Solliton-like order parameter distributions in the critical region

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Some exact one-component order parameter distributions for the Michelson thermodynamic potential are obtained. The phase transition of second kind in Ginzburg-Landau type model is investigated. The exact partial distribution of the order parameter in the form of Jacobian elliptic function is obtained. The energy of this distribution is lower at some temperature interval than for the best known models.

Key words: symmetry, phase transition, order parameter, soliton, modulated structure

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1. Introduction

The problem of spontaneous symmetry breaking (SSB) at the second kind phase transition (PT), which is the basis of many theories and processes in physics is notable for the fact that even in an $\theta$-approximation it cannot be reduced to the model of free field [1]. This is conditioned by the strong interacation and correlation in the system of anomalously growing strong critical fluctuations – the system with infinite number of degrees of freedom.

In most physical systems, the number of order parameter (OP) components in the critical region decreases, and many PT can be described by a scalar one-component OP (as in ferroelectrics, numerous magnetics, superconductors without magnetic fields, etc.). In this case SSB for the discrete group of reflection symmetry ($\varphi \rightarrow -\varphi$, where $\varphi(x)$-OP field) takes place in the system [2,3].

We consider the following model which describes the PT to the commensurate phase through the phases with modulated structures:

$$\Phi_0 = \tilde{\Phi}_0 \int \left[ \nu_0'' + g_0^2 \tilde{\varphi}_0' - \gamma_0 \tilde{\varphi}_0 - r_0 \tilde{\varphi}_0' - s_0 \tilde{\varphi}_0'' + \frac{h_0}{3} \tilde{\varphi}_0^6 \right] d\tilde{x},$$

where $\tilde{\varphi}_0(x)$ is the scalar OP field ($\tilde{\varphi}_0'' = d^2\tilde{\varphi}_0/d\tilde{x}^2$); $\tilde{\Phi}_0, g_0, \gamma_0, r_0, s_0, h_0$ are material constants [4–6].

Without the gradient terms this model describes the second kind PT from phase $\tilde{\varphi}_0 = 0$ to phase $\tilde{\varphi}_0 = \sqrt{-r_0/s_0}$, occurring at $r_0$ sign inversion. In this case the system has one degree of freedom. The inclusion into $\Phi_0$ of only one gradient term $\sim (\nabla \tilde{\varphi}_0)^2$ (Ginzburg-Landau-Wilson model) adds to $\tilde{\varphi}_0$ the continuum of freedom degrees, interconnected by shot-range interaction, and that allows us to take into account both the correlation and the interaction of fluctuation $\tilde{\varphi}_0(x)$. However, the stable $\tilde{\varphi}_0(x)$ structure can arise only as a result of the competition between several gradient terms or due to the specific boundary condition.

Using the scale transform of $\tilde{\varphi}_0(x)$, $\tilde{x}$ as well as by relationship:

$$\tilde{\varphi}_0 (\tilde{x}) = \sqrt{|\gamma_0|} \varphi \left( x \sqrt{|\gamma_0|} \right), \quad r_0 = q \gamma_0^2, \quad s_0 = p \gamma_0, \tilde{x} \sqrt{|\gamma_0|} = x,$$

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121
we reduce the number of parameters by 3, clearing coefficient under \((\varphi^2)\), \((\varphi^6)\) and \(\varphi^2\):

\[
\Phi = \Phi_0 \int \left[ \varphi'' + g^2 \varphi^2 \varphi' - \varphi^2 + q \varphi^4 + \frac{p}{2} \varphi^4 + \frac{1}{3} \varphi^6 \right] dx.
\]

(3)

The Euler-Poisson variational equation for functional (1) has the following form:

\[
\varphi^{IV} + g \left( \varphi^2 \varphi'' + \varphi \varphi^2 \right) + \varphi'' + q \varphi + p \varphi^3 + \varphi^5 = 0.
\]

(4)

2. Analysis of order parameter asymptotic behavior

The first integral of equation (4) has the form:

\[
2 \varphi' \varphi''' - \varphi''^2 + g^2 \varphi^2 \varphi'' + q \varphi^2 + \frac{p}{2} \varphi^4 + \frac{1}{3} \varphi^6 = D,
\]

(5)

where \(D\) – integration constant.

Equation (5) by means of substitution:

\[
\varphi' = 2A(\varphi),
\]

(6)

can be easily led to the form [7]:

\[
4AA'' - A^2 + 2gA \varphi^2 + 2B \varphi^2 + \frac{p}{2} \varphi^4 + \frac{1}{3} \varphi^6 = 0.
\]

(7)

One can introduce the function \(U(\varphi) = A(\varphi)\) and obtain the following equation:

\[
\frac{16}{3} U^* U'' + 2g \varphi^2 U^* + 2B U^* + q \varphi^2 + \frac{p}{2} \varphi^4 + \frac{1}{3} \varphi^6 = 0.
\]

(8)

In case \(\varphi(x) \to 0\) one can eject the terms with infinitesimal order more than \(\varphi^2\):

\[
U^* U'' + \frac{3}{8} \gamma U^* + \frac{3}{16} q \varphi^2 = 0.
\]

(9)

Equation (9) is equivalent to the linearized initial equation (4), it makes the scale invariance possible. Anomalous scale dimension of equation (9) is \(\Delta = 3/2\). Due to this fact by means of substitution:

\[
U(\varphi) = \varphi^{-\Delta} z(t), \quad \varphi = e^t,
\]

(10)

it is possible to dispose of obvious dependence on the argument:

\[
z'' + 2z' + \frac{3}{4} z + \frac{3}{8} \gamma z^{-2} + \frac{3}{16} q z^{-2} = 0.
\]

(11)

The analysis of equation (11) shows that if \(\varphi(x) \to 0\) then \(z(t) \to \text{const}\), and therefore \(A(\varphi) \sim \varphi^2\). The analysis of \(\varphi(x)\) pole shows that if \(\varphi(x) \to \infty\), then \(A(\varphi) \sim \varphi^4\).

3. Exact partial solution of variational equation

Consider the solutions of (5), which satisfy the following relation:

\[
\varphi''^2 = a + b \varphi^2 + c \varphi^4 \ldots,
\]

(12)

where \(b \gg 0, c \ll 0\).

This approach allows us to investigate the set of PT from the ordered to the disordered state through the phases with modulated OP structures.
In this case the condition of compatibility (5) and (12) will be:

\[ 24c^2 + 3gc + 1 = 0, \quad b^2 + b + q + 12ac + ag = 0, \]
\[ 10bc + gb + c + \frac{p}{2} = 0, \quad 2ab + a = D, \]  

(13)

the set (5) is overdetermined, therefore it is possible to find the relationship between \( g, q, p \), which is the condition of compatibility (5) and (12).

Consider the case \( D = p = 0 \), in this case \( a = 0 \), and the set of equation (13) will take the following form:

\[ 24c^2 + 3gc + 1 = 0, \quad b^2 + b + q = 0, \quad 10bc + gb + c = 0. \]  

(14)

In this case the condition of compatibility (5) and (12) will be:

\[ g^2(q) = \frac{(10b(q) + 1)^2}{3b(q)(2b(q) + 1)}, \]  

(15)

where \( b \) is the solution of the second equation (14). The graph of function \( g^2(q) \) is the line of PT. This function has minimum \( g^2(-7/16) = 32/3 \) and asymptote \( q \to \infty, g^2(q) \to 50/3 \). The corresponding solution of (5) has the following form:

\[ \varphi(x) = \frac{\alpha}{\cosh(\beta x)}, \]  

(16)

and describes the bell-soliton state of OP field [9].

Consider the solution of variational equation (4) of the following form:

\[ \varphi(x) = a \cdot sn(b(x-x_0), k). \]  

(17)

After substituting (17) into (4) we obtain the set of equations for \( a, b, k \):

\[ a^4 + 3g(abk)^2 + 24(bk)^4 = 0, \]
\[ -2(1 + k^2)(ga^2b^2 + 10k^2b^4) + 2(bk)^2 + pa^2 = 0, \]
\[ (1 + 14k^2 + k^4)b^4 + g(ab)^2 - (1 + k^2)b^2 + q = 0. \]  

(18)

The solution of set (18) has the following form [10]:

\[ a^2 = g_s(bk)^2, \quad g_s = 3 \left( \frac{2}{-g - \sqrt{g^2 - \frac{32}{3}}} \right), \]  

(19)

\[ b^2 = \frac{\alpha_s}{1 + k^2}, \quad \alpha_s = \frac{2 + pg_s}{2(10 + gg_s)}, \]  

(20)

\[ k^2 = \left\{ \frac{-[(14 + gg_s)\alpha_s^2 + 2(q - \alpha_s)]}{2[\alpha_s^2 + (q - \alpha_s)]} \right\}, \]  

(21)

\[ g \leq -\frac{10}{\sqrt{6}} q \geq -\frac{1}{4}(16 + gg_s)\alpha_s^2 + \alpha_s . \]  

(22)

Consider the solution of variational equation (4) of the following form:

\[ \varphi(x) = a \cdot cn(b(x-x_0), k). \]  

(23)

After substituting (23) into (4) we obtain the set of equations for \( a, b, k \) [11,12]:

\[ a^4 - 3g(abk)^2 + 24(bk)^4 = 0, \]
\[ 2(1 - 2k^2)(10k^2b^4 - ga^2b^2) - 2(bk)^2 + pa^2 = 0, \]
\[ (1 + 16k^2 + 16k^4)b^4 + g(1 - k^2)(ab)^2 - (1 - 2k^2)b^2 + q = 0. \]  

(24)
If $k^2 \neq 1/2$ then the solutions of set (24) has the following form:

$$a^2 = g_c(kb)^2, \quad b^2 = \frac{\alpha_c}{1 - 2k^2}, \quad k^2 = \frac{1}{2} \left\{ 1 \pm \sqrt{\frac{\alpha_c^2(12 - gg_c)}{\alpha_c^2(16 - gg_c) + 4(q - \alpha_c)}} \right\}, \quad (25)$$

$$g_c \equiv \frac{3}{2} \left\{ g \pm \sqrt{g^2 - \frac{32}{3}} \right\}, \quad \alpha_c \equiv \frac{2 - pg_c}{2(10 - gg_c)}, \quad (26)$$

In case $k^2 = 1/2$ :

$$b^4 = \frac{4g}{12 - gg_c}, \quad a^2 = \frac{g_c}{2} \sqrt{\frac{4g}{12 - gg_c}}. \quad (27)$$

Consider the solution of variational equation (4) of the following form:

$$\varphi(x) = a \cdot \text{dn}(b(x - x_0), k). \quad (28)$$

After substituting (28) into (4) we obtain the set of equations for $a, b, k$:

$$a^4 - 3g(ab)^2 + 24b^4 = 0,$$
$$2(k^2 - 2)(10b^4 - ga^2b^2) - 2b^2 + pa^2 = 0,$$
$$(16 - 16k^2 + k^4)b^4 + g(k^2 - 1)(ab)^2 - (2 - k^2)b^2 + q = 0. \quad (29)$$

Solutions of set (29) has the following form:

$$a^2 = 2\sqrt{3}(kb)^2, \quad b^2 = \frac{\alpha_d}{k^2 - 2}, \quad \alpha_d = \frac{p\sqrt{3} - 1}{2}. \quad (30)$$

4. Conclusion

In the present paper we investigate the PT in Ginzburg-Landau type model with high gradients. New OP distributions were found. These distributions make it possible to describe the set of PT from the ordered to the disordered phase through the set of states with modulated OP structures. We also investigated soliton-like OP distributions. The presence of soliton-like solution in the model considered is determined by variational symmetry of initial Euler-Poisson equation.
References


Солітонно-подібні розподіли параметра порядку в критичній області

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Ключові слова: симетрія, фазовий перехід, параметр порядку, солітон, модульована структура

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125