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Non-equilibrium relaxations: ageing and finite-size effects[*]

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The long-time behaviour of spin-spin correlators in the slow relaxation of systems undergoing phase-ordering kinetics is studied in geometries of finite size. A phenomenological finite-size scaling ansatz is formulated and tested through the exact solution of the kinetic spherical model, quenched to below the critical temperature, in $2 < d < 4$ dimensions.

Key words: *phase transitions, finite-size scaling, dynamic critical phenomena, ageing, exactly solvable models*

1. Critical relaxations in finite-size systems

Collective phenomena arise in many-body systems with dynamically created long-range interactions and thereby often show new qualitative properties which cannot be obtained in systems with a small number of degrees of freedom. An important class are *critical phenomena*, characterised by *scale-invariance*. We are interested here in time-dependent phenomena with time-dependent or 'dynamical' scaling. As a physical example, we consider many-body spin systems, initially prepared in a disordered state with at most short-ranged correlations and then suddenly quenched to a temperature $0 < T < T_c$ below the critical temperature $T_c > 0$, with at least two physically distinct phases. Such a quenched spin system is then said to undergo *phase-ordering kinetics* [1]. For a spatially infinite geometry, observables such as correlation functions are then expected to be invariant under the time-space dilatation

$$t \mapsto t' = \kappa^z t, \quad \mathbf{r} \mapsto \mathbf{r}' = \kappa \mathbf{r}, \tag{1.1}$$

where κ is a constant rescaling factor and the *dynamical exponent z* serves to distinguish the scaling between time and space. The relaxation of the system after the quench can be measured through the connected correlators of the time- and space-dependent spin variables $S_{\mathbf{r}}(t)$, namely

$$C(t; \mathbf{r}) := \langle S_{\mathbf{r}}(t) S_{\mathbf{0}}(t) \rangle - \langle S_{\mathbf{r}}(t) \rangle \langle S_{\mathbf{0}}(t) \rangle = F_C \left(\frac{|\mathbf{r}|}{t^{1/z}} \right), \tag{1.2a}$$

$$C(t, s) := \langle S_{\mathbf{r}}(t) S_{\mathbf{r}}(s) \rangle - \langle S_{\mathbf{r}}(t) \rangle \langle S_{\mathbf{r}}(s) \rangle = f_C \left(\frac{t}{s} \right), \tag{1.2b}$$

where the quoted scaling forms are meant to hold in the limit of large times and large distances, such that $|\mathbf{r}|^z / t$ and t/s are kept fixed. In (1.2b), t is the *observation time* and s is the *waiting time*. Asymptotically, the scaling function $f_C(y)$ in (1.2b) should be algebraic

$$f_C(y) \sim y^{-\lambda/z}, \quad \text{as } y \to \infty, \tag{1.3}$$

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where $\lambda = \lambda_C$ is the *autocorrelation exponent*. A many-body non-stationary system whose slow relaxation dynamics also breaks time-translation-invariance and is such that the single-time correlator $C(t, \mathbf{r})$ and the two-time auto-correlator $C(t, s)$ obey the dynamical scaling (1.2), is said to be *ageing* [2–5].

For phase-ordering, with a non-conserved order parameter, some general exact results exist for models with short-ranged interactions. First, the dynamical exponent $z = 2$ for a non-conserved order parameter [6, 7].[1] Second, the Yeung-Rao-Desai inequality states that $\lambda \geqslant d/2$ [8]. Third, for the 2D Ising model one has the Fisher-Huse inequality $\lambda \leqslant 5/4$ [9]. Some typical values for z and λ are listed in table 1. They illustrate the sharpness of these exact bounds and permit a comparison between short-ranged and long-ranged interactions. The agreement of the available experiments [10, 11] with the 2D universality classes is very satisfying. For more detailed tables, see [4].

Table 1. Dynamical exponent z and autocorrelation exponent λ, as measured experimentally or found in some spin models. Long-range (LR) behaviour occurs in the Ising model for $\sigma < 1$ and in the spherical model for $\sigma < 2$. The spherical model is considered for dimensions $d > z$.

Material/model			z	λ	References
Merck (CCH-501)			1.94(5)	1.246(79)	[10]
nematic TNLC			2.01(1)	1.28(11)	[11]
Ising	1D	LR	$1 + \sigma$	0.5	[12, 13]
Ising	2D	LR	$1 + \sigma$	1	[14, 15]
Ising	2D	SR	2	1.24(2)	[16]
			2	1.25	[14, 15]
			2	1.3	[17–21]
Ising	3D	SR	2	1.60(2)	[22]
			2	1.6	[17–20]
Potts-3	2D	SR	2	1.19(3)	[16]
			2	1.22(2)	[23]
Potts-8	2D	SR	2	1.25(1)	[16]
XY	3D	SR	2	1.7(1)	[24]
			2	1.6	[17–19]
spherical		SR	2	$d/2$	[25]
spherical		LR	σ	$d/2$	[26, 27]

How is the scaling behaviour, encoded in the scaling forms (1.2), modified in a system confined to a domain of finite size, e.g., being placed into a box?

For a phenomenological answer, consider figure 1. For a fully finite hyper-cubic lattice with N^d sites and periodic boundary conditions, the single-time correlator $C(t; \mathbf{r})$ is shown in figure 1a, where $\mathbf{r}$ is oriented along one of coordinate axes. If the spatial distances $r = |\mathbf{r}|$ are not too large, the shape of the correlator does not depend sensitively on N. Only if $r \lesssim N/2$, does the correlator also receive contributions 'from around the world', such that for $r \approx N/2$ it no longer tends towards zero, but rather saturates at a N-dependent constant $C_{\lim}^{(1)}(N) > 0$. Figure 1b shows the two-time auto-correlator $C(ys, s)$. For large s, but y small enough, there is a clear data collapse. However, for larger values of y, C begins to decrease more rapidly than the infinite-size curve (1.2b).[2] As $y \gg 1$, C finally saturates at the limit value $C_{\infty}^{(2)}(N) > 0$.

Although the single-time correlator does not display strong finite-size effects, this is different for the

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[1]In this work, we restrict to this *model-A dynamics*.

[2]Since for lattices large enough that the system is just leaving the effective finite-size regime, the local exponent estimates $\lambda_{\mathrm{eff}}(y)$ may slightly over-estimate λ. In certain cases this might lead to claims of violation of exact upper bounds such as the Fisher-Huse inequality.
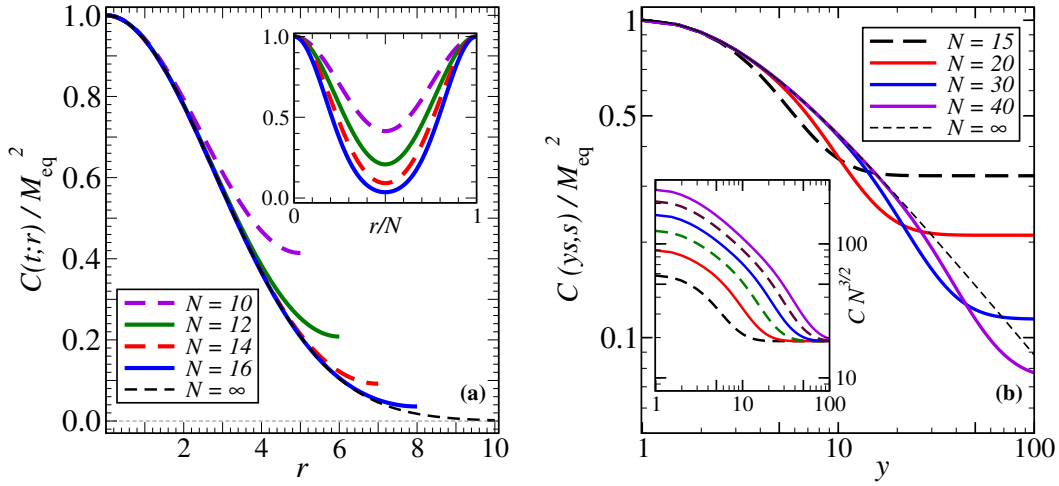
Figure 1. (Colour online) (a) Finite-size effects for the single-time correlator $C(t, \mathbf{r})$ in the fully finite 3D spherical model at $T < T_c$, with $t = 50$ and $N = 10, 12, 14, 16, \infty$ from top to bottom. The inset shows the periodicity over the interval $0 \leqslant r/N \leqslant 1$. (b) Finite-size effects for the two-time auto-correlator $C(ys, s)$ in the 3D fully finite spherical model at $T < T_c$, for $N = 15, 20, 30, 40$ from top to bottom (at the right) and s fixed. The thin dashed line gives the infinite-size auto-correlator. The inset shows the data collapse of the rescaled correlator $CN^{3/2}$ for $y = t/s$ large, with $N = 15, 20, 25, 30, 35, 40$ from left to right (arbitrary units).

length scale $L = L(t)$ of the growing clusters, estimated from the second moment

$$L^2(t) = \frac{\sum_{\mathbf{r}} |\mathbf{r}|^2 C(t; \mathbf{r})}{\sum_{\mathbf{r}} C(t; \mathbf{r})}. \tag{1.4}$$

The precise extent of the sums is specified below. Figure 2 shows that for sufficiently short times, the length $L^2(t) \sim t$ behaves as for the infinite system, but as t grows further, finally there occurs a cross-over towards a finite constant $L_\infty(N)$. We can see how to explain the findings of figures 1 and 2 in terms of phenomenological finite-size scaling. The resulting predictions are tested in the exactly solved kinetic spherical model, for dimensions $2 < d < 4$.

The *spherical model* of a ferromagnet [28–30] served as an exactly solvable, yet non-trivial, model for the detailed analysis of general concepts of critical phenomena, see [31] for a historical perspective. Its non-equilibrium behaviour after a quench was also thoroughly analysed, see [4, 25–27, 32–40]. The related Arcetri model provides a qualitative description of the dynamics in the non-equilibrium growth of interfaces [41, 42]. Finite-size effects at equilibrium were also analysed at great depth in the spherical model and were of value to test the theory of finite-size scaling derived from the renormalisation group, see [43–52] and references therein. For dimensions $d > d_c = 4$, that is above the upper critical dimension, the standard finite-size scaling *ansatz* must be considerably modified [49, 53–58].

Finite-size scaling techniques were applied in studies of phase-ordering kinetics [20, 21], the ageing of polymer collapse [59–62] or the dynamics of mitochondrial networks [63]. Explicit studies of finite-size scaling in an ageing system were carried out in Ising spin glasses [64] and notably on the dimensional cross-over between the 3D and 2D Edwards-Anderson spin glass [65] motivated by extremely accurate experiments on CuMn films [66, 67]. In addition, finite-size effects analogous to figures 1 and 2 are clearly visible in the time-evolution of characteristic cluster sizes in long-ranged Ising models quenched to $T < T_c$ [14] or in the auto-correlator [15]. Since the bulk 3D spherical model and the bulk ($p = 2$) spherical spherical spin glass are in the same dynamic universality class [36], one might hope that finite-size effects could be similar as well. Not so! Rather, detailed studies of the ($p = 2$) spherical spin glass [68, 69] show that this equivalence only holds in the spin glass for times $t \ll t_{\text{cross}} \sim N^{2/3}$. For time
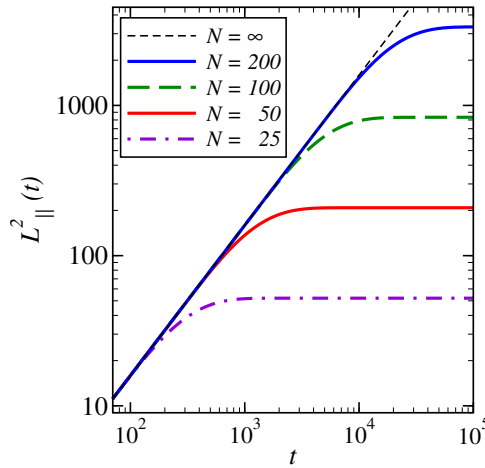
Figure 2. (Colour online) Finite-size effects for the longitudinal characteristic length $L_\parallel^2(t)$, measured along a coordinate axis, in the fully finite spherical model with lattice sizes $N = 25, 50, 100, 200$ from bottom to top. The thin dashed line indicates the infinite-size behaviour $L(t) \sim t^{1/2}$.

scales $t \gg t_{\text{cross}}$, ageing still holds with a new set of universal exponents [69], to be followed by a second cross-over to a regime of exponential decay at extremely large times [68].

This work is organised as follows. In section 2, we recall the main features of dynamical scaling in ageing phase-ordering kinetics. In section 3, we extend this phenomenological treatment to finite systems, using the hyper-cubic geometry $\overbrace{N \times \cdots \times N}^{d^* \text{ factors}} \times \overbrace{\infty \times \cdots \times \infty}^{d - d^* \text{ factors}}$, where the first $d^* \leqslant d$ directions are finite and periodic and the other $d - d^*$ directions are infinite. The finite-size forms so obtained are checked in section 4 using the exact solution of the kinetic spherical model in $2 < d < 4$ dimensions, quenched to $T > T_c$ from a totally disordered state and in section 5 we conclude. Technical details of the exact solution are given in the appendix.

2. Dynamical scaling description

A central ingredient of ageing is dynamical scaling. For the general two-time and spatial bulk correlator, our starting point is (below the upper critical dimension $d < d_c$; for short-ranged interactions usually $d_c^{(\text{short})} = 4$)

$$C\left(\kappa^z t, \kappa^z s; \kappa \mathbf{r}\right) = \kappa^\phi C\left(t, s; \mathbf{r}\right), \tag{2.1}$$

where t, s are the observation and the waiting time, z is the dynamical exponent, ϕ is a scaling exponent and $\mathbf{r}$ is the spatial distance. Writing (2.1) means that we assume negligible all finite-time and finite-distance corrections to scaling. Choosing $\kappa = s^{-1/z}$, this gives

$$C\left(t, s; \mathbf{r}\right) = s^{\phi/z} C\left(\frac{t}{s}, 1; \frac{\mathbf{r}}{s^{1/z}}\right). \tag{2.2}$$

In phase-ordering, the single-time correlator at $\mathbf{r} = \mathbf{0}$ is finite; namely either $C(t; \mathbf{0}) = 1$ in Ising-like systems or else $C(t; \mathbf{0}) = M_{\text{eq}}^2$ for order parameters with a continuous global symmetry. Setting $s = t$ in (2.2), this leads to $\phi = 0$ and[3] further to $C(t; \mathbf{r}) = C\left(1, 1; |\mathbf{r}| t^{-/z}\right) =: F_C\left(|\mathbf{r}| t^{-/z}\right)$. On the other hand, setting now $\mathbf{r} = \mathbf{0}$, the two-time auto-correlator is $C(t, s) = C(t, s; \mathbf{0}) = C(t/s, 1; \mathbf{0}) =: f_C(t/s)$. These results fully reproduce (1.2).

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[3] If more generally, one would expect $C(t, s) = s^{-b} f_C(t/s)$, this would lead to the identification $b = -\phi/z$, but for $\phi \neq 0$, this is incompatible with $C(t; \mathbf{0})$ being finite and constant for $t \to \infty$.

3. Dynamical finite-size scaling

According to the original definition, *finite-size scaling* [70] is the scaling behaviour in a nearly critical system confined to a *geometry of finite linear extent* N. For finite geometries, the natural generalisation of (2.1) consists, as at equilibrium [43, 44, 49, 71], in considering $1/N$ as a further relevant scaling field[4]. While this hypothesis was originally specified for the order parameter at the critical point [71], we adapt this to the situation at hand and write down the finite-size scaling (FSS) *ansatz* for the full two-time correlator

$$C\left(\kappa^z t, \kappa^z s; \kappa \mathbf{r}; \kappa^{-1}\frac{1}{N}\right) = \kappa^\phi C\left(t, s; \mathbf{r}; \frac{1}{N}\right),$$

(3.1)

meant to hold in the hyper-cubic geometry $\overbrace{N \times \cdots \times N}^{d^* \text{ factors}} \times \overbrace{\infty \times \cdots \times \infty}^{d-d^* \text{ factors}}$, where N describes the finite length in the system. For simplicity, we consider a single length of this kind[5]. Of course, for $N \to \infty$, one is back to the bulk scaling form (2.1), and hence (1.2).

Choose the rescaling factor $\kappa = s^{-1/z}$. For phase-ordering kinetics, recall from section 2 that $\phi = 0$. Then, (3.1) can be equivalently expressed as

$$C\left(t, s; \mathbf{r}; \frac{1}{N}\right) = C\left(\frac{t}{s}, 1; \frac{\mathbf{r}}{s^{1/z}}; \frac{s^{1/z}}{N}\right).$$

(3.2)

As above in section 2, we then expect for the correlators (provided spatial rotation-invariance can be assumed)

$$C(t; \mathbf{r}; N^{-1}) = F_C\left(\frac{|\mathbf{r}|^z}{t}; \frac{N^z}{t}\right), \quad C(t, s; N^{-1}) = f_C\left(\frac{t}{s}; \frac{N^z}{t}\right)$$

(3.3)

such that the corresponding scaling functions are now functions of two variables. Finite-size scaling in ageing can be analysed in the asymptotic FSS limit where $t \to \infty$, $s \to \infty$, $|\mathbf{r}| \to \infty$ and $N \to \infty$ such that the three scaling variables

$$y = \frac{t}{s}, \quad \varrho = \frac{\mathbf{r}}{t^{1/z}}, \quad Z = \frac{N^z}{t}$$

(3.4)

are kept fixed. The precise form of the finite-size scaling functions (3.3) will depend on the universality class under study, and on the boundary conditions [44, 45, 49, 53].

As a first consequence, consider the characteristic length $L(t)$ of the clusters. From (1.4) and (3.3), we derive the finite-size scaling form

$$L^2(t; N^{-1}) = \frac{\sum_{\mathbf{r}} |\mathbf{r}|^2 C(t; \mathbf{r}; N^{-1})}{\sum_{\mathbf{r}} C(t; \mathbf{r}; N^{-1})} \simeq t^{2/z} \frac{\int d\mathbf{r}\, (|\mathbf{r}|t^{-1/z})^2 F_C(|\mathbf{r}|^z/t; N^z/t)}{\int d\mathbf{r}\, F_C(|\mathbf{r}|^z/t; N^z/t)} = t^{2/z} f_L\left(\frac{N^z}{t}\right).$$

(3.5)

For $Z \gg 1$, the behaviour of an effectively infinite system requires that $f_L(Z) \overset{Z \gg 1}{\simeq} f_0 = \text{const}$ and for $Z \ll 1$, the time-independent saturation in figure 2 is captured by $f_L(Z) \overset{Z \ll 1}{\simeq} Z^{2/z}$ such that $L_\infty(N) \sim N$, as would have been expected from dimensional analysis.

Next, we consider the plateau in the two-time auto-correlator $C(ys, s)$ for $y \gg 1$, see figure 1b. Recall that for the infinite system, we expect from (1.2b) and (1.3) that $C(t, s; \mathbf{0}; 0) = f_C(t/s) \sim (t/s)^{-\lambda/z}$. For $N < \infty$, we reformulate (3.2) as follows

$$C(t, s; N^{-1}) = C\left(t, s; \mathbf{0}; \frac{1}{N}\right) = C\left(\frac{t}{s}, 1; \mathbf{0}; \frac{s^{1/z}}{N}\right) = \left(\frac{t}{s}\right)^{-\lambda/z} \mathscr{F}_C\left(\left(\frac{t}{s}\right)^{1/z}, \frac{s^{1/z}}{N}\right).$$

(3.6)

Herein, the first argument in the scaling function $\mathscr{F}_C = \mathscr{F}_C(y, u)$ will be considered large and be kept fixed, $y \gg 1$. In that case, the scaling function will describe the cross-over between (i) the infinite-system

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[4]Very interesting adaptations of this idea were brought forward in the study of the kinetics of polymer collapse, where N is now the finite number of monomers, but the spatial geometry of the system was not specified [61, 62].

[5]Spatially anisotropic finite-size effects could be taken into account by introducing distinct finite sizes N_j in different spatial directions.

behaviour (when $u = s^{1/z}/N \to 0$) $f_C(y) = \mathscr{F}_C(y, 0) \sim y^{-\lambda/z}$, which is independent of s, and (ii) the fully finite-system behaviour (when $u = s^{1/z}/N \to \infty$), when $C \xrightarrow{y \gg 1} C_\infty^{(2)}$ no longer depends on $y = t/s$. The first limit case is taken into account by admitting $\mathscr{F}_C(y, u) \simeq \mathsf{F}(yu)$ and $\mathsf{F}(0) = \text{const}$. Then, the second limit case leads to

$$C\left(t, s; \mathbf{0}; \frac{1}{N}\right) \overset{t/s \gg 1}{\simeq} \left(\frac{t}{s}\right)^{-\lambda/z} \mathsf{F}\left(\left(\frac{t}{s}\right)^{1/z} \cdot \frac{s^{1/z}}{N}\right) \sim \left(\frac{t}{s}\right)^{-\lambda/z} \left[\left(\frac{t}{s}\right)^{1/z} \cdot \frac{s^{1/z}}{N}\right]^\omega, \tag{3.7}$$

where in the last step, we assumed a power-law form of $\mathsf{F}(yu) \sim (yu)^\omega$ for $yu \gg 1$. The y-independent plateau $C_\infty^{(2)}$ observed for fully finite systems (see figure 1b for s fixed) is reproduced if we choose $\omega = \lambda$. Hence, for finite systems with $y = t/s \gg 1$

$$C\left(t, s; \mathbf{0}; \frac{1}{N}\right) \overset{t/s \gg 1}{\longrightarrow} C_\infty^{(2)} \sim \left(\frac{s^{1/z}}{N}\right)^\lambda. \tag{3.8}$$

Herein, s is still kept fixed whereas N must be taken large enough such that the system under study is indeed in its finite-size scaling regime (in other word, $Ns^{-1/z}$ must be large enough).

Hence, for fully finite systems, quenched to $T < T_c$, the auto-correlator $C(ys, s) = f_C(y) \xrightarrow{y \gg 1} C_\infty^{(2)}$, such that the plateau value $C_\infty^{(2)} = C_\infty^{(2)}(s, N)$ should obey the scalings

$$C_\infty^{(2)} \sim N^{-\lambda} \quad \text{with } s \text{ fixed}, \qquad C_\infty^{(2)} \sim s^{\lambda/z} \quad \text{with } N \text{ fixed}. \tag{3.9}$$

These are the sought scalings for the plateau of the auto-correlator and the main result of this section.

The inset in figure 1b shows the data collapse of $N^\lambda C(ys, s)$ to a y-independent constant for y large enough and s fixed, in the 3D spherical model, where $\lambda = \frac{3}{2}$. In the next section, (3.9) is verified analytically from the exact solution of the quenched kinetic spherical model in dimensions $2 < d < 4$.

A simple heuristic argument to establish (3.8) goes as follows. For widely different times $t \gg s \gg \tau_{\text{mic}}$, the asymptotic form of the auto-correlator is expressed through the cluster sizes L as $C(t, s) \sim [L(t)/L(s)]^{-\lambda}$. If furthermore t is so large that $L(t) \sim N$ while s is small enough such that still $L(s) \sim s^{1/z}$, the scaling (3.8) of the plateau $C_\infty^{(2)}$ follows.

4. The kinetic spherical model

Following standard developments [25, 34, 35, 42], the kinetic spherical model is defined in terms of real spin variables $S_{\mathbf{n}} = S_{\mathbf{n}}(t) \in \mathbb{R}$ at each lattice site $\mathbf{n} \in \Lambda \subset \mathbb{Z}^d$, subject to the *spherical constraint* $\sum_{\mathbf{n} \in \Lambda} S_n^2(t) = |\Lambda|$, where $|\Lambda| = \prod_{j=1}^d N_j$ is the number of sites of the lattice $\Lambda \subset \mathbb{Z}^d$. Its dynamics is given by the Langevin equation

$$\partial_t S_{\mathbf{n}}(t) = D\Delta_{\mathbf{n}} S_{\mathbf{n}}(t) - \mathfrak{z}(t)S_{\mathbf{n}}(t) + \eta_{\mathbf{n}}(t), \tag{4.1}$$

with the spatial laplacian $\Delta_{\mathbf{n}}$ and the thermal white noise $\eta_{\mathbf{n}} = \eta_{\mathbf{n}}(t)$. It has the first two moments

$$\langle \eta_{\mathbf{n}}(t) \rangle = 0, \quad \langle \eta_{\mathbf{n}}(t)\eta_{\mathbf{m}}(t') \rangle = 2DT\delta(t - t')\delta_{\mathbf{n}, \mathbf{m}}, \tag{4.2}$$

where T is the bath temperature and D is a kinetic coefficient. The Lagrange multiplier $\mathfrak{z}(t)$ is fixed from the spherical constraint. The Fourier representation

$$S_{\mathbf{n}}(t) = \frac{1}{|\Lambda|} \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_d=0}^{N_d-1} \exp\left(2\pi\mathrm{i} \sum_{j=1}^d \frac{k_j}{N_j} n_j\right) \widehat{S}(t, \mathbf{k}) \tag{4.3}$$

achieves a formal solution of the model which reads

$$\widehat{S}(t, \mathbf{k}) = \widehat{S}(0, \mathbf{k}) \frac{\exp[-2D\omega(\mathbf{k})t]}{\sqrt{g(t)}} + \int_0^t \mathrm{d}\tau\, \widehat{\eta}(\tau, \mathbf{k}) \sqrt{\frac{g(\tau)}{g(t)}} \exp[-2D\omega(\mathbf{k})(t - \tau)], \tag{4.4a}$$

with the abbreviations (nearest-neighbour interactions assumed)

$$\omega(\mathbf{k}) = \sum_{j=1}^{d} \left(1 - \cos\frac{2\pi}{N_j}k_j\right), \quad g(t) = \exp\left[2\int_{0}^{t} d\tau\, \mathfrak{z}(\tau)\right]. \tag{4.4b}$$

In what follows, we restrict ourselves to a totally disordered initial state, such that $\langle S_{\mathbf{n}}(0)\rangle = 0$ and $\langle S_{\mathbf{n}}(0)S_{\mathbf{m}}(0)\rangle = \delta_{\mathbf{n},\mathbf{m}}$. In momentum space, the second moments of initial and thermal noises become

$$\left\langle \widehat{S}(0,\mathbf{k})\widehat{S}(0,\mathbf{k}')\right\rangle = |\Lambda|\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}, \quad \left\langle \widehat{\eta}(t,\mathbf{k})\widehat{\eta}(t',\mathbf{k}')\right\rangle = 2DT|\Lambda|\delta(t-t')\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}. \tag{4.4c}$$

Then, the spherical constraint can be cast into a Volterra integral equation for $g = g(t)$

$$g(t) = f(t) + 2DT\int_{0}^{t} d\tau\, g(\tau)f(t-\tau), \quad f(t) := \frac{1}{|\Lambda|}\sum_{\mathbf{k}} \exp\left[-4D\omega(\mathbf{k})t\right]. \tag{4.4d}$$

Here and below, we abbreviate $\sum_{\mathbf{k}} := \sum_{k_1=0}^{N_1-1} \cdots \sum_{k_d=0}^{N_d-1}$. Equations (4.4) specify the exact solution of the kinetic spherical model. We are interested in

(I) the two-time correlation function $\widehat{C}(t,s;\mathbf{k})$ in momentum space, defined by

$$\left\langle \widehat{S}(t,\mathbf{k})\widehat{S}(s,\mathbf{k}')\right\rangle =: |\Lambda|\delta_{\mathbf{k}+\mathbf{k}',\mathbf{0}}\, \widehat{C}(t,s;\mathbf{k}), \tag{4.5a}$$

$$\widehat{C}(t,s;\mathbf{k}) = \frac{e^{-2D\omega(\mathbf{k})(t+s)}}{\sqrt{g(t)g(s)}} + 2DT\int_{0}^{\min(t,s)} d\tau\, \frac{g(\tau)}{\sqrt{g(t)g(s)}} e^{-2D\omega(\mathbf{k})(t+s-2\tau)}, \tag{4.5b}$$

and especially the two-time auto-correlator

$$C(t,s) := \frac{1}{|\Lambda|}\sum_{\mathbf{k}} \widehat{C}(t,s;\mathbf{k}) = C(s,t). \tag{4.6}$$

(II) The single-time correlator in momentum space $\widehat{C}(t;\mathbf{k}) := \widehat{C}(t,t;\mathbf{k})$, obtained from (4.5) by setting $s = t$. The time-space correlator reads

$$C(t;\mathbf{n}) = \frac{1}{|\Lambda|}\sum_{\mathbf{k}} \exp\left(2\pi i \sum_{j=1}^{d} \frac{k_j}{N_j}n_j\right)\widehat{C}(t;\mathbf{k}). \tag{4.7}$$

The well-known bulk critical temperature [28] ($I_0(u)$ is a modified Bessel function [72])

$$\frac{1}{T_c(d)} = \int_{0}^{\infty} du \left[e^{-2u}I_0(2u)\right]^d \tag{4.8}$$

is finite and positive for $d > 2$. Explicitly [73, 74]

$$\frac{1}{T_c(3)} = \frac{\sqrt{3}-1}{192\pi^3}\left[\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{11}{24}\right)\right]^2 \approx 0.25273\ldots \tag{4.9}$$

In what follows, we consider a hyper-cubic geometry $\overbrace{N \times \cdots \times N}^{d^* \text{ factors}} \times \overbrace{\infty \times \cdots \times \infty}^{d-d^* \text{ factors}}$, where the first $d^* \leqslant d$ directions are finite and periodic and the other $d - d^*$ directions are infinite. We also restrict

ourselves to $2 < d < 4$ and rescale the temporal units such that $8\pi D \stackrel{!}{=} 1$. After a quench from the disordered initial state (4.4c) to a temperature $T < T_c(d)$, we find in the FSS limit (3.4) (see the appendix for calculations):

(A) the single-time temporal-spatial correlator, namely

$$C(t; \mathbf{n}) = M_{\text{eq}}^2 \exp\left(-\pi \sum_{j=1}^{d} \frac{n_j^2}{t}\right) \prod_{j=1}^{d^*} \frac{\vartheta_3\left(i\pi \frac{Nn_j}{t}, e^{-\pi Z}\right)}{\vartheta_3\left(0, e^{-\pi Z}\right)} \tag{4.10a}$$

$$= M_{\text{eq}}^2 \exp\left(-\pi \sum_{j=d^*+1}^{d} \frac{n_j^2}{t}\right) \prod_{j=1}^{d^*} \frac{\vartheta_3\left(\pi n_j/N, e^{-\pi/Z}\right)}{\vartheta_3\left(0, e^{-\pi/Z}\right)}, \tag{4.10b}$$

where $M_{\text{eq}}^2 = 1 - T/T_c(d)$ is the squared equilibrium magnetisation and Z was defined in (3.4) with $z = 2$. Finally, $\vartheta_3(z, q) = \sum_{p=-\infty}^{\infty} q^{p^2} \cos(2pz)$ is a Jacobi theta function [72][6]. See figure 1a for illustration. From (4.10a) we identify the finite-size scaling function $F_C = F_C(\varrho, Z)$ in (3.3). The shape of this function is temperature-independent. Indeed, an universal shape of F_C is expected, since the temperature T should be irrelevant in phase-ordering kinetics [1].

Equation (4.10a) gives a factorisation of $C(t, \mathbf{n}) = C_{\text{bulk}}(t; \mathbf{n}) \cdot C_{\text{red}}(t; \mathbf{n}; N)$ into a size-independent 'bulk' part and a 'reduced' part which contains the finite-size effects. Because of the identity $\vartheta_3(z+\pi, q) = \vartheta_3(z, q)$, it is seen from (4.10b) that the correlator repeats periodically when $n_j \mapsto n_j + N$ is in the finite directions, as illustrated in the inset of figure 1a. For Z large enough[7], the central peak of the correlator around $\mathbf{n} = \mathbf{0}$ decays as in the bulk with a length scale $L(t) \sim t^{1/2}$ such that the system decomposes into separate and independent clusters of linear size $L(t)$, as expected. The bulk gaussian decay $\sim \exp(-\mathbf{n}^2/t)$, rather than an exponential $\sim \exp(-|\mathbf{n}|/\sqrt{t})$, is a peculiar property of the spherical model which distinguishes it from the Ising universality class.

(B) the two-time auto-correlator, for all $T < T_c(d)$, reads

$$C(ys, s) = M_{\text{eq}}^2 \left(\frac{2\sqrt{y}}{1+y}\right)^{d/2} \left[\frac{\vartheta_3\left(0, \exp\left(-\pi \frac{2Z}{1+1/y}\right)\right)^2}{\vartheta_3\left(0, \exp(-\pi Z)\right)\vartheta_3\left(0, \exp(-\pi Zy)\right)}\right]^{d^*/2} \tag{4.11a}$$

$$= M_{\text{eq}}^2 \left(\frac{2\sqrt{y}}{1+y}\right)^{(d-d^*)/2} \left[\frac{\vartheta_3\left(0, \exp\left(-\pi \frac{1+1/y}{2Z}\right)\right)^2}{\vartheta_3\left(0, \exp(-\pi/Z)\right)\vartheta_3\left(0, \exp(-\pi/Zy)\right)}\right]^{d^*/2}, \tag{4.11b}$$

as illustrated in figure 1b. We identify from (4.11a) the finite-size scaling function $f_C = f_C(y, Z)$ in (3.3), whose shape is once more temperature-independent. As above for the single-time correlator, (4.11a) displays a natural factorisation into the bulk two-time auto-correlator $C_{\text{bulk}}(ys, s) = M_{\text{eq}}^2 \left[2\sqrt{y}/(1+y)\right]^{d/2}$ and a 'reduced' factor which alone contains all finite-size effects. Equation (4.11a) shows that for $Z \gg 1$, finite-size corrections with respect to the bulk behaviour are exponentially small. On the other hand, equation (4.11b) shows that for $Z \ll 1$, the system behaves effectively as if it had only $d - d^*$ dimensions, up to exponentially small corrections[8].

Having verified the generic finite-size scaling forms (3.3), we now test the validity of the finite-size scaling predictions (3.9) for the plateau values $C_\infty^{(2)}$. To be specific, we consider a fully finite system, with $d^* = d$. Fix the system size N and the waiting time s and consider the changes in $y = t/s$ by varying the observation time t. Physically, finite-size effects will be felt first by the larger length $L(t) \sim t^{1/2}$.

---

[6]Analogous expressions of the finite-size scaling functions in terms of Jacobi Theta functions are known for the particle density in several 1D reaction-diffusion processes for both periodic and open boundary conditions [75, 76] and for the single-time correlator in the periodic 1D Glauber-Ising model at temperature $T = 0$ [77].

[7]Actually for $Z \gtrsim 25$, which in physical units corresponds to $L(t) \lesssim 5N$.

[8]Finite-temperature and finite-time effects merely give a corrective factor $1 + O\left(Ts^{1-d/2}\right)$, negligible for large waiting times $s \to \infty$, if $d > 2$.

Since $t \gg s$, we expect that $L(t) \gg L(s)$. The limit $y \gg 1$ is realised by taking $t \gg 1$. With the identity $\vartheta_3\left(0, e^{-\pi y}\right) = y^{-1/2}\,\vartheta_3\left(0, e^{-\pi/y}\right)$, we have

$$C(t,s) = M_{\text{eq}}^2 \left(\frac{t\,(s/t)^{1/2}}{(t+s)/2}\right)^{d/2} \left[\frac{\left(2\frac{N^2}{t+s}\right)^{-1/2} \vartheta_3\left(0, \exp\left(-\frac{\pi}{2}\frac{t+s}{N^2}\right)\right)}{\sqrt{\left(\frac{N^2}{t}\right)^{-1/2} \vartheta_3\left(0, \exp\left(-\pi\frac{t}{N^2}\right)\right) \vartheta_3\left(0, \exp\left(-\pi\frac{N^2}{s}\right)\right)}}\right]^{d^*}. \qquad (4.12)$$

For N^2/s finite but large enough (such that the plateaux in figure 1b is reached), the last of the theta functions in (4.12) are very close to unity. Because of the condition $t/N^2 \gg 1$, the other two theta-functions in (4.12) are also close to unity. Up to constants, we obtain

$$C(t,s) \overset{t\to\infty}{\sim} \left(\frac{s}{t}\right)^{d/4} \left[\left(\frac{t+s}{N^2}\right)^{1/2} \left(\frac{N^2}{t}\right)^{1/4} \text{const}\right]^{d^*} \sim \left(\frac{s}{t}\right)^{d/4} \left[\frac{t(1+s/t)}{t^{1/2}}\right]^{d^*/2} \left(N^{-2\frac{1}{2}+2\frac{1}{4}}\right)^{d^*}. \qquad (4.13)$$

Finally, now admitting a fully finite system such that $d = d^*$, we have (for $2 < d < 4$)

$$C(t,s) \sim \left(\frac{s}{t}\right)^{d/4} t^{d/4} N^{-d/2} = s^{d/4} N^{-d/2}, \qquad (4.14)$$

which in view of the well-known results $\lambda = d/2$ [25] and $z = 2$ [1] does indeed reproduce equation (3.8), or (3.9) if either s or N is kept fixed.

(C) Characteristic time-dependent length scales $L(t)$ of the ordered clusters can be measured as second moments of the single-time correlator

$$L^2(t) := \frac{\sum_{\mathbf{n}} \mathbf{n}^2 C(t; \mathbf{n})}{\sum_{\mathbf{n}} C(t; \mathbf{n})}. \qquad (4.15)$$

Precise expressions follow from (4.10a) once the range of summation of the distances $|\mathbf{n}|$ is fixed. For example, if one measures the distances along one of the coordinate axes of one of the infinite directions, one obtains the *'transverse'* length scale $L_\perp^2(t) = 4Dt$, as for a fully infinite system [78]. On the other hand, if the distances are measured along the coordinates axes of one of the finite directions, we find a *'longitudinal'* length scale, which reads for sufficiently thick films, and in agreement with (3.5)

$$L_\parallel^2(t) = \frac{1}{\pi} t f_L(Z), \quad f_L(Z) = \frac{\pi}{6} Z \left(1 + \frac{12}{\pi^2} \sum_{p=1}^{\infty} \frac{(-1)^p}{p^2} e^{-\pi p^2/Z}\right) \simeq \begin{cases} \frac{\pi}{6} Z; & \text{if } Z \ll 1, \\ 1; & \text{if } Z \gg 1. \end{cases} \qquad (4.16)$$

The scaling function f_L is temperature-independent. This describes the cross-over shown in figure 2, such that for $Z = N^2/t$ small enough, we obtain saturation at $L_\parallel^2(t) \to L_\infty^2 \sim N^2$, but on the other hand one has $L_\parallel^2(t) \sim t$ of an effectively infinite system for Z large enough.

5. Conclusions

We studied finite-size scaling in the ageing relaxation of phase-ordering kinetics after a quench from a disordered initial state into the two-phase coexistence regime with temperature $0 < T < T_c$. The finite-size scaling *ansatz* (3.1) is the natural extension of dynamic finite-size scaling at equilibrium [71]. Phenomenologically, the observations to be gleaned from figure 1 for the single-time and two-correlations and figure 2 for the characteristic length scale are captured by the finite-size scaling forms (3.3). The *form* of the associated scaling functions is temperature-independent, which confirms the expectation that the temperature should be irrelevant in phase-ordering kinetics [1]. From these, the finite-size scaling (3.5) for the length scale $L_\parallel(t)$ and especially (3.9) for the plateaux $C_\infty^{(2)}$ in the two-time auto-correlator of a fully finite system were derived. We checked that these predictions are fully bourne out in the phase-ordering of the exactly solved kinetic spherical model, for $2 < d < 4$ dimensions.

Clearly, several open questions remain, including:

1. Do the FSS predictions (3.3), (3.5), (3.9) also hold for other universality classes? For kinetic Ising models with either short-ranged or long-ranged interactions, detailed tests on all these have been carried out recently and will be reported elsewhere [79].

2. Although the discussion was entirely formulated here in terms of classical dynamics, a finite-size scaling ansatz such as (3.3) should *a priori* also work for relaxations in quantum systems, either closed or open.

3. Our analysis is restricted to below the upper critical dimension $d < d_c$. At equilibrium, it is well-known that dangerous irrelevant variables lead to essential modifications of the finite-size scaling ansatz (3.1), (3.3) [49, 53–58]. Such modifications should also become necessary for the dynamics.

   Considerations of this kind might become crucial either for long-ranged interactions, where d_c is lowered with respect to the value $d_c^{(\text{short})} = 4$ of short-ranged systems or else for d-dimensional *quantum* systems (possibly with long-ranged interactions as well), for which at least the equilibrium quantum phase transitions at $T = 0$ are known to be in the same universality class as the corresponding $(d + \theta)$-dimensional classical universality class at finite temperature, where the anisotropy exponent $\theta \geqslant 1$ [80].

4. From figure 1b it appears that finite-size effects might create a spurious regime where the auto-correlator $C(ys, s) \sim y^{-\lambda_{\text{eff}}}$ might look algebraic in a certain window; but rather the system already is the transition region between the rapid fall-off after having left the infinite-size behaviour of $C_{\text{bulk}}(ys, s)$ and the turn-around towards the saturation plateau $C_\infty^{(2)}$. Since $\lambda_{\text{eff}} > \lambda$, not recognising that this effect carries the risk of systematic over-estimation of the auto-correlation exponent λ, in simulations or in experiments.

5. One may generalise dynamical FSS to critical quenches and to two-time response functions as well. The theory and numerical tests thereof will be presented elsewhere [79].

6. Can one use (3.9) to devise improved methods for the measurement of λ?

Acknowledgement

Appendix. Analytical derivations

The exact solution of the kinetic spherical model at $T < T_c(d)$, starting from (4.4), is described.

A.1. Spherical constraint

The Volterra integral equation (4.4d) gives the long-time behaviour of $g(t)$ in a large, but finite system, as follows. The first part retraces the steps used at equilibrium [51, 52], with the notation adapted for dynamics. The second part gives the new ingredients needed for non-equilibrium dynamics.

1. Through a Laplace transform one formally solves (4.4d)

$$\overline{g}(p) = \mathscr{L}(g)(p) := \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-pt} g(t) = \frac{\overline{f}(p)}{1 - 2DT\overline{f}(p)}. \tag{A.1}$$

Standard Tauberian theorems [81] relate the behaviour of $\overline{g}(p)$ in the $p \to 0$ limit to the asymptotic long-time behaviour of $g(t)$ for $t \to \infty$. One needs the leading terms of $\overline{f}(p)$ as $p \to 0$. Recall the generalised Poisson resummation formula [82]

$$\sum_{n=a}^{b} f(n) = \sum_{q=-\infty}^{\infty} \int_a^b \mathrm{d}x\, \mathrm{e}^{2\pi\mathrm{i}qx} f(x) + \frac{1}{2}f(a) + \frac{1}{2}f(b) \tag{A.2}$$

and use this to deduce the important identity, for $m \in \mathbb{Z}$ and $x \in \mathbb{R}$

$$\sum_{k=0}^{N-1} \exp\left(\frac{2\pi\mathrm{i}}{N}km + x\cos\frac{2\pi k}{N}\right) = N \sum_{q=-\infty}^{\infty} I_{qN+m}(x), \tag{A.3}$$

where $I_n(x)$ is a modified Bessel function [72].

Now, one writes as in [51], using equation (A.3) with $m = 0$ in the second line d times

$$
\begin{aligned}
2D\overline{f}(p) &= \frac{2D}{|\Lambda|} \sum_{\mathbf{k}} \int_0^\infty \mathrm{d}t\, \exp\left\{-\left[p + 4D\sum_{j=1}^d \left(1 - \cos\frac{2\pi}{N_j}k_j\right)\right]t\right\} \\
&= 2D \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-(p+4Dd)t} \sum_{q_1,\ldots,q_d\in\mathbb{Z}} \prod_{j=1}^d I_{N_j q_j}(4Dt) \\
&= \frac{1}{2}\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-\frac{1}{2}\phi u}\left[\mathrm{e}^{-u}I_0(u)\right]^d + \frac{1}{2}\sideset{}{'}\sum_{\mathbf{q}\in\mathbb{Z}^d} \int_0^\infty \mathrm{d}u\, \mathrm{e}^{-\frac{1}{2}\phi u}\prod_{j=1}^d \left[\mathrm{e}^{-u}I_{N_j q_j}(u)\right], \tag{A.4}
\end{aligned}
$$

where one sets $\phi := p/2D$. In the last line, the bulk contribution which arises from $\mathbf{q} = \mathbf{0}$, is separated from the finite-size terms which have $\mathbf{q} \neq \mathbf{0}$ (indicated by $\sum'$).

In what follows, restrict throughout to dimensions $2 < d < 4$. First, standard techniques [25, 43, 46, 48] give the leading order of the Watson function $W_d(\phi)$ for $\phi \ll 1$, as follows

$$
\begin{aligned}
W_d(\phi) &:= \frac{1}{2}\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-\frac{1}{2}\phi u}\left[\mathrm{e}^{-u}I_0(u)\right]^d \\
&\simeq W_d(0) - (4\pi)^{-d/2}\left|\Gamma\left(1 - \frac{d}{2}\right)\right|\phi^{(d-2)/2}\left[1 + \mathrm{o}(\phi)\right], \tag{A.5}
\end{aligned}
$$

with an implied analytic continuation in d. Next, the finite-size terms are evaluated in the hyper-cubic geometry, such that the first d^* dimensions are finite ($0 < d^* \leqslant d$), with periodic boundary conditions (for simplicity, set $N_j = N$ for all $j = 1, \ldots, d^*$). The remaining $d - d^*$ dimensions are assumed to be infinite, formally $N_j = \infty$. With the asymptotic identity [51] $I_\nu(x) = (2\pi x)^{-1/2}\mathrm{e}^{x - \nu^2/2x}\left[1 + \mathrm{O}(1/x)\right]$ one has

$$
\begin{aligned}
\frac{1}{2}\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-\frac{1}{2}\phi u}\prod_{j=1}^d \left[\mathrm{e}^{-u}I_{N_j q_j}(u)\right] &\simeq \frac{1}{2}\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-\frac{1}{2}\phi u}(2\pi u)^{-d/2}\prod_{j=1}^{d^*} \mathrm{e}^{-(Nq_j)^2/2u} \\
&= (4\pi)^{-d/2}\phi^{d/2-1}\int_0^\infty \mathrm{d}v\, v^{-d/2}\exp\left(-v - \frac{1}{v}\frac{\phi}{4}\sum_{j=1}^{d^*} N^2 q_j^2\right) \\
&= \frac{2}{(4\pi)^{d/2}}\left(\frac{2\psi}{N}\right)^{d-2}\left(\frac{1}{\psi|\mathbf{q}|}\right)^{(d-2)/2} K_{(d-2)/2}\big(2\psi|\mathbf{q}|\big), \tag{A.6}
\end{aligned}
$$

with the *thermo-geometric parameter* $\psi := \frac{1}{2}N\phi^{1/2}$, the short-hand $|\mathbf{q}|^2 := \sum_{j=1}^{d^*} q_j^2$, the other modified Bessel function $K_\nu(x)$ [72] and where the identity [51]

$$\int_0^\infty \mathrm{d}x\, x^{\nu-1}\mathrm{e}^{-\beta x - \alpha/x} = 2\left(\frac{\alpha}{\beta}\right)^{\nu/2} K_\nu\big(2\sqrt{\alpha\beta}\,\big) \tag{A.7}$$

was used in the last line. In the infinite directions, only the terms with $q_j = 0$ contribute in (A.6), for $j = d^* + 1, \ldots, d$. The final result of the first part is, for $2 < d < 4$ [51, 52]

$$2D\overline{f}(p) = W_d(0) - \frac{1}{(4\pi)^{d/2}} \left[\left| \Gamma\left(1 - \frac{d}{2}\right) \right| - 2 \sum_{\mathbf{q} \in \mathbb{Z}^{d^*}}' \frac{K_{(d-2)/2}(2\psi|\mathbf{q}|)}{(\psi|\mathbf{q}|)^{(d-2)/2}} \right] \left(\frac{2\psi}{N}\right)^{d-2} + \ldots \quad (A.8)$$

2. We define the abbreviation

$$H_\alpha(\psi) := \frac{1}{(4\pi)^{d/2}} \left[\left| \Gamma(-\alpha) \right| - 2 \sum_{\mathbf{q}^*}' \frac{K_\alpha(2\psi|\mathbf{q}|)}{(\psi|\mathbf{q}|)^\alpha} \right], \quad (A.9)$$

where $\sum_{\mathbf{q}^*} = \sum_{\mathbf{q} \in \mathbb{Z}^{d^*}}$ is only extended over the finite directions. In the spherical model, the equilibrium magnetisation $M_{\text{eq}}^2 = 1 - T/T_c$, where the critical temperature $1/T_c = W_d(0)$ [25, 28, 34, 43, 51]. For quenches to $T < T_c$ one has $M_{\text{eq}}^2 > 0$. Then, using (A.1) and (A.8)

$$\begin{aligned}
\overline{g}(p) &\simeq \frac{1}{2D} \frac{W_d(0) - H_{(d-2)/2}(\psi)\left(\frac{2\psi}{N}\right)^{d-2} + \ldots}{1 - TW_d(0) + TH_{(d-2)/2}(\psi)\left(\frac{2\psi}{N}\right)^{d-2} + \ldots} \\
&\simeq \frac{1}{2DT_c} \frac{1}{M_{\text{eq}}^2} - \frac{1}{2D} \frac{1}{M_{\text{eq}}^4} H_{(d-2)/2}(\psi) \left(\frac{2\psi}{N}\right)^{d-2} + \ldots \\
&= \frac{1}{2DT_c} \frac{1}{M_{\text{eq}}^2} - \frac{1}{2DM_{\text{eq}}^4} \frac{|\Gamma(1 - d/2)|}{(4\pi)^{d/2}} \left(\frac{p}{2D}\right)^{(d-2)/2} \\
&\quad + \frac{2}{2DM_{\text{eq}}^4} \frac{1}{(4\pi)^{d/2}} \left(\frac{p}{2D}\right)^{(d-2)/4} \sum_{\mathbf{q}^*}' \left(\frac{N|\mathbf{q}|}{2}\right)^{(2-d)/2} K_{(d-2)/2}\left(\frac{N|\mathbf{q}|}{\sqrt{2D}} p^{1/2}\right) \quad (A.10)
\end{aligned}$$

gives the leading terms of $\overline{g}(p)$ for small values of p. The first two of these terms are the bulk contributions, while the remaining ones give the leading finite-size effects.

The leading long-time behaviour of $g(t)$ is then obtained via the identities [83]

$$\mathcal{L}^{-1}\left(p^{\nu/2} K_\nu(2ap^{1/2})\right)(t) = \frac{1}{2} \frac{a^\nu}{t^{\nu+1}} e^{-a^2/t}, \quad (A.11a)$$

$$\mathcal{L}^{-1}\left(p^{-\nu}\right)(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}, \quad (A.11b)$$

and we find, where from now on both d and d^* can be considered as continuous parameters

$$\begin{aligned}
g(t) &= \frac{1}{2DT_c} \frac{1}{M_{\text{eq}}^2} \delta(t) + \frac{1}{M_{\text{eq}}^4} \frac{1}{(8\pi Dt)^{d/2}} + \frac{1}{M_{\text{eq}}^4 (8\pi Dt)^{d/2}} \sum_{\mathbf{q}^*}' \exp\left(-\pi \frac{N^2}{8\pi Dt} |\mathbf{q}|^2\right) \\
&= \frac{1}{2DT_c} \frac{1}{M_{\text{eq}}^2} \delta(t) + \frac{(8\pi Dt)^{-d/2}}{M_{\text{eq}}^4} \vartheta_3\left(0, \exp\left(-\pi \frac{N^2}{8\pi Dt}\right)\right)^{d^*} \quad (A.12)
\end{aligned}$$

with the Jacobi Theta function ϑ_3 [72], which obeys the functional identity

$$\vartheta_3\left(0, e^{-\pi y}\right) = y^{-1/2} \vartheta_3\left(0, e^{-\pi/y}\right). \quad (A.13)$$

Figure 3 illustrates the rapid cross-over (essentially in the interval $\frac{1}{2} \lesssim y \lesssim 2$) between the two asymptotic regimes. Therefore, we have the following asymptotic limits, for $2 < d < 4$ and $T < T_c$

$$g(t) = \frac{1}{2DT_c} \frac{1}{M_{\text{eq}}^2} \delta(t) + \begin{cases} \frac{(8\pi Dt)^{-d/2}}{M_{\text{eq}}^4}; & \text{if } N^2/t \gg 1 \quad \text{infinite-size system,} \\ \frac{(8\pi Dt)^{-(d-d^*)/2}}{M_{\text{eq}}^4} N^{-d^*}; & \text{if } N^2/t \ll 1 \quad \text{finite-size system.} \end{cases} \quad (A.14)$$

This shows that the long-time behaviour of the spherical constraint in a finite geometry is effectively $(d - d^*)$-dimensional. The singular terms in (A.12) and (A.14) will become very important for the calculation of the correlators, as we shall see below.

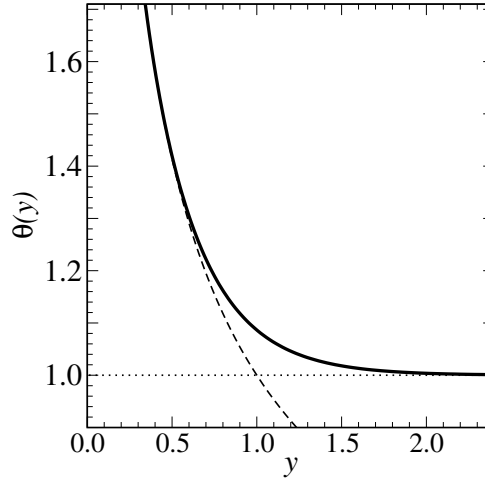Equation (A.12) is the main result of this subsection.

Figure 3. The function $\theta(y) := \vartheta_3\left(0, e^{-\pi y}\right)$ and its cross-over between the regimes where $y \gg 1$ and $\theta(y) \simeq 1$ (dotted line) and $y \ll 1$ and $\theta(y) \simeq y^{-1/2}$ (dashed line).

A.2. Two-time auto-correlator

We decompose in (A.12) $g(t) = g_{\text{sing}}(t) + g_{\text{reg}}(t)$, where $g_{\text{sing}}(t) = \frac{1}{2DT_c} \frac{1}{M_{\text{eq}}^2} \delta(t)$. In momentum space, with the convention $t > s$, we have from (4.5), for large times, the decomposition

$$
\begin{aligned}
\widehat{C}(t, s; \mathbf{k}) &= \frac{e^{-2D\omega(\mathbf{k})(t+s)}}{\sqrt{g_{\text{reg}}(t) g_{\text{reg}}(s)}} \left[1 + \frac{2DT}{2DT_c} \frac{1}{M_{\text{eq}}^2} \int_0^s d\tau\, \delta(\tau) e^{2D\omega(\mathbf{k})2\tau} + 2DT \int_0^s d\tau\, g_{\text{reg}}(\tau) e^{2D\omega(\mathbf{k})2\tau} \right] \\
&= \frac{1}{M_{\text{eq}}^2} \frac{e^{-2D\omega(\mathbf{k})(t+s)}}{\sqrt{g_{\text{reg}}(t) g_{\text{reg}}(s)}} + 2DT \int_0^s d\tau\, \frac{g_{\text{reg}}(\tau)}{\sqrt{g_{\text{reg}}(t) g_{\text{reg}}(s)}} e^{-2D\omega(\mathbf{k})(t+s-2\tau)} \quad \text{(A.15)}
\end{aligned}
$$

for all temperatures $T < T_c$. With (4.6), this gives the two-time auto-correlator $C(t, s) = C^{[1]}(t, s) + C^{[2]}(t, s)$. The first term in (A.15) leads to

$$
\begin{aligned}
C^{[1]}(t, s) &= \frac{|\Lambda|^{-1} M_{\text{eq}}^{-2}}{\sqrt{g_{\text{reg}}(t) g_{\text{reg}}(s)}} \sum_{\mathbf{k}} \exp\left[-2D \sum_{j=1}^d \left(1 - \cos\frac{2\pi}{N_j} k_j \right) (t+s) \right] \\
&= \frac{M_{\text{eq}}^{-2}}{\sqrt{g_{\text{reg}}(t) g_{\text{reg}}(s)}} \prod_{j=1}^d \sum_{q_j \in \mathbb{Z}} e^{-2D(t+s)} I_{N_j q_j}(2D(t+s)) \\
&\simeq \frac{M_{\text{eq}}^{-2}}{\sqrt{g_{\text{reg}}(t) g_{\text{reg}}(s)}} \frac{1}{[4\pi D(t+s)]^{d/2}} \prod_{j=1}^{d^*} \sum_{q_j \in \mathbb{Z}} \exp\left[-\frac{(Nq_j)^2}{4D(t+s)} \right] \left[1 + \mathrm{O}\left((t+s)^{-1}\right) \right] \\
&= M_{\text{eq}}^2 \left(\frac{t^{d/2} s^{d/2}}{[(t+s)/2]^d} \right)^{1/2} \left[\frac{\vartheta_3\left(0, \exp\left(-\pi \frac{N^2}{4\pi D(t+s)}\right)\right)}{\sqrt{\vartheta_3\left(0, \exp\left(-\pi \frac{N^2}{8\pi Dt}\right)\right) \vartheta_3\left(0, \exp\left(-\pi \frac{N^2}{8\pi Ds}\right)\right)}} \right]^{d^*}, \quad \text{(A.16)}
\end{aligned}
$$

where in the first line (A.3) with $m = 0$ was used once more. In the second line, we use the asymptotic expansion of the modified Bessel function $I_n(x)$. In the third and forth lines, $g_{\text{reg}}(t)$ was inserted with

$N_j = N$ for $j = 1, \ldots, d^*$ from (A.12) and the sums in the same line were expressed in terms of the Jacobi theta function ϑ_3. Both d and d^* can be taken as continuous variables.

The second term in (A.15) can be expressed as a convolution

$$C^{[2]}(ys, s) = \frac{2DT}{\sqrt{g_{\text{reg}}(ys)g_{\text{reg}}(s)}} \, \mathscr{L}^{-1}\left(\overline{g_{\text{reg}}}(p) \left(\overline{\left[e^{-4D((y+1)s/2)}I_0(4D(y+1)s/2)\right]^d}\right)(p)\right)(s). \quad \text{(A.17)}$$

For $s \to \infty$, a Tauberian theorem relates the leading behaviour to the one of the Laplace transform at $p \to 0$ [81]. In turn, the behaviour of the two factors should be dominated by the long-time behaviour of the original functions. Therefore, one expects the leading contribution to be of the order [g_0 is the amplitude of $g_{\text{reg}}(\tau)$]

$$\begin{aligned}
C^{[2]}(ys, s) &\simeq \frac{2DT}{\sqrt{g_{\text{reg}}(ys)g_{\text{reg}}(s)}} \int_0^s \mathrm{d}\tau \, g_0 \tau^{-d/2} \left(8\pi D \frac{y+1}{2}(t + s - 2\tau)\right)^{-d/2} \\
&\simeq 2DT(ys^2)^{d/4} s^{1-d} \int_0^1 \mathrm{d}v \, v^{-d/2} \left(4\pi D(y+1)(y+1-2v)\right)^{-d/2} \\
&= \mathrm{O}\left(Ts^{1-d/2}\right)
\end{aligned} \quad \text{(A.18)}$$

up to an s-independent amplitude. For $d > 2$, $C^{[2]}(ys, s)$ is negligible in the scaling limit where $s \to \infty$. Hence, for all temperatures $T < T_c$, the leading term of the auto-correlator is $C(t, s) = C^{[1]}(t, s)$.

Finally, introducing the scaling variables Z and y in (A.15), and with the scaling $8\pi D \overset{!}{=} 1$, we arrive at (4.11a). With (A.13), the equivalent form (4.11b) is obtained.

A.3. Single-time correlator

We reuse the decomposition $g(t) = g_{\text{sing}}(t) + g_{\text{reg}}(t)$ from above. In momentum space, we decompose $\widehat{C}(t; \mathbf{k}) = \widehat{C}^{[1]}(t; \mathbf{k}) + \widehat{C}^{[2]}(t; \mathbf{k})$ and have for all $T < T_c$

$$\begin{aligned}
\widehat{C}(t; \mathbf{k}) &= \frac{e^{-4D\omega(\mathbf{k})t}}{g_{\text{reg}}(t)} + \frac{2DT}{g_{\text{reg}}(t)} \int_0^t \mathrm{d}\tau \left[\frac{1}{2DT_c}\frac{1}{M_{\text{eq}}^2}\delta(\tau) + g_{\text{reg}}(\tau)\right] e^{-4D\omega(\mathbf{k})(t-\tau)} \\
&= \frac{e^{-4D\omega(\mathbf{k})t}}{M_{\text{eq}}^2 \, g_{\text{reg}}(t)} + 2DT \int_0^t \mathrm{d}\tau \, \frac{g_{\text{reg}}(\tau)}{g_{\text{reg}}(t)} e^{-4D\omega(\mathbf{k})(t-\tau)}.
\end{aligned} \quad \text{(A.19)}$$

Herein, the first term is analysed as follows

$$\begin{aligned}
C^{[1]}(t; \mathbf{n}) &= \frac{|\Lambda|^{-1}}{M_{\text{eq}}^2 \, g_{\text{reg}}(t)} \sum_{\mathbf{k}} \exp\left[\sum_{j=1}^d \frac{2\pi \mathrm{i}}{N_j} k_j n_j - 4D\left(1 - \cos\frac{2\pi}{N_j}k_j\right)t\right] \\
&= \frac{e^{-4Ddt}}{M_{\text{eq}}^2 \, g_{\text{reg}}(t)} \sum_{\mathbf{q} \in \mathbb{Z}^d} \prod_{j=1}^d I_{N_j q_j + n_j}(4Dt) \\
&\simeq \frac{M_{\text{eq}}^2}{\vartheta_3\left(0, e^{-\pi N^2/(8\pi Dt)}\right)^{d^*}} \prod_{j=1}^d \sum_{q_j \in \mathbb{Z}} e^{-(q_j N_j + n_j)^2/(8Dt)},
\end{aligned} \quad \text{(A.20)}$$

where first the full identity (A.3) is used d times, then the asymptotic form of the modified Bessel function $I_n(x)$ is used for $x \gg 1$ and finally, in the chosen finite-size geometry, the asymptotic form (A.12) is inserted. The product over the sums in the last line of (A.20) is evaluated as follows: (i) in the $d - d^*$

infinite directions where formally $N_j = \infty$, only the terms with $q_j = 0$ contribute and lead to a factor $\exp\left[-\frac{1}{8Dt}\sum_{j=d^*+1}^{d} n_j^2\right]$. (ii) The d^* finite directions with $N_j = N$ produce d^* factors, each of the form

$$\sum_{q_j \in \mathbb{Z}} \exp\left[-\frac{(q_j N + n_j)^2}{8Dt}\right] = \mathrm{e}^{-n_j^2/(8Dt)} \sum_{q_j \in \mathbb{Z}} \exp\left[-\frac{N n_j}{4Dt} q_j - \frac{N^2}{8Dt} q_j^2\right]. \tag{A.21}$$

With the identity

$$\mathrm{e}^{-n_j^2/(8Dt)}\, \vartheta_3\left(\mathrm{i}\pi\frac{N n_j}{8\pi Dt}, \mathrm{e}^{-\pi N^2/(8\pi Dt)}\right) \;=\; \frac{\sqrt{8\pi Dt}}{N}\vartheta_3\left(\pi\frac{n_j}{N}, \mathrm{e}^{-\pi[N^2/(8\pi Dt)]^{-1}}\right) \tag{A.22}$$

we finally obtain [and used again (A.13)]

$$\begin{aligned}
C^{[1]}(t;\mathbf{n}) &= M_{\mathrm{eq}}^2 \exp\left(-\pi\sum_{j=1}^{d}\frac{n_j^2}{8\pi Dt}\right)\prod_{j=1}^{d^*}\frac{\vartheta_3\left(\mathrm{i}\pi\frac{N n_j}{8\pi Dt}, \mathrm{e}^{-\pi N^2/(8\pi Dt)}\right)}{\vartheta_3\left(0, \mathrm{e}^{-\pi N^2/(8\pi Dt)}\right)} \\
&= M_{\mathrm{eq}}^2 \exp\left(-\pi\sum_{j=d^*+1}^{d}\frac{n_j^2}{8\pi Dt}\right)\prod_{j=1}^{d^*}\frac{\vartheta_3\left(\pi\frac{n_j}{N}, \mathrm{e}^{-\pi[N^2/(8\pi Dt)]^{-1}}\right)}{\vartheta_3\left(0, \mathrm{e}^{-\pi[N^2/(8\pi Dt)]^{-1}}\right)}.
\end{aligned} \tag{A.23}$$

The second term can be rewritten as follows

$$C^{[2]}(t;\mathbf{n}) = 2DT\sum_{\mathbf{q}}\int_0^t \mathrm{d}\tau\,\frac{g_{\mathrm{reg}}(\tau)}{g_{\mathrm{reg}}(t)}\prod_{j=1}^{d}\mathrm{e}^{-4D(t-\tau)}I_{q_j N_j + n_j}(4D(t-\tau)) \tag{A.24}$$

and takes on the form of a convolution. For large times $t \to \infty$, we estimate this asymptotically by appealing to Tauberian theorems [81]. Then, the leading term should become

$$\begin{aligned}
C^{[2]}(t;\mathbf{n}) &\simeq \frac{2DT}{(8\pi D)^{d/2}}\int_0^t \mathrm{d}\tau\, t^{-d/2}\left(1 - \frac{\tau}{t}\right)^{-d/2}\exp\left[-\pi\sum_{j=1}^{d}\frac{n_j^2}{8\pi D(t-\tau)}\right] \\
&\quad \times \prod_{j=1}^{d^*}\frac{\vartheta_3\left(\mathrm{i}\pi\frac{N n_j}{8\pi D(t-\tau)}, \mathrm{e}^{-\pi N^2/(8\pi D(t-\tau))}\right)\vartheta_3\left(0, \mathrm{e}^{-\pi N^2/(8\pi D\tau)}\right)}{\vartheta_3\left(0, \mathrm{e}^{-\pi N^2/(8\pi Dt)}\right)} \\
&\sim \mathrm{O}\left(Tt^{1-d/2}\right),
\end{aligned} \tag{A.25}$$

which becomes negligible in the long-time limit $t \to \infty$ for $d > 2$.

Therefore, in the long-time limit $t \to \infty$, $C(t;\mathbf{n}) = C^{[1]}(t;\mathbf{n})$. Introducing the scaling variables (3.5) into (A.23), and reusing (A.13) and (A.22) and scaling $8\pi D \stackrel{!}{=} 1$, we arrive at equations (4.10).

A.4. Characteristic length

The characteristic lengths $L(t)$ are defined from (4.15), with the single-time correlator given by (A.23). If the distances are calculated along the coordinate axes in one of the d^* finite directions, i.e., $\mathbf{n} = (n, 0, \ldots, 0)$, we find a *longitudinal length* $L_\parallel$. If n is measured along one of the infinite directions, we find a *transverse length* $L_\perp(t)$.

The most simple example of a transverse length arises if the distances are measured along one of the coordinate axes in one of the infinite directions [i.e., $\mathbf{n} = (0, 0, \ldots, n)$ with $d^* \leqslant d - 1$]

$$L_\perp^2(t) = \frac{\sum_{n=-\infty}^{\infty} n^2 \exp\left[-\pi\frac{n^2}{8\pi Dt}\right]}{\sum_{n=-\infty}^{\infty} \exp\left[-\pi\frac{n^2}{8\pi Dt}\right]} \simeq 8\pi Dt\,\frac{\int_{-\infty}^{\infty}\mathrm{d}n\, n^2\,\mathrm{e}^{-\pi n^2}}{\int_{-\infty}^{\infty}\mathrm{d}n\,\mathrm{e}^{-\pi n^2}} = 4Dt, \tag{A.26}$$

which is identical to the known result for the bulk system [78].

A longitudinal length is found when $\mathbf{n} = (n, 0, \ldots, 0)$ with $d^* \geqslant 1$ is measured along one of the coordinate axes in a finite direction. If $N = 2M$ is even, we have

$$L_\parallel^2(t) = \frac{\sum_{n=-M+1}^{M} n^2 \, \vartheta_3\left(\pi \frac{n}{2M}, \mathrm{e}^{-\pi/Z}\right)}{\sum_{n=-M+1}^{M} \vartheta_3\left(\pi \frac{n}{2M}, \mathrm{e}^{-\pi/Z}\right)}. \tag{A.27}$$

Using the definition of the Jacobi Theta function ϑ_3, we have

$$\sum_{n=-M+1}^{M} \vartheta_3\left(\pi \frac{n}{2M}, \mathrm{e}^{-\pi/Z}\right) = \sum_{p \in \mathbb{Z}} \sum_{n=-M+1}^{M} \exp\left[-\pi\mathrm{i}\frac{n}{M}p - \frac{\pi p^2}{Z}\right]$$

$$= \; 2M + \sum_{p \neq 0} \mathrm{e}^{-\pi p^2/Z} \left(1 + \mathrm{e}^{-\pi\mathrm{i}p} + \sum_{n=1}^{M-1} \mathrm{e}^{-\pi\mathrm{i}(n/M)p} + \sum_{n=1}^{M-1} \mathrm{e}^{\pi\mathrm{i}(n/M)p}\right) = \; 2M \tag{A.28}$$

and

$$\sum_{n=-M+1}^{M} n^2 \, \vartheta_3\left(\pi \frac{n}{2M}, \mathrm{e}^{-\pi/Z}\right) = \sum_{p \in \mathbb{Z}} \sum_{n=-M+1}^{M} n^2 \exp\left[-\pi\mathrm{i}\frac{n}{M}p - \frac{\pi p^2}{Z}\right]$$

$$= \; \sum_{n=-M+1}^{M} n^2 + \sum_{p \neq 0} \mathrm{e}^{-\pi p^2/Z} \left[0 + M^2 \mathrm{e}^{-\pi\mathrm{i}p} + \sum_{n=1}^{M-1} n^2 \, \mathrm{e}^{-\pi\mathrm{i}(n/M)p} + \sum_{n=1}^{M-1} n^2 \, \mathrm{e}^{\pi\mathrm{i}(n/M)p}\right]$$

$$\simeq \; \frac{2}{3}M^3 + M^2 + \sum_{p \neq 0} \mathrm{e}^{-\pi p^2/Z} \left[M^2(-1)^p + \frac{4(-1)^p}{\pi^2 p^2}M^3 + (-1)^p M^2\right] + \mathrm{O}(M)$$

$$\simeq \; \frac{2}{3}M^3 + \frac{8M^3}{\pi^2} \sum_{p=1}^{\infty} \mathrm{e}^{-\pi p^2/Z} \frac{(-1)^p}{p^2} + \mathrm{O}(M^2), \tag{A.29}$$

where in the third line, an asymptotic expansion for M large was made. Inserting (A.28) and (A.29) into (A.27) and fixing $8\pi D = 1$ gives (4.16). The same leading result also holds if $N = 2M + 1$ is odd.

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Нерівноважні релаксації: явища старіння та скінченно-вимірні ефекти

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Досліджено довгочасову поведінку кореляторів "спін-спін" при повільній релаксації системи в умовах кінетики фазового впорядкування та геометрій скінченного розміру. Феноменологічний скінченно-вимірний скейлінговий анзац сформульовано та перевірено через точний розв'язок кінетичної сферичної моделі для системи, миттєво охолодженої до температур нижче критичної та при розмірностях $2 < d < 4$.

Ключові слова: *фазові переходи, скінченно-вимірний скейлінг, динамічні критичні явища, ефекти старіння, точні моделі*