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Novel phase transition in two-dimensional xy-models with long-range interaction

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The purpose of this article is to give an overview of results concerning ordering and critical properties of two-dimensional ferromagnets including the dipolar interaction. We investigate a two-dimensional xy-model extended by the dipolar interaction. Describing our system by a nonlinear σ-model and using renormalization group methods we predict a phase transition to an ordered state. This transition is due to the long-range dipolar interaction. The ferromagnetic phase is governed by a low temperature fixed-point with infinite dipolar coupling. In the critical regime we find exponential behavior for the correlation length and the order parameter in contrast to the usual power laws. The nature of the transition shows a striking similarity to the Kosterlitz-Thouless transition. We show that there is a whole class of long-range xy-models leading to such non-standard behavior. Parameterizing the divergencies in terms of the correlation length we are able to calculate the critical exponents. These exponents are correct in any loop order.

Key words: *critical phenomena, magnetic properties, low-dimensional systems*

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Dedicated to Reinhard Folk on the occasion of his 60th birthday

1. Introduction

Magnetic order in two dimensions has been investigated from various authors. In a two-dimensional ferromagnet with isotropic exchange interaction the order parameter has to vanish. On the other hand, a finite order parameter is possible when rotational invariance is broken, in case the interaction contains uniaxial contributions. Even for the magnetic dipolar interaction, which does not favor any particular direction, magnetic fluctuations are sufficiently reduced, leading to a finite order parameter below a transition temperature. In this article we want to describe the critical properties in the ordered phase [1].

The low temperature phase of the two-dimensional xy-model shows critical behavior, but long-range order is suppressed by strong fluctuations. The absence of spontaneous order for isotropic, short-range systems was rigorously proven by Mermin and Wagner [2]. The high temperature phase with exponentially decaying correlations is, according to Kosterlitz and Thouless, due to the existence of free vortices [3–5]. On the contrary, for low temperatures, these topological excitations are bound closely in pairs of charge zero. Introducing a long-range force alters the low temperature properties of the xy-model in a crucial way. Due to the long-range interaction the fluctuations are reduced and the system orders at a certain temperature, thereby showing a very special critical behavior.

Situations described by a dipolar xy-model also appear in experimental systems. For example consider a two-dimensional layer of magnetic moments. Such layers are created by depositing magnetic atoms on a non-magnetic substrate (e.g. [6–8]). In the simplest picture the magnetic properties are described by a 2d Heisenberg-model with isotropic short-range interaction. But for a more realistic model one also has to be aware of the dipolar interaction between the magnetic moments and the existence of magneto-crystalline anisotropies due to spin-orbit coupling [9,10]. Under certain conditions, when the spins prefer to order in the plane of the magnetic atoms, it is sufficient to concentrate on the isotropic exchange and the dipolar interaction and to deal with a two-component spin (for details see [1]). In this case it is possible to describe the large scale properties of the system by a continuum model

$$\frac{\mathcal{H}[\boldsymbol{S}]}{t} = \frac{1}{2t}\int \mathrm{d}^2x\,(\partial_\mu \boldsymbol{S}(\boldsymbol{x}))^2 - \frac{H}{t}\int \mathrm{d}^2x\,S^x(\boldsymbol{x}) + \frac{G}{4\pi t}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\frac{(\boldsymbol{\partial S}(\boldsymbol{x}))(\boldsymbol{\partial' S}(\boldsymbol{x}'))}{|\boldsymbol{x}-\boldsymbol{x}'|}\,, \tag{1}$$

where $\boldsymbol{S}$ is a two-component spin constrained by $\boldsymbol{S}^2 = 1$, t is an effective temperature, H is the external field and G measures the strength of the dipolar interaction.

Due to the dipolar interaction this model no longer becomes trivial by introducing polar coordinates. So we prefer to renormalize (1) in the coordinates of the nonlinear σ-model [11,13,14,16]. As the dipolar interaction violates the $O(2)$-symmetry the proof of renormalizability of Brézin, Zinn-Justin and Le Guillou is no longer applicable [12]. The model described by the Ginzburg-Landau functional (1) and which we call dxy-model is instead invariant only if we rotate spin and coordinate-vectors simultaneously. With the help of this invariance we show in [1] that the dxy-model is renormalizable.

The next step is to express the spins by independent coordinates of the nonlinear σ-model

$$(S_x, S_y) = (\sigma, \pi), \qquad \sigma = \sqrt{1-\pi^2}, \tag{2}$$

and to expand the functional in terms of π. According to symmetry considerations it is not necessary to renormalize each arising term. Instead, it is sufficient to renormalize the field π, the temperature t and the dipolar coupling G. Usually one renormalizes the nonlinear σ-model at the lower critical dimension. By including the long-range forces this lower critical dimension becomes $d_{\mathrm{L}} = 1$ [15]. One of the constraints arising from the form of the dipolar interaction in (1) is the restriction for the number of components N to $N = d$, where d is the dimension of the lattice. Hence working at the lower critical dimension reduces the number of components to $N = d_{\mathrm{L}} = 1$. As one component is eliminated via the condition $\boldsymbol{S}^2 = 1$ one effectively deals with a zero component field. Therefore an ϵ-expansion about the lower critical dimension extrapolates from a trivial situation. To avoid this problem we work in the fixed dimension $d = N = 2$ and refrain from using an ϵ-expansion. Moreover, we use a cutoff regularization and work out the renormalization-group equations for the non-renormalized correlation functions following the ideas of Zinn-Justin [17]. Using this method one reduces the two-dimensional integrals to a single integration. For more details see [1]. Finally we derive the Callan-Symanzik equations

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta_t(t,g)\frac{\partial}{\partial t} + \beta_g(t,g)\frac{\partial}{\partial g} + \frac{n}{2}\zeta(t,g) + \rho(t,g)H\frac{\partial}{\partial H}\right)G^{(n)}(\{\boldsymbol{q}_i\},t,g,H,\Lambda) = 0, \tag{3}$$

where Λ is the cutoff, g is the dimensionless dipolar coupling G/Λ and $G^{(n)}$ are the n-point correlation functions. Doing a one loop calculation and expanding the flow functions β_t and β_g in the vicinity of the xy-line ($t = t_0, g = 0$) one obtains for the linear order

$$\beta_t(t,g) = \frac{5t_0^2}{32\pi}\,g, \tag{4a}$$

$$\beta_g(t,g) = \left(\frac{t_0}{2\pi} - 1\right)g\ . \tag{4b}$$

Obviously any point on the xy-line is a fixed point. But the stability properties change at $t_{\mathrm{c}} \equiv 2\pi$. While the xy-line is stable for $t > t_{\mathrm{c}}$ it becomes unstable for $t < t_{\mathrm{c}}$ due to the dipolar interaction[1]. At the critical point ($t_{\mathrm{c}} = 2\pi, g_{\mathrm{c}} = 0$) the linear order vanishes and the critical behavior is determined by the nonlinear contributions to (4b).

The global flow predicted by the renormalization group is depicted in figure 1. One observes that for temperatures below t_{c}, where the xy-line is unstable, there is a flow towards a low temperature fixed point with infinite dipolar coupling. This fixed point is attractive for the regions I and II of the flow diagram and determines the asymptotic properties. In region III the long-range interaction is irrelevant and the system shows classical xy-behavior. As we used the low temperature form of the Hamiltonian and did not consider any topological excitations, the xy-line persists to be stable in region III. Our physical predictions are therefore restricted to the low-temperature and to the critical part of the flow diagram. In the vicinity of the critical point the flow diagram shows a striking similarity to the flow diagram of the Kosterlitz-Thouless transition [4]. But while in the Kosterlitz-Thouless model the vortices destabilize the high-temperature part of the xy-line here it is the low temperature part which becomes unstable due to the long-range interaction.

[1] The change of stability at $t = 2\pi$ is in agreement with the results of [15,18]. Our general scaling form (13) reduces to the results of [15,18] in the limit of vanishing g.

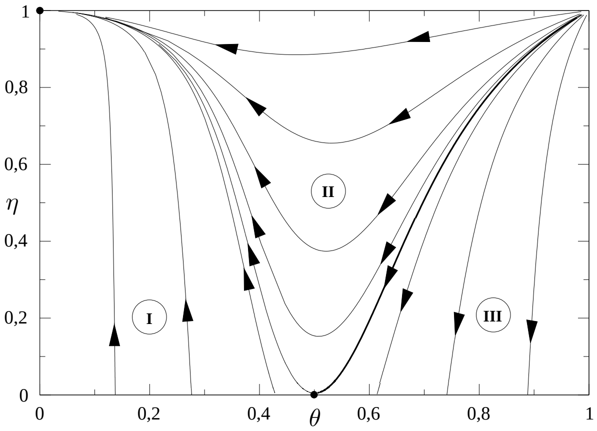

Figure 1. Renormalization group flow of the dxy-model with $\theta = t/(t_c+t)$ and $\eta = g/(1+g)$. The trajectory crossing the critical point ($t_c = 2\pi$, $g_c = 0$) divides the diagram into three different regions. In region III the xy-line (g=0) is stable against the perturbation from the long-range interaction, while it is unstable in region I. For region I and II the low temperature fixed point with infinite dipolar coupling is attractive. Bold printed is the critical trajectory of the system.

In order to find an interpretation for the flow diagram we solve the Callan-Symanzik equations for the correlation length ξ and the spontaneous magnetization M_0. From the numerical solution one obtains a finite correlation length ξ that diverges by approaching the xy-line or the critical trajectory. Hence as in the Kosterlitz-Thouless transition, the destabilization of the xy-line leads to a finite correlation length. On the other hand, the evaluation of the spontaneous magnetization shows that there exists a finite order parameter in the regions I and II of the diagram. This order parameter vanishes at the critical trajectory and on the xy-line. Therefore it is well justified to call the bold printed curve in figure 1 the critical trajectory. At a first glance it is surprising that the critical point has the fixed point value $g_c = 0$, although it is just the long-range interaction which is responsible for the ferromagnetic phase. But one has to bear in mind, that in this description every non-vanishing value of g produces a finite magnetization. As the order parameter has to vanish at the critical point, the fixed point has to be located at the xy-line.

In the vicinity of the critical point it is possible to obtain some analytic results. Expanding around the critical point, the lowest orders of the flow equations are

$$s\,\frac{\mathrm{d}\vartheta(s)}{\mathrm{d}s} = \lambda g(s), \tag{5a}$$

$$s\,\frac{\mathrm{d}g(s)}{\mathrm{d}s} = -\mu g^2(s) + \frac{1}{2\pi}\,g(s)\vartheta(s), \tag{5b}$$

where $\vartheta = t - t_c$ and λ and μ are non-universal constants. In order to label the different trajectories we define τ to be the value of the trajectory at $t = t_c$. As concerns the explicit form of the trajectories $g(t,\tau)$ a positive value of τ means that the trajectory lies in region II, while negative values indicate region I and III. Using this convention an explicit solution of (5) is given by

$$g(t,\tau) = g_{\rm c}(t) + \tau \exp\left[\frac{\mu}{\lambda}\,(t - t_{\rm c})\right] . \tag{6}$$

Here $g_{\rm c}(t)$ is the trajectory traversing the critical point and the integration constant τ measures the distance from the critical trajectory according to the above definition. In this approximation the diverging parts of the correlation length are

$$\xi \propto \exp\left[\int_{t_0}^{t_{\rm c}} {\rm d}t' \, \frac{1}{\beta_t(t', g(t',\tau))}\right] . \tag{7}$$

Expanding (6) around the critical point one can determine the asymptotic behavior of the correlation length to be

$$\xi(\tau) \propto \exp\left(\frac{b}{\sqrt{\tau}}\right) \tag{8}$$

with the non-universal constant $b = \sqrt{\pi^3/\lambda}$. Following Kosterlitz we interpret τ as the relative temperature. Hence the correlation length diverges exponentially.

Solving the Callan-Symanzik equations for the spontaneous magnetization one derives for the behavior of M_0 in the vicinity of the critical point

$$M_0(\tau) \propto \exp\left[-\frac{1}{2}\int_{t_0}^{t_{\rm c}} {\rm d}t' \, \frac{\zeta(t', g(t',\tau))}{\beta_t(t', g(t',\tau))}\right] . \tag{9}$$

As the ζ function takes the finite value $\zeta^\star = t_{\rm c}/(2\pi)$ at the critical point the asymptotic behavior is given by

$$M_0(\tau) \propto \xi^{-\zeta^\star/2} , \tag{10}$$

defining the critical exponent $\tilde{\beta} \equiv \beta/\nu$ to equal $\zeta^\star/2$.

The asymptotic forms of the critical correlation function and the magnetization on the critical isotherm can be calculated in the usual manner to give

$$G^{(2)}(q) \propto q^{-2+\zeta^\star}, \tag{11}$$

$$M(H) \propto H^{\frac{\zeta^\star}{4-\zeta^\star}}, \tag{12}$$

which yields the exponents $\eta = \zeta^\star$ and $\delta = (4-\zeta^\star)/\zeta^\star$, in accordance with the results of the short-range xy-model. In order to find the exponents α and γ we solve the Callan-Symanzik equations (3), obtaining the general scaling form of the correlation functions

$$G^{(n)}\left(\{\boldsymbol{q}_i\}, t, g, H, \Lambda\right) = M_0^n \, \xi^{2(n-1)} \, g^{(n)}\left(\{\boldsymbol{q}_i \xi\}, H\,\frac{M_0\,\xi^2}{t}, \tau\right) . \tag{13}$$

For the following we assume that the critical behavior is contained in the first two arguments, while we can use $\tau = 0$ in the vicinity of the critical point for the last argument. Then the above scaling form yields for the free energy, which is just $G^{(0)}$

$$F(\xi, H) = \xi^{-2} f\left(H\xi^{2-\zeta^\star/2}\right) . \tag{14}$$

Comparing this with the usual scaling form

$$F \propto \tau^{\beta(1+\delta)} f(H\tau^{-\delta\beta}) \propto \xi^{-\tilde{\beta}(1+\delta)} f(H\xi^{\delta\tilde{\beta}}) \tag{15}$$

one obtains the two equations $\tilde{\beta}(1+\delta) = 2$ and $\delta\tilde{\beta} = 2 - \zeta^\star/2$ which are satisfied by the exponents resulting from (10) and (12). The specific heat can be calculated by transforming the derivative with respect to τ into a derivative with respect to the correlation length by means of equation (8). Neglecting logarithms it follows

$$C_H \propto \xi^{-2} \ , \tag{16}$$

predicting an exponent $\tilde{\alpha} \equiv \alpha/\nu = -2$, which is the same as the exponent of the Kosterlitz-Thouless transition. The exponent γ can be obtained by considering the scaling form of the uniform susceptibility χ

$$\chi \propto \xi^{2-\zeta^\star} \, \hat{\chi}(H\xi^{2-\zeta^\star/2}) \ . \tag{17}$$

Comparing this result with the usual scaling form leads to $\tilde{\gamma} \equiv \gamma/\nu = 2 - \zeta^\star$.

In the framework of the ϕ^4-model it is known that nonanalytic interactions like the dipolar interaction renormalize trivially, that is to say that there are no contributions from perturbation theory [15]. Within the nonlinear σ-model the situation is less clear, because there exist vertices of a nonanalytic structure. However, a careful analysis shows that the dipolar interaction behaves trivially also in the nonlinear σ-model. The same is true for a generalized isotropic long-range interaction of the form

$$g\Lambda^{2-\sigma} \int_q |\boldsymbol{q}|^\sigma \boldsymbol{S}(\boldsymbol{q})\boldsymbol{S}(-\boldsymbol{q}) \ . \tag{18}$$

As long as $\sigma < 2$, this interaction is long-ranged and nonanalytic. For the considered models the flow function β_g is due to the trivial renormalization given by

$$\beta_g(t,g) = -g\left(2 - \sigma - \zeta(t,g) - \frac{\beta_t(t,g)}{t}\right) . \tag{19}$$

with $\sigma = 1$ for the dxy-model. Using the known properties of the short-range 2d xy-model [2] and expanding the flow functions around the critical point one again obtains the flow equations (5) [3]. Therefore, the basic structure of the flow equations (5) and the location of the critical point remain unchanged in any loop order. For the interaction (18) the critical fixed point is now given by $(t_c = 2\pi(2-\sigma), g_c = 0)$, which is shifted towards the Polyakov fixed point [19] $(t_c = 0, g_c = 0)$ for $\sigma \to 2$. Since the fixed point is located at the xy-line one can use the value [11]

$$\zeta^\star = \zeta(t_c, 0) = \frac{t_c}{2\pi} \tag{20}$$

[2] One has to use the properties $\beta_t \to c_1 t g$ and $\zeta \to t/(2\pi) + c_2 g$ in the vicinity of the xy-line.

[3] For the generalized long-range isotropic interaction this can be verified by comparing with the results of Sak [20]. Although he excluded the case $N = d = 2$ his flow functions are still valid and result in flow equations having the structure (5) with $\mu = 0$. But also for $\mu = 0$ renormalization group leads to the exponential behavior (8) of the correlation length.

Table 1. Critical exponents for the generalized isotropic long-range xy-model (first row). The fixed point value of ζ is given by $\zeta^\star = t_c/(2\pi) = 2 - \sigma$. The second and the third row show the exponents for the dipolar xy-model ($t_c = 2\pi$) and the Kosterlitz-Thouless transition ($t_c = \pi/2$). Exponents marked by a tilde parameterize the divergence in terms of the correlation length.

	$\tilde{\beta}$	$\tilde{\gamma}$	$\tilde{\alpha}$	δ	η
long-range xy-model	$\zeta^\star/2$	$2 - \zeta^\star$	-2	$(4 - \zeta^\star)/\zeta^\star$	$\zeta^\star$
dipolar xy-mode	$1/2$	1	-2	3	1
Kosterlitz-Thouless		$7/4$	-2	15	$1/4$

for the ζ function, which is exact in the absence of topological excitations. Hence we conclude that the analytic structure of the divergence (8) and the derived values for the critical exponents are correct in any loop order.

The calculated exponents for the different models are summarized in table 1. As for the Kosterlitz-Thouless transition the exponents fulfill the usual scaling laws if one uses $\nu = \infty$ [4]. The exponents for the dxy-model agree with the isotropic long-range model for $\sigma = 1$. The value of $\sigma = 1$ corresponds to an interaction that has the same scaling behavior as the dipolar interaction. Therefore, the anisotropy of the dxy-model is not important here for the critical behavior. Surprisingly the exponents of the Kosterlitz-Thouless transition fit into the structure of the generalized long-range model, as one can see by using $\zeta^\star = 1/4$. The reason is that the asymptotic properties are determined by the nature of the critical point. As for the long-range models the fixed point of the Kosterlitz-Thouless transition is located at the xy-line and this enforces the same nonlinear behavior. This can also be verified by comparing (5) with the flow equations of the Kosterlitz-Thouless transition [4]. The value $\zeta^\star = 1/4$ is in accordance with the critical temperature $t_c = \pi/2$ of the unbinding transition. As concerns the critical temperature and the exponents, the vortex interaction behaves similar to a long-range force scaling like $q^{7/4}$.

Finally we want to make some remarks concerning the role of topological excitations. As the transition temperature of the Kosterlitz-Thouless transition ($t_v = \pi/2$) is lower than the temperature of the ferromagnetic transition ($t_f = 2\pi$) in the dxy-model, one might speculate about an unbinding transition taking place in the dipolar system. A detailed investigation of the influence of the dipolar interaction on the behaviour of topological excitations is beyond the scope of this contribution. But staying in the vortex picture [4] one can try to estimate the main effect of the dipolar interaction on the behavior of the vortices. Using some approximations this leads to

[4] Taking the dipolar forces into account the topological excitations would no longer be vortices, but would be modified especially on large scales.

the following vortex part of the Hamiltonian [1]

$$\frac{\mathcal{H}_v}{t} = -\frac{1}{2t}\sum_{i,j}^{i\neq j} q_i q_j \left(2\pi \ln\left(\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{a}\right) + G\alpha|\boldsymbol{x}_i - \boldsymbol{x}_j|\right) + \frac{\mu}{t}\sum_i q_i^2 \,, \qquad (21)$$

where the q_i and $\boldsymbol{x}_i$ are charge and position of the vortices, μ is the chemical potential of the vortices and α is a positive constant. The most important part of the Hamiltonian is the linear attractive force between the vortices. By means of the methods of Kosterlitz and Thouless [3] it becomes clear that there cannot be an unbinding transition as long as the dipolar interaction is finite. Therefore the vortices are bound closely and do not effect the large-scale behavior. Summarizing we conclude that our renormalization group results describe the asymptotic behavior of the system correctly below the transition temperature. Above the transition one expects a high temperature phase with disordered spins and exponential correlations.

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Новий фазовий перехід у двовимірній xy-моделі з далекосяжною взаємодією

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Метою цієї статті є дати огляд результатів, що стосуються впорядкування і критичних властивостей двовимірних феромагнетиків, які включають дипольні взаємодії. Ми досліджуємо двовимірну xy-модель розширену дипольними взаємодіями. Описуючи нашу систему нелінійною σ-моделлю і використовуючи ренормалізаційно-групові методи ми передбачаємо фазовий перехід до впорядкованого стану. Цей перехід з'являється в результаті далекосяжної дипольної взаємодії. В критичному режимі ми знаходимо експоненційну поведінку для кореляційної довжини та параметра порядку на відміну від звичайних степеневих законів. Природа переходу виявляє разючу подібність з переходом Костерліца-Таулеса. Ми показуємо, що існує цілий клас далекосяжних xy-моделей, які приводять до такої нестандартної поведінки. Параметризуючи розбіжності в термінах кореляційної довжини, ми здатні обчислити критичні показники.Ці показники є точними в будь-якому петлевому порядку.

Ключові слова: *критичні явища, магнітні властивості, низьковимірні системи*

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