limit of strong on-site repulsion $U_0, U_2 \gg t$, the charge degrees of freedom are strongly gapped, and at low excitation energies the system can be described by the effective Hamiltonian involving only spin degrees of freedom. The antiferromagnetic $SU(4)$-symmetric point corresponds[22] to the limit $U_2 \to \infty$ and its Hamiltonian can be written in terms of the on-site spin-3/2 operators $\widehat{S}_i$ as follows [23]:

$$\widehat{\mathcal{H}}_{SU(4)} = J \sum_{\langle ij \rangle} \left\{ -\frac{31}{24}(\widehat{S}_i \widehat{S}_j) + \frac{10}{36}(\widehat{S}_i \widehat{S}_j)^2 + \frac{2}{9}(\widehat{S}_i \widehat{S}_j)^3 \right\}, \tag{1.2}$$

with $J = t^2/U_0$. For general values of U_0, U_2, the effective spin Hamiltonian still exhibits an enhanced $Sp(4)$ symmetry [6], which is special for spin 3/2. The effects of symmetry reduction from $SU(4)$ to $Sp(4)$ were studied in [22], the corresponding perturbation has been shown to be dangerously irrelevant and is not of interest to us in the present work. Thus, we consider perturbing the $SU(4)$-invariant model (1.2)

$$\widehat{\mathcal{H}}_{SU(4)} \mapsto \widehat{\mathcal{H}}_{SU(4)} + \lambda \sum_{\langle ij \rangle} (\widehat{S}_i \widehat{S}_j), \qquad \lambda > 0 \tag{1.3}$$

by the term that breaks the symmetry down to $SU(2)$ and favors the antiferromagnetic spin ordering. According to the mean-field study [23], the other sign of λ would favor an exotic phase characterized by the presence of octupolar and quadrupolar orders and will not be considered here. Such a perturbation is not possible if only the s-wave scattering is taken into account, but will naturally arise due to the contribution from the p-wave scattering. Normally, the p-wave scattering is neglected because the corresponding contributions to the interaction are about a few percent compared to the s-wave ones [24], but in the present case this is sufficient to break the enhanced symmetry. Beside that, the effective strength of the p-wave scattering can be controlled by quasi-low-dimensional confinement [25].

2. $SU(2)$-perturbed CP^3 model: effective theory

The effective low-energy continuum theory for the $SU(4)$ antiferromagnet (1.2) is the well-known CP^3 model described by the following euclidean action [12, 13]

$$\mathcal{A}_{CP^3} = \frac{\Lambda^{d-1}}{2g_0} \int d^{d+1}x |\mathcal{D}_\mu z|^2 + \mathcal{A}_{\text{top}}. \tag{2.1}$$

Here, the Planck constant and the lattice spacing are set to unity, Λ is the ultraviolet momentum cutoff, the 4-component complex vector field z is subjected to the unit length constraint $z^{\dagger} z = 1$, $\mathcal{D}_\mu = \partial_\mu - iA_\mu$ is the gauge covariant derivative, and $A_\mu = -i(z^{\dagger}\partial_\mu z)$ is the gauge field, $x^0 = c\tau$, $\tau = it$ is the imaginary time. Assuming for definiteness that the lattice is hypercubic, one obtains the limiting velocity $c = 2J\sqrt{d}$, and the bare coupling constant $g_0 = \sqrt{d}$. The topological term in the action

$$\mathcal{A}_{\text{top}}[z] = \int d\tau \sum_{j} \eta_j z_j^{\dagger} \partial_\tau z_j, \tag{2.2}$$

where the phase factors $\eta_j = \pm 1$ take opposite signs at lattice sites belonging to A and B sublattices, can be cast in the continuum form only for $d = 1$ [12, 13, 26]. Without the topological term, the action (2.1) can be viewed as the energy of the static $(d + 1)$ dimensional "classical" spin texture.

The CP^{N-1} model [27–31] has been extensively studied as an effective theory for $SU(N)$ antiferromagnets [12, 13]. In $d = 1$, there is no long-range spin order and for $N > 2$ excitations are gapped at any value of the coupling g_0 even in the presence of the topological term (2.2), while in $d = 2$ the disordered phase appears if the coupling g_0 exceeds some N-dependent critical value g_c [29, 30]. Numerical work [15, 16, 32] suggests that for $d = 2$ the value $g_c N/g_0$ lies somewhere between 4 and 5.

The topological term becomes important in the disordered phase: it drives a spontaneous breaking of the translational invariance of the underlying lattice, leading to the twofold degenerate (dimerized) ground state in $d = 1$, and in $d = 2$ the ground state degeneracy and the pattern of the resulting "valence bond solid" is lattice-dependent [12, 13, 26].

The leading contribution to the continuum action from the perturbation (1.3) is given by the gradient-free term, so the perturbed action takes the form

$$\mathcal{A}_{AF} = \frac{\Lambda^{d-1}}{2g_0} \int d^{d+1}x \left\{ |\partial_\mu z|^2 - |z^\dagger \partial_\mu z|^2 - m_0^2 \langle S \rangle^2 \right\} + \mathcal{A}_{top}, \qquad (2.3)$$

where $m_0^2 = 2g_0\lambda/(c\Lambda^{d-1}) > 0$ is proportional to the perturbation strength, $\langle S \rangle = z^\dagger S^a z$ is the spin average, and S^a are spin-3/2 matrices, $a = 1, 2, 3$. Here, we assume that four components z_m of the complex vector field z are directly related to the amplitudes of the four spin-3/2 basis states $|\frac{3}{2}m\rangle$, $m = -\frac{3}{2} \dots \frac{3}{2}$

To analyze the behavior of the perturbed theory, it is convenient to separate the modes becoming massive under the perturbation. We parameterize the 4-component field z in the following way:

$$z_m = D_{mm'}^{(3/2)}(\alpha, \theta, \varphi)\psi_{m'}(\beta, \vartheta, \phi). \qquad (2.4)$$

Here, ψ is the spin-3/2 state [23] taken in the principal axes of the spin-quadrupolar tensor

$$\psi_{3/2} = \cos(\beta)\cos\frac{\vartheta}{2}, \quad \psi_{1/2} = \sin\beta\sin\frac{\vartheta}{2}e^{i\phi}, \quad \psi_{-1/2} = \sin\beta\cos\frac{\vartheta}{2}, \quad \psi_{-3/2} = \cos\beta\sin\frac{\vartheta}{2}e^{i\phi}. \quad (2.5)$$

Such a choice ensures that the tensor $\psi^\dagger(S^a S^b + S^b S^a)\psi$ is diagonal. The spin average in the state (2.5) is given by

$$\langle S^z \rangle = \frac{1}{2}\cos\vartheta(4\cos^2\beta - 1), \quad \langle S^+ \rangle = \sin\vartheta\sin\beta(\sqrt{3}\cos\beta e^{i\phi} + \sin\beta e^{-i\phi}). \qquad (2.6)$$

The Wigner matrix $D^{(j)}$ is the standard $(2j+1)$-dimensional representation of a rotation [33] and depends on three Euler angles $(\alpha, \theta, \varphi)$:

$$d_{m'm}^{(j)}(\theta) = \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{1/2}\left(\cos\frac{\theta}{2}\right)^{m'+m}\left(\sin\frac{\theta}{2}\right)^{m'-m}P_{j-m'}^{(m'-m,m'+m)}(\cos\theta),$$

$$D_{m'm}^{(j)}(\alpha, \theta, \varphi) = e^{im'\varphi}d_{m'm}^{(j)}(\theta)e^{im\alpha}, \qquad (2.7)$$

$P_n^{(a,b)}(\cos\theta)$ being the Jacobi polynomials. Rotating the state (2.5), we obtain the general normalized spin-3/2 state characterized by six real parameters (we omit the overall phase that depends on the gauge).

Antiferromagnetic perturbation (1.3) favors field configurations with small ϑ, β, so it is convenient to introduce three-component real field $h = (h_x, h_y, h_z)$:

$$h_x + ih_y = \sin\frac{\vartheta}{2}e^{i\phi}, \qquad h_z = \sin\beta. \qquad (2.8)$$

The perturbation in (2.3) amounts to making h massive, as $\langle S \rangle^2 = \frac{9}{4} - 9(h_x^2 + h_y^2) - 6h_z^2 + O(h^2)$.

Substituting the ansatz (2.4), (2.8) into the action (2.3) and retaining up to quadratic terms in powers of $h_{x,y,z}$, one obtains the action in the following form:

$$\mathcal{A}_{\text{AF}} = \mathcal{A}_{\text{NLSM}}[\theta, \varphi] + \mathcal{A}_{\text{m}}[\boldsymbol{h}] + \mathcal{A}_{\text{int}}[\boldsymbol{h}, \theta, \varphi, \alpha] + \mathcal{A}_{\text{top}}, \qquad (2.9)$$

where $\mathcal{A}_{\text{NLSM}}$ is the action of the $O(3)$ nonlinear sigma-model,

$$\mathcal{A}_{\text{NLSM}} = \frac{3\Lambda^{d-1}}{8g_0} \int \mathrm{d}^{d+1}x \left\{ (\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \varphi)^2 \right\}, \qquad (2.10)$$

$\mathcal{A}_{\text{m}}$ is the quadratic action of the massive field,

$$\mathcal{A}_{\text{m}} = \frac{\Lambda^{d-1}}{2g_0} \int \mathrm{d}^{d+1}x \left\{ (\partial_\mu \boldsymbol{h})^2 + 9m_0^2(h_x^2 + h_y^2) + 6m_0^2 h_z^2 \right\}, \qquad (2.11)$$

and $\mathcal{A}_{\text{int}}$ describes the interaction,

$$\mathcal{A}_{\text{int}} = \frac{\Lambda^{d-1}}{2g_0} \int \mathrm{d}^{d+1}x \left\{ h_z^2 \left[(\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \varphi)^2 \right] + (4h_z^2 + 9h_x^2 + 9h_y^2)(\partial_\mu \alpha + \cos \theta \partial_\mu \varphi)^2 + \ldots \right\}. \quad (2.12)$$

We integrate out the "fast" field $\boldsymbol{h}$ and obtain the effective action that depends only on the "slow" filelds θ, φ, α:

$$\mathcal{A}_{\text{eff}} = \frac{\Lambda^{d-1}}{2\Gamma} \int \mathrm{d}^{d+1}x \left\{ (\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \varphi)^2 \right\} + \frac{\Lambda^{d-1}}{2G} \int \mathrm{d}^{d+1}x (\partial_\mu \alpha + \cos \theta \partial_\mu \varphi)^2 + \mathcal{A}_{\text{itop}}, \qquad (2.13)$$

where the renormalized couplings Γ, G are determined by the equations

$$\frac{\Lambda^{d-1}}{\Gamma} = \frac{\Lambda^{d-1}}{\Gamma_0} + \frac{1}{(2\pi)^{d+1}} \int\limits_{k<\Lambda} \frac{\mathrm{d}^{d+1}k}{k^2 + 6m_0^2}, \qquad \Gamma_0 = \frac{4g_0}{3},$$

$$\frac{\Lambda^{d-1}}{G} = \frac{1}{(2\pi)^{d+1}} \int\limits_{k<\Lambda} \mathrm{d}^{d+1}k \left\{ \frac{4}{k^2 + 6m_0^2} + \frac{18}{k^2 + 9m_0^2} \right\}. \qquad (2.14)$$

Beside terms of higher than quadratic order in $\boldsymbol{h}$, in the interaction (2.12) we have omitted several types of terms that will not contribute to the renormalized action at the one-loop level. Namely, terms proportional to $h_z h_{x,y}(\partial_\mu \Phi)^2$, $h_z(\partial_\mu \Phi)^2$, where Φ denotes any of the slow fields, would generate terms of the fourth and higher order in gradients of Φ. The other omitted term, of the structure $(h_x \partial_\mu h_y - h_y \partial_\mu h_x)(\partial_\mu \Phi)$, yield contributions that vanish after integration over the wave vector.

For small AF perturbations, $m_0/\Lambda \ll 1$, from (2.14) one has

$$\Gamma \simeq \frac{2\pi}{\ln \frac{\Lambda}{m_0}}, \qquad G \simeq \frac{\pi}{11 \ln \frac{\Lambda}{m_0}} \quad \text{for } d = 1,$$

$$\Gamma \simeq \frac{\Gamma_0}{1 + \Gamma_0/2\pi^2} + O(m_0/\Lambda), \qquad G \simeq \frac{11}{\pi^2} + O(m_0/\Lambda) \quad \text{for } d = 2. \qquad (2.15)$$

The action (2.13) describes the $O(3) \times O(2)$ NLSM that is encountered as an effective theory of frustrated antiferromagnets [34, 35]. It can be recast in the form

$$\mathcal{A}_{\text{eff}} = \frac{\Lambda^{d-1}}{2\Gamma} \int \mathrm{d}^{d+1}x \mathrm{Tr} \left(\partial_\mu R^T P \partial_\mu R \right), \qquad P = \mathrm{diag}(1, 1, \zeta), \qquad \zeta = \Gamma/G, \qquad (2.16)$$

where the matrix field $R \in SO(3)$ is the rotation matrix, and filelds θ, φ, α are connected to R via the standard relations

$$\Omega_\mu = \begin{pmatrix} 0 & -\omega_{\mu 3} & \omega_{\mu 2} \\ \omega_{\mu 3} & 0 & -\omega_{\mu 1} \\ -\omega_{\mu 2} & \omega_{\mu 1} & 0 \end{pmatrix} = \partial_\mu R R^{\text{T}}, \qquad (2.17)$$

where $\omega_{\mu a}$ are the rotation "frequencies" in the rest frame

$$\omega_{\mu 1} = -\sin\alpha\,\partial_\mu\theta + \cos\alpha\sin\theta\,\partial_\mu\varphi, \quad \omega_{\mu 2} = \cos\alpha\,\partial_\mu\theta + \sin\alpha\sin\theta\,\partial_\mu\varphi, \quad \omega_{\mu 3} = \partial_\mu\alpha + \cos\theta\,\partial_\mu\varphi. \quad (2.18)$$

The field R may be visualized as rotating axisymmetric top. In the standard $O(3)$ NLSM, $\zeta = 0$ and this top is an "arrow" (a unit vector defined by the polar and azimuthal angles θ, φ) with one inertia momentum equal to zero. One can see that fluctuations of massive fields lead to a dynamic generation of the third inertia momentum, so the effective theory of the AF-perturbed model is not the standard $O(3)$ NLSM as one might naively guess, but rather the $O(3) \times O(2)$ NLSM. Properties of the latter model are well known [35–37]. In one dimension, ζ flows to the $O(4)$ fixed point $\zeta = 1$, while Γ flows to infinity, indicating the dynamic generation of a finite correlation length. For $d = 2$, the the $O(3) \times O(2)$ NLSM has long-range AF order, couplings Γ, ζ get renormalized but stay finite. We have checked that the similar dynamic generation of the third inertia momentum occurs for spin-1 systems close to the antiferromagnetic $SU(3)$ point, resulting in $O(3) \times O(2)$ NLSM as the effective model, so one may expect this result to be valid for general S.

3. Multiplet binding of topological excitations

Consider the fate of topologically nontrivial excitations of the $SU(4)$ -symmetric model (2.1) under AF perturbation (2.3) breaking the symmetry down to $SU(2)$. In $(1 + 1)$ dimensions, such excitations ("skyrmions") in CP^{N-1} model are characterized by the nonzero topological charge [29] that is essentially the winding number of the overall phase taken over a contour at infinity,

$$q_{CP^{N-1}} = -\frac{1}{2\pi}\oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = -\frac{1}{2\pi}\int \mathrm{d}^2 x \epsilon_{\mu\nu}(\partial_\mu A_\nu) = -\frac{\mathrm{i}}{2\pi}\int \mathrm{d}^2 x \epsilon_{\mu\nu}(\partial_\mu z^\dagger \partial_\nu z) \quad (3.1)$$

and for $d = 1$ this charge is directly related to the topological term in the action (2.1), $\mathcal{A}_{\mathrm{top}} = \mathrm{i}\pi q_{CP(3)}$. The topological charge density is proportional to the dimerization order parameter; the ground state of the $SU(4)$ -symmetric model (2.1) has finite topological charge density and thus is spontaneously dimerized [13].

The effect of the $SU(4) \mapsto SU(2)$ perturbation on the topological charge can be illustrated by the following simple observation: finite λ favors field configurations with the maximum spin length, i.e., with $\boldsymbol{h} = 0$. Such field configurations are given by

$$z_m = D^{(3/2)}_{mm'}(\alpha, \theta, \varphi)\psi^{(0)}_{m'}, \qquad \psi^{(0)}_m = \delta_{3/2,m}. \quad (3.2)$$

Substituting the above ansatz into (3.1), one straightforwardly obtains

$$q_{CP^3} \;\; = \;\; \frac{3}{4\pi}\int \mathrm{d}^2 x \sin\theta\,\epsilon_{\mu\nu}(\partial_\mu\theta)(\partial_\nu\varphi) = 3Q_{O(3)}, \quad (3.3)$$

where the topological charge $Q_{O(3)}$ is the winding number of the $S^2 \mapsto S^2$ mapping characterizing the space-time distribution of the unit vector $\boldsymbol{n}(\theta, \varphi)$. It should be remarked that although the homotopy group $\pi_2(SO(3)) = 0$, one can still define the $O(3)$-NLSM topological charge of the spin texture $Q_{O(3)}$ in the usual way via the unit vector $\boldsymbol{n}(\theta, \varphi)$ that corresponds to the local direction of the Néel vector. The AF perturbation thus favors z-field configurations with charge q_{CP^3} being a multiple of 3. One may conclude that unit-charge skyrmions of the CP^3 model bind into *triplets* under the influence of the AF perturbation. Such a triplet is the well-known unit-charge skyrmion (Belavin-Polyakov soliton [38]) of the $O(3)$ NLSM.

This is completely analogous to the formerly noted effect[20, 21] of topological binding of skyrmions into *pairs* in the $SU(3)$ spin-1 antiferromagnet under the AF perturbation lowering the symmetry from $SU(3)$ to $SU(2)$.

This statement is easily generalized: consider the $SU(2S + 1)$ antiferromagnet with the underlying spin S. Assume that the enhanced symmetry is broken down to $SU(2)$ by the perturbation that favors the field configuration with the maximal spin length, as in (2.3):

$$z = D^{(S)}(\alpha, \theta, \varphi)\psi^{(0)} = \mathrm{e}^{\mathrm{i}\varphi\hat{S}_3}\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}\mathrm{e}^{\mathrm{i}\alpha\hat{S}_3}\psi^{(0)}, \qquad \psi^{(0)}_m = \delta_{S,m}. \quad (3.4)$$

Then,

$$
\begin{aligned}
\partial_\mu z &= \mathrm{i}(\partial_\mu\varphi)\hat{S}_3 z + \mathrm{i}(\partial_\mu\theta)\mathrm{e}^{\mathrm{i}\varphi\hat{S}_3}\hat{S}_2\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}\mathrm{e}^{\mathrm{i}\alpha\hat{S}_3}\psi^{(0)} + \mathrm{i}(\partial_\mu\alpha)Sz\,, \\
\epsilon_{\mu\nu}\partial_\mu z^\dagger\partial_\nu z &= \varepsilon_{\mu\nu}(\partial_\mu\theta)(\partial_\nu\varphi)\left(\psi^{(0)}\right)^\dagger \mathrm{e}^{-\mathrm{i}\theta\hat{S}_2}(\hat{S}_2\hat{S}_3 - \hat{S}_3\hat{S}_2)\mathrm{e}^{\mathrm{i}\theta\hat{S}_2}\psi^{(0)} \\
&= \mathrm{i}S\sin\theta\varepsilon_{\mu\nu}(\partial_\mu\theta)(\partial_\nu\varphi)\,.
\end{aligned}
\tag{3.5}
$$

Substituting this into (3.1), we see that the CP^{2S} charge takes the form

$$
q_{CP^{2S}} = 2SQ_{O(3)}\,.
\tag{3.6}
$$

Thus, in $SU(2S + 1)$ antiferromagnets with underlying spin S, AF perturbation that breaks the enhanced symmetry down to $SU(2)$ leads to the binding of unit-charge skyrmions of the CP^{2S} model to $2S$-*multiplets*.

Strictly speaking, in $(1 + 1)$ dimensions, skyrmions considered above, are not excitations, but instanton events. The same effect obviously holds for "monopoles" in $(2 + 1)$ dimensions (instanton events changing the skyrmion topological quantum number $q_{CP^{2S}}$).

For the $(2 + 1)$ dimensional case, skyrmions may be viewed as static solitons in two spatial dimensions, and similarly, in $d = 3$ monopoles they may be viewed as static solitons ("hedgehogs"); the same reasoning on multiplet binding applies.

To show that explicitly, one may look at the "skyrmion current" $j = (2\pi)^{-1}\nabla \times A$, whose flux through a closed surface surrounding the monopole, $\widetilde{q} = \oint j \cdot \mathrm{d}S$, determines the monopole charge $\widetilde{q}$. A calculation essentially following (3.5) shows that

$$
j_a = -\frac{\mathrm{i}}{2\pi}\varepsilon_{abc}(\partial_b z^\dagger\partial_c z) = \frac{S}{2\pi}\varepsilon_{abc}\sin\theta(\partial_b\theta)(\partial_c\varphi) = 2SJ_a\,,
\tag{3.7}
$$

where J is the corresponding skyrmion current of the $O(3)$ NLSM. Thus, $\widetilde{q} = \pm 1$ monopoles bind into $2S$ multiplets under the influence of the perturbation.

The topological term in the action in the $(2 + 1)$ dimensional case is determined by monopole events and can not be expressed in the continuum limit as it is lattice-dependent [13, 26]. On the square lattice, it is given by

$$
\mathscr{A}_{\text{top}} = \frac{\mathrm{i}}{2}\pi n_{\mathrm{c}}\sum_{r_i}\zeta(r_i)\widetilde{q}_i\,,
\tag{3.8}
$$

where the sum is over the locations r_i of monopoles having the charge $\widetilde{q}_i$, and factors $\zeta(r_i)$ take on values 0, 1, 2, 3 for r_i belonging to the four dual sublattices W, X, Y, Z respectively (see figure 7 of [13]), and n_{c} is the "colour number" that in our case is equal to 1.

In $(2 + 1)$ dimensions, the ground state of the CP^{N-1} model can have the long-range order or can be disordered, depending on the value of the coupling g_0. In the ordered phase, the topological term is ineffective. However, if for some reason the coupling gets driven over the critical value (e.g., due to the presence of next-nearest neighbor interactions) and we land in the disordered phase, the topological term becomes important: it leads to the ground state with nonzero monopole density and thus to the spontaneous dimerization (i.e., breaking of the translation symmetry) [13]. The dimerization pattern of the ground state depends on the value of n_{c}: it is twofold degenerate for $n_{\mathrm{c}} = 2 \bmod 4$, fourfold degenerate for $n_{\mathrm{c}} = 1 \bmod 4$ or $n_{\mathrm{c}} = 3 \bmod 4$, and non-degenerate (with unbroken translational invariance) for $n_{\mathrm{c}} = 0 \bmod 4$. Thus, when the $SU(N)$-symmetric antiferromagnet gets perturbed as in (2.3) by the $SU(2)$ term favoring the Néel order, $2S$-multipletting of unit-charge monopoles leads to $\widetilde{q}_i$ in (3.8) multiplied by $2S$, which is equivalent to changing the number of "colours" n_{c} from 1 to $2S$. The ground state becomes respectively twofold degenerate for odd-integer spins $S = 2n + 1$, stays fourfold degenerate for half-integer S, and is non-degenerate for even-ineger $S = 2n$. This result coincides with the conclusion obtained by Haldane [26] in the framework of the $O(3)$ NLSM analysis.

4. Summary

We have considered the low-dimensional spin-S antiferromagnet on a bipartite lattice, close to the point with the enhanced $SU(2S + 1)$ symmetry which is decribed in the continuum field approximation by

the CP^{2S} model, and studied the consequences of explicit symmetry breaking from $SU(2S + 1)$ to $SU(2)$ (with the appropriate sign that favors the Nèel order). This model is motivated by the physics of cold spinor bosonic atoms in optical lattices, and the symmetry-breaking perturbation can be associated with weak interactions such as p-wave scattering that are usually neglected. We derive the effective theory for the perturbed system, and show that it is not the standard $O(3)$ nonlinear sigma model (NLSM) as one might naively guess, but rather the $O(3) \times O(2)$ NLSM. This occurs due to the dynamic generation of the third inertia momentum for the "spin arrow", caused by fluctuations of massive fields that correspond to non-axisymmetric deformations of the quadrupolar tensor. We have further shown that under the influence of the $SU(2S + 1) \mapsto SU(2)$ perturbation unit-charge topological excitations (skyrmions and monopoles) of the CP^{2S} model bind into $2S$ multiplets that correspond to excitations with unit $O(3)$ topological charge defined in terms of the unit Nèel vector.

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Порушення симетрії $SU(N) \to SU(2)$ в квантових антиферомагнетиках

О.К. Колежук [1,2], Т.Л. Завертаний[1]

[1] Інститут високих технологій, Київський національний університет імені Тараса Шевченка, МОН України, Київ 03022, Україна

[2] Інститут магнетизму НАН України та МОН України, просп. Вернадського 36–Б, Київ 03142, Україна

Ми досліджуємо $SU(2)$-симетричну систему спіну 3/2 на двороздільній ґратці, поблизу антиферомагнітної $SU(4)$-симетричної точки, що може бути описана CP^3 моделлю зі збуренням, що порушує симетрію до $SU(2)$ і заохочує неєлівське впорядкування. Показано, що ефективною теорією для збуреної моделі є не звичайна $O(3)$ нелінійна сігма-модель (НЛСМ), а $O(3) \times O(2)$ НЛСМ. Разом з тим, топологічний заряд Q $O(3)$-НЛСМ типу для спінової текстури одиничного вектора антиферомагнетизму може бути введений звичайним чином. Показано, що в присутності збурення топологічний заряд q поля CP^3 пов'язаний з зарядом Q $O(3)$-НЛСМ типу (для спінової текстури одиничного вектора антиферомагнетизму) співвідношенням $3Q = q$, тому під дією збурення скірміони моделі CP^3 з одиничним зарядом зв'язуються в триплети. Також показано, що у загальному випадку спіну S, порушення симетрії з $SU(2S + 1)$ до $SU(2)$ приводить до співвідношення $2SQ_{O(3)} = q_{CP^{2S}}$ між топологічними зарядами CP^{2S} та $O(3)$ типу, тобто можна очікувати що скірміони будуть у цьому випадку зв'язуватися в $2S$-мультиплети.

Ключові слова: *фрустровані магнетики, скірміони, холодні гази в оптичних ґратках*