

Revisiting (logarithmic) scaling relations using renormalization group

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We explicitly compute the critical exponents associated with logarithmic corrections (the so-called hatted exponents) starting from the renormalization group equations and the mean field behavior for a wide class of models at the upper critical behavior (for short and long range ϕ^n -theories) and below it. This allows us to check the scaling relations among these critical exponents obtained by analysing the complex singularities (Lee-Yang and Fisher zeroes) of these models. Moreover, we have obtained an explicit method to compute the $\hat{\nu}$ exponent [defined by $\xi \sim L(\log L)^{\hat{\nu}}$] and, finally, we have found a new derivation of the scaling law associated with it.

Key words: *renormalization group, scaling, logarithms, mean field*

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1. Introduction

One of the main achievements of Wilson's [1] renormalization group (RG) was the definition of universality class by means of a finite number of critical exponents. These critical exponents determine the divergences of some observables at the critical point [2–6].

In particular circumstances, logarithmic corrections arise multiplicatively in these critical laws. These logarithms are of a paramount importance in some materials (for example, dipolar magnets in three dimensions, which is the upper critical dimension of the system [7]) and can be accessed experimentally [8]. Moreover, their effects are very important in the non-perturbative definition of quantum field theories in four dimensions (the so-called trivality problem) [9].

In [10–12], the scaling relations of the exponents which characterize the logarithmic corrections were derived using the Lee-Yang [13] and Fisher zeroes [14] techniques in a model-independent manner. In this paper, we will explicitly compute, using RG and field theory, the value of these exponents and then check the (scaling) relations among them. We have done this for a wide class of models [ϕ^n models at their upper critical dimensions with short (SR) and long range (LR) interactions] and can also be applied to the models in low dimensions (as the four-state Potts model in two dimensions).

In the presence of logarithmic corrections, the scaling laws for the observables near the critical point must be modified as [2–6]¹

$$\xi \sim |t|^{-\nu} |\log |t||^{\hat{\nu}}, \quad (1.1)$$

$$C \sim |t|^{-\alpha} |\log |t||^{\hat{\alpha}}, \quad (1.2)$$

$$m \sim |t|^{\beta} |\log |t||^{\hat{\beta}} \text{ for } t < 0, \quad (1.3)$$

$$\chi \sim |t|^{-\gamma} |\log |t||^{\hat{\gamma}}, \quad (1.4)$$

¹We use in the definition of the critical exponents the standard notation, see, for example, [2–6, 10, 11].

$$m \sim h^{1/\delta} |\log h|^{\hat{\delta}} \text{ for } t = 0, \quad (1.5)$$

$$r_{\text{LY}} \sim |t|^{\Delta} |\log |t||^{\hat{\Delta}}, \quad (1.6)$$

$$G(r) \sim \frac{(\log r)^{\hat{\eta}}}{r^{d-2+\eta}} \text{ for } t = 0, \quad (1.7)$$

which define the so-called hatted exponents (d being the dimension). The standard critical exponents (e.g., α , β , γ , etc.) satisfy the classic scaling laws (see, for example, [2–6]). In addition, in [10–12], it was shown that the hatted exponents satisfy the following scaling relations:²

$$\hat{\Delta} = \hat{\beta} - \hat{\gamma}, \quad (1.8)$$

$$\hat{\beta}(\delta - 1) = \delta \hat{\delta} - \hat{\gamma}, \quad (1.9)$$

$$\hat{\eta} = \hat{\gamma} - \hat{\nu}(2 - \eta). \quad (1.10)$$

Finally, see [10–12], at the infinite volume critical point, the correlation length of the system defined on a finite box of size L behaves as³

$$\xi \sim L(\log L)^{\hat{\varrho}} \quad (1.11)$$

and the associated related scaling relation is

$$\hat{\alpha} = d\hat{\varrho} - d\hat{\nu}. \quad (1.12)$$

When $\alpha = 0$ and the impact angle of the Fisher zeros satisfies $\phi \neq \pi/4$, the previous relation should be modified as [11]

$$\hat{\alpha} = 1 + d\hat{\varrho} - d\hat{\nu}. \quad (1.13)$$

Finally, an additional scaling relation can be written [12]

$$2\hat{\beta} - \hat{\gamma} = d\hat{\varrho} - d\hat{\nu}. \quad (1.14)$$

In this paper we will mainly analyze generic ϕ^n theories, with Hamiltonian (for simplicity we write the scalar version for the short range model):⁴

$$\mathcal{H} = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{1}{n} g_n \phi^n \right]. \quad (1.15)$$

2. Some mean field results

We will use RG to analyze the critical behavior of the models, and after a finite number of RG step we will finish in the parameter region in which we can apply mean field results. In this section we will briefly review the basic facts of the scaling in this mean field region [4, 5].

We start with the free energy per spin for a ϕ^n -theory:

$$f(m) = \frac{r_0}{2} m^2 + \frac{g_n}{n} m^n. \quad (2.1)$$

Minimizing $f(m)$, for $r_0 < 0$, we obtain magnetization as:

$$m = \left(\frac{|r_0|}{g_n} \right)^{1/(n-2)} \sim \frac{1}{g_n^{p_m}}, \quad (2.2)$$

²Recall $\delta = (d + 2 - \eta)/(d - 2 + \eta)$.

³In this paper we avoid the mean field region by working at and below the upper critical region.

⁴Using power counting, we can compute when the coupling g_n is marginal, obtaining the so-called upper critical dimension, that for short range models is

$$d_u = \frac{2n}{n-2}$$

and for long range models (with propagator $1/q^\sigma$)

$$d_u = \frac{n\sigma}{n-2}.$$

For $\sigma = 2$, we recover the short range result.

where $p_m = 1/(n-2)$ and

$$f_{\min} \propto \frac{r_0^{n/(n-2)}}{g_n^{2/(n-2)}}. \quad (2.3)$$

The susceptibility is

$$\chi \propto |r_0|, \quad (2.4)$$

and the specific heat

$$C \propto \frac{r_0^{2/(n-2)}}{g_n^{2/(n-2)}} \sim \frac{1}{g_n^{p_c}}, \quad (2.5)$$

where $p_c = 2/(n-2)$. Finally, we can add a magnetic field [which induces a term $-hm$ in equation (2.1)] and compute the minimum of the free energy just at the critical point, $r_0 = 0$ (which is relevant in the computation of the critical isotherm)

$$f_{\min}(r_0 = 0, h) \propto \frac{h^{n/(n-1)}}{g_n^{1/(n-1)}}, \quad (2.6)$$

and the magnetization at criticality is

$$m(r_0 = 0, h) \propto \left(\frac{h}{g_n}\right)^{p_h}, \quad (2.7)$$

where $p_h = 1/(n-1)$.⁵ Hence, since $n > 2$, g_n is an irrelevant dangerous variable for magnetization, critical isotherm and specific heat, yet, χ is free of this problem.

3. Revisiting logarithmic corrections

The starting point is the behavior of the singular part of the free energy density (that we denote simply as f and denoting g_n by g) under a RG transformation

$$f(t_0, h_0, g_0) = \frac{1}{b^d} f[t(b), g(b), h(b)], \quad (3.1)$$

where b is the RG scaling factor and $t(b)$, $h(b)$ and $g(b)$ (the running couplings) denote the evolution of different couplings under a RG transformation, which are obtained solving the following differential equations (we write them for the LR model)

$$\frac{dt}{d \log b} = t[\sigma + \bar{\gamma}(g)], \quad (3.2)$$

$$\frac{d \log h}{d \log b} = \frac{d}{2} + 1 - \frac{\gamma}{2}, \quad (3.3)$$

$$\frac{dg}{d \log b} = \beta_W(g), \quad (3.4)$$

which define the functions β_W , γ and $\bar{\gamma}$.⁶ For further use we define two functions $F(b)$ and $\zeta(b)$ and we assume the following asymptotic behavior [$g_0 \equiv g(1)$]

$$F(g) \equiv \exp \left[\int_{g_0}^{g(b)} dg \frac{\bar{\gamma}(g)}{\beta_W(g)} \right] \sim b^a (\log b)^p, \quad (3.5)$$

$$\zeta(g) \equiv \exp \left[-\frac{1}{2} \int_{g_0}^{g(b)} dg \frac{\gamma(g)}{\beta_W(g)} \right] \sim b^c (\log b)^x. \quad (3.6)$$

⁵The introduction of p_m , p_h and p_c will be useful at the upper critical dimension to collect the extra logs yielded by the g renormalizing to zero in a logarithmic way. Below the upper critical dimension, g_n is not a dangerous irrelevant variable: in this situation, we will use $p_m = p_c = p_h = 0$, i.e., there will be no extra logs from the $g_n(b)$ in the mean field region.

⁶We can compute the thermal and magnetic critical exponents by means of $\eta = \gamma(g^*)$ and $1/\nu = \sigma + \bar{\gamma}(g^*)$, where g^* satisfies $\beta_W(g^*) = 0$ [3, 15].

The solutions are (we also write the asymptotic behavior as $b \rightarrow \infty$) as follows:

$$t(b) = t_0 b^\sigma \exp \left[\int_{g_0}^{g(b)} dg \frac{\bar{\gamma}(g)}{\beta_W(g)} \right] \sim t_0 b^{\sigma+a} (\log b)^p, \quad (3.7)$$

$$h(b) = h_0 b^{\frac{d}{2}+1} \exp \left[-\frac{1}{2} \int_{g_0}^{g(b)} dg \frac{\gamma(g)}{\beta_W(g)} \right] \sim h_0 b^{\frac{d}{2}+1+c} (\log b)^x, \quad (3.8)$$

$$\log b = \exp \left[\int_{g_0}^{g(b)} dg \frac{1}{\beta_W(g)} \right]. \quad (3.9)$$

In the asymptotic regime (and for the models under consideration in this paper where $\beta_W \propto g^s$), the last equation can be written as

$$g(b) \sim (\log b)^{-r}, \quad (3.10)$$

and this defines the r exponent ($1/r = s - 1$). In particular, the useful relation $t(b^*) = 1$ can be written as

$$b^* \sim t_0^{-1/(\sigma+a)} (\log t_0)^{-p/(\sigma+a)}. \quad (3.11)$$

Therefore, we can identify $\nu = 1/(\sigma + a)$ and $\hat{\nu} = -p/(\sigma + a) = -p\nu$. From the form of $h(b)$, one can obtain $c = -\eta/2$. By computing suitable derivatives of the free energy per spin [see equation (3.1)] and using the renormalized couplings given by equations (3.7)–(3.9), and in the case of the upper critical dimension using the expression of the intensive free energy in the mean field regime [equations (2.2), (2.4)–(2.6)], we can obtain the following relations for the exponents which control the logarithmic corrections (see the appendix for more details)

$$\hat{\alpha} = -d\hat{\nu} + r p_c, \quad (3.12)$$

$$\hat{\gamma} = \hat{\nu}(2 - \eta) + 2x, \quad (3.13)$$

$$\hat{\beta} = -\hat{\nu} \left(\frac{d}{2} - 1 + \frac{\eta}{2} \right) + x + r p_m, \quad (3.14)$$

$$\hat{\delta} = \frac{2xd}{d+2-\eta} + r p_h, \quad (3.15)$$

$$\hat{\Delta} = -\hat{\nu} \left(\frac{d}{2} + 1 - \frac{\eta}{2} \right) - x + r p_m, \quad (3.16)$$

$$\hat{\eta} = 2x. \quad (3.17)$$

These equation must be read at the upper critical dimension with $\eta = 0$ (SR) or $\eta = 2 - \sigma$ (LR) and $d = d_u$, otherwise, below d_u , all the p 's from the mean field are zero ($p_m = p_c = p_h = 0$). With these explicit expressions for the hatted exponents, it is easy to re-derive the scaling relations given by equations (1.8)–(1.10), (1.14).

In models with $\alpha = 0$ and impact angle of the Fisher zeroes $\phi \neq \pi/4$, a circumstance equivalent to $A_-/A_+ = 1$ (being A_\pm the critical amplitudes of the specific heat) [16], the scaling of the free energy is modified as⁷

$$f(t_0, h_0, g_0) = \frac{1}{b^d} f(t(b), g(b), h(b)) + \frac{1}{b^d} (\log b) f_l(t(b), g(b), h(b)), \quad (3.18)$$

where the functions f and f_l satisfy additional constraints to generate the right logarithmic corrections (for more details see [16] and references therein). This decomposition of the free energy can be

⁷As described in [16] the appearance of this extra log term in the free energy can be explained either as a resonance between the thermal and the identity operators or as an interplay between the singular and regular parts of the free energy.

also understood in terms of a Lee-Yang and Fisher zeros analysis, see [11, 12]. For instance, in the two-dimensional pure Ising model, only the “energy”-sector develops logarithmic corrections, and these corrections (for the free energy, energy and specific heat) are provided by the term proportional to f_i . However, the scaling of the “magnetic”-sector is given by the standard term, proportional to f . In the two dimensional diluted Ising model, the magnetic sector also shows logarithmic corrections, provided by the (standard) term proportional to f , whereas the corrections for the energy-sector are given by the term proportional to f_i . Hence, only the relation of $\hat{\alpha}$ (which is computed with the f_i -term) should be modified

$$\hat{\alpha} = -d\hat{\nu} + r p_c + 1. \quad (3.19)$$

We have checked that these equations provide correct hatted exponents in $O(N)$ - ϕ^4 models⁸ in the short range and long range interactions, tensor (short range) ϕ^3 (which includes percolation, m -component spin glasses and Lee-Yang singularities, and can also be related with lattice animals), all of them at their upper critical dimension and in the four-state Potts models, pure Ising model and diluted Ising model in two dimensions [12, 15, 17–20]. The logarithmic scaling relations for all these models were thoroughly checked in [12].⁹

Finally, using this theoretical framework we have been able to compute $\hat{\Delta}$ for the four-state two-dimensional Potts model, $\hat{\Delta}$, $\hat{\beta}$, $\hat{\eta}$ and $\hat{\delta}$ for SR tensor ϕ^3 -theories and $\hat{\Delta}$ for the LR $O(N)$ ϕ^4 -theories. Finally, $\hat{\nu}$ has been computed for the LR $O(N)$ ϕ^4 -theories. The numerical values for all these exponents were derived in references [10–12] using the logarithmic scaling relations (1.8)–(1.10), (1.12), (1.14). See [12] for the values of these hatted exponents.

4. A re-derivation of $\hat{\alpha} = d\hat{\nu} - d\hat{\nu}$

We start with the dependence of a singular part of the intensive free energy on L

$$f_{\text{sing}} \propto L^{-d}. \quad (4.1)$$

This is the key point of the derivation. Below the upper critical dimension, one has $L \sim \xi$ and one can write $f_{\text{sing}} \propto \xi^{-d}$, but due to the logarithmic corrections which appear at the upper critical dimension this is no longer true.

We can also write the singular part of the free energy, using the scaling of the specific heat [see equation (1.2)], as

$$f_{\text{sing}} \propto L^{-d} \propto t^{2-\alpha} (\log t)^{\hat{\alpha}}. \quad (4.2)$$

Using equations (1.1) and (1.11) one can write

$$L^{-d} \sim \xi^{-d} (\log \xi)^{d\hat{\nu}} \sim t^{\nu d} (\log t)^{d\hat{\nu}-d\hat{\nu}} \sim t^{2-\alpha} (\log t)^{\hat{\alpha}}. \quad (4.3)$$

Identifying the exponents of $\log t$ of the last two expressions we obtain the scaling relation given by equation (1.12).

When $\alpha = 0$ and $\phi \neq \pi/4$ [11], the free energy scales as $f \propto L^{-d} \log L$ [see equation (3.18) and the discussion of section 3]. This extra-log, using the previous arguments, provides the following scaling law:

$$\hat{\alpha} = 1 + d(\hat{\nu} - \hat{\nu}), \quad (4.4)$$

obtaining equation (1.13).

⁸Where N is the number of components of the field.

⁹In [12] other exponents were defined (e.g., $\hat{\epsilon}$, $\hat{\nu}_c$ and $\hat{\alpha}_c$). It is straightforward to compute them using the theoretical framework of this paper.

5. Computation of the $\hat{\phi}$ -exponent

We will compute the exponent $\hat{\phi}$ for a generic ϕ^n theory at its upper critical dimension for both short and long range models. The starting point is the expression of χ in terms of the free energy¹⁰

$$\chi \sim b^2 \zeta^2 \left. \frac{\partial^2 f(t(b), g(b), h)}{\partial h^2} \right|_{h=0}. \quad (5.1)$$

This can be written as [using $t(b^*) = 1$ and $b^* \sim \xi$]

$$\chi \sim \zeta(\xi)^2 \xi^2 \propto \xi^{2+2c} (\log \xi)^{2x}. \quad (5.2)$$

In a ϕ^n theory we can rescale the field via $\phi' = g^{1/n} \phi$ [21], and the free energy per spin verifies

$$f(t_0, g_0, h_0) = L^{-d} G \left(\frac{t(L)}{g^{2/n}}, \frac{h(L)}{g^{1/n}} \right). \quad (5.3)$$

Differentiating twice equation (5.3) with respect to the magnetic field (h_0), we obtain

$$\chi \propto L^{-d} \left[\frac{\partial h(L)}{\partial h_0} \right]^2 \left. \frac{\partial^2}{\partial h(L)^2} G \left(\frac{t(L)}{g(L)^{2/n}}, \frac{h(L)}{g(L)^{1/n}} \right) \right|_{h_0=t_0=0} \sim L^2 \zeta(L)^2 \frac{1}{g(L)^{2/n}}. \quad (5.4)$$

Comparing with equation (5.2), we finally obtain

$$\xi \sim \frac{L}{g(L)^{2/[n(2+2c)]}}, \quad (5.5)$$

and assuming the asymptotic behavior of $g(L)$ given in equation (3.10) we finally get

$$\hat{\phi} = \frac{2r}{n(2+2c)} = \frac{2r}{n(2-\eta)}. \quad (5.6)$$

For the short range ϕ^4 theory ($\sigma = 2$, $n = 4$, $\eta = -2c = 0$ and $r = 1$) we obtain $\hat{\phi} = 1/4$. For the short range ϕ^3 theory ($\sigma = 2$, $n = 3$, $\eta = -2c = 0$ and $r = 1/2$) we get $\hat{\phi} = 1/6$. In addition, for the long range ϕ^4 model [$n = 4$, $\eta = -2c = (2 - \sigma)$ and $r = 1$], $\hat{\phi} = 1/(2\sigma)$.

Another way to obtain $\hat{\phi}$ is to use the scaling relation provided by equation (1.12) and equation (3.12)

$$\hat{\phi} = \frac{\hat{\alpha}}{d} + \hat{\nu} = \frac{r p_c}{d} \quad (5.7)$$

or for $\alpha = 0$ and $\phi \neq \pi/4$, equations (1.13), (3.19)

$$\hat{\phi} = \frac{\hat{\alpha}}{d} + \hat{\nu} - \frac{1}{d} = \frac{r p_c}{d}, \quad (5.8)$$

obtaining the same final result irrespectively of the value of α and the impact angle ϕ .

So, $\hat{\phi} = 0$ below d_u since $p_c = 0$ therein; at the upper critical dimension (SR models) $d = d_u = 2n/(n-2)$, then $\hat{\phi} = r/n$ as computed before. For LR models, $d_u = n\sigma/(n-2)$ and then we recover the result given by equation (5.6).

In [15, 22], the $\hat{\phi}$ exponent was computed using a misidentification of the correlation length for a lattice of size $L = 1$ [see equations (4.1), (4.2) and (3.11), (3.12) of [22] and [15], respectively], providing, however, with the correct value of $\hat{\phi}$ in general ϕ^4 theories (and, in particular, for the four dimensional diluted model, see reference [22], where the right value of $\hat{\phi} = 1/8$ was obtained [22]) but not in ϕ^3 ones [15]. In this section we have developed a new general method which avoids the previous misidentification of ξ . In particular, we have obtained the correct value of $\hat{\phi} = 1/6$ for the general class of ϕ^3 theories, see above.

¹⁰Since, in this section, we work with the susceptibility, we take into account only the term proportional to f in equation (3.18) independently of the value of α and ϕ . See discussion of section 3.

In [23] it was conjectured that there is a relationship between $\hat{\nu}$ and $1/d_u$ which is frequently an equality but not always so. Indeed, It was already known [24] that $\hat{\nu} = 1/8$ for the four dimensional Ising model which is described by a ϕ^4 theory which has $d_u = 4$. In this paper we have provided the general relation between $\hat{\nu}$ and d_u .

To finish this section, we present two examples in which $\hat{\nu} \neq 1/d_u$ to understand the reasons behind the modification of this behavior. The first one is based on the study of ϕ^{2k} -theories with $k > 2$ and the second one is the two parameter ϕ^4 -theory which describes the four dimensional diluted Ising model.

5.1. ϕ^{2k} -theories with $k > 2$ and short range interactions

The upper critical dimension for these models is $d_u = 2k/(k-1)$. One can compute the RG equations at d_u obtaining [5]

$$\frac{dg_{2k}}{d\log b} \propto g_{2k}^2 \quad (5.9)$$

and so $g_{2k} \propto 1/\log L$ ($r = 1$), that using equation (5.6) provides $\hat{\nu} = 1/(2k)$ which is different to $\hat{\nu} = (k-1)/(2k)$ (only works for $k = 2$).¹¹

5.2. Diluted Ising model

One can obtain an effective field theoretical version of the diluted Ising model by using the replica trick, with effective Hamiltonian given by [22, 25]

$$\mathcal{H}_{\text{eff}}[\phi_i] = \int d^d x \left[\frac{1}{2} \sum_{i=1}^n (\partial_\mu \phi_i)^2 + \frac{r}{2} \sum_{i=1}^n \phi_i^2 + \frac{u}{4!} \left(\sum_{i=1}^n \phi_i^2 \right)^2 + \frac{v}{4!} \sum_{i=1}^n \phi_i^4 \right], \quad (5.10)$$

where v is related with the original Ising coupling and u is a function of the disorder strength. In the replica trick it is mandatory to take the limit of the number of replicas, n , to zero ($i = 1, \dots, n$). The RG equations are, in $d = 4$ and $n = 0$,

$$\frac{dr}{d\log b} = 2r + 4(2u + 3v)(1 - r), \quad (5.11)$$

$$\frac{dv}{d\log b} = -12v(4u + 3v), \quad (5.12)$$

$$\frac{du}{d\log b} = -8u(4u + 3v). \quad (5.13)$$

In the standard ϕ^4 theory one gets $\beta \propto g^2$. Hence, $g \propto 1/\log L$ and $\hat{\nu} = 1/4$. However, the RG flow of the diluted model asymptotically finishes on the line $4u + 3v = O(u^2)$, so we need to include the next (cubic) terms in the perturbative expression and the RG β -functions are no longer quadratic in the couplings. Finally, one finds that $u(b)^2 \sim v(b)^2 \sim 1/\log b$: hence, $\hat{\nu} = 1/8$ as derived in [22, 24].

6. Conclusions

By explicitly computing the hatted critical exponents for a wide family of models we have been able to check the scaling relations among them using the RG framework and the behavior in the mean field regime. Some of these hatted exponents (for some of the models) have been previously derived by using the logarithm scaling relations.

In addition, we have generalized a conjecture regarding a relationship between $\hat{\nu}$ and d_u and derived it.

Finally, we have found a new method to derive the scaling relation associated with $\hat{\nu}$ and we have briefly discussed the logarithmic corrections to the free energy when the Fisher zeros have an impact angle other than $\pi/4$ and $\alpha = 0$.

¹¹In addition, working at d_u for short range models, $\eta = 0$ and so $c = 0$.

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A. Appendix

In this appendix we give additional details of the computation of the hatted exponents, see section 3.

By differentiating once the free energy (3.1) with respect to the magnetic field, then renormalizing to $t(b^*) = 1$, and finally evaluating the magnetization using the mean field behavior (2.2), we obtain

$$m \propto (b^*)^{-d} (b^*)^{d/2+1} \exp \left[-\frac{1}{2} \int_{g_0}^{g(b^*)} dg \frac{\gamma(g)}{\beta_W(g)} \right] \frac{1}{g(b^*)^{p_m}} \sim (b^*)^{-d/2+1+c} (\log b^*)^{x+p_m r}. \quad (\text{A.1})$$

The susceptibility is obtained by differentiating twice the free energy with respect to the magnetic field [notice that there is no dependence on g in the mean field region (2.4)]:

$$\chi \propto (b^*)^{-d} (b^*)^{d+2} \exp \left[-\int_{g_0}^{g(b^*)} dg \frac{\gamma(g)}{\beta_W(g)} \right] \sim (b^*)^{2+2c} (\log b^*)^{2x}. \quad (\text{A.2})$$

To obtain the specific heat, we differentiate twice the free energy with respect to the temperature, renormalize to $t(b^*) = 1$, and evaluate the specific heat using the mean field behavior (1.2), obtaining

$$C \propto (b^*)^{-d} (b^*)^{2\sigma} \exp \left[2 \int_{g_0}^{g(b^*)} dg \frac{\bar{\gamma}(g)}{\beta_W(g)} \right] \frac{1}{g(b^*)^{p_c}} \sim (b^*)^{-d+2\sigma+2a} (\log b^*)^{2p+r p_c}. \quad (\text{A.3})$$

The correlation length is obtained from $t(b^*) = 1$

$$\xi \propto b^* \sim t_0^{-1/(\sigma+a)} (\log t_0)^{-p/(\sigma+a)}. \quad (\text{A.4})$$

By putting the previous relation between b^* and t_0 in equations (A.1)–(A.3) and matching the l.h.s. logarithms [given by equations (1.2)–(1.4)] with the r.h.s. ones [given by equations (A.1)–(A.3)] we obtain equations (3.12)–(3.14).

To compute the Lee-Yang edge, the starting point is the renormalized potential [5]

$$V(t(b), g(b), h(b)) = \frac{t(b)}{2} m^2 + \frac{g(b)}{n} m^n - h(b) m. \quad (\text{A.5})$$

From the constraints $\partial V / \partial m = 0$ and $\partial^2 V / \partial m^2 = 0$ and working in the broken phase with $t(b^*) = -1$, it is possible to show that

$$h(b^*) \sim m(b^*) \sim 1/g(b^*)^{p_m}, \quad (\text{A.6})$$

that can be written as

$$h(b^*) = h_0 (b^*)^{\frac{d}{2}+1} \exp \left[-\frac{1}{2} \int_{g_0}^{g(b^*)} dg \frac{\gamma(g)}{\beta_W(g)} \right] \sim h_0 (b^*)^{\frac{d}{2}+1+c} (\log b^*)^x \sim (\log b^*)^{r p_m}, \quad (\text{A.7})$$

which allows us to compute h_0 as a function of b^* , and knowing $b^*(t_0)$ (A.4), we can easily obtain $h_0(t_0)$. The comparison of the logarithm of $h_0(t_0)$ with that of equation (1.6) provides us with relation (3.16).

Relation (3.17) can be obtained taking the Fourier transform of equation (1.7) at $b \sim 1/q$ (q being the momentum), and comparing this with the renormalized propagator in momentum space (see [6]).

Finally, for the critical isotherm, we start with the free energy computed at the critical point $f(0, h_0, g_0)$, differentiate once with respect to h_0 to compute the critical magnetization, then renormalize to $h(b^*) = 1$ and use the mean field behavior of the critical magnetization (2.7), obtaining

$$m \propto (b^*)^{-d} \frac{h(b^*)}{h_0} \left[\frac{h(b^*)}{g(b^*)} \right]^{p_h} \sim (b^*)^{-d} \frac{1}{h_0} g(b^*)^{-p_h} \sim (b^*)^{-d} \frac{1}{h_0} (\log b^*)^{r p_h}, \quad (\text{A.8})$$

where $h(b^*) = 1$

$$b^* \sim h_0^{-2/(d+2+2c)} (\log h_0)^{-2x/(d+2+2c)}. \quad (\text{A.9})$$

By matching the l.h.s. and r.h.s. logarithms of equations (1.5) and (A.8), respectively, we obtain the relation (3.15).

Remember that $2c = -\eta$, $\nu = 1/(\sigma + a)$ and $\hat{\nu} = -p\nu$.

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Перегляд (логарифмічних) співвідношень скейлінгу з використанням ренормгрупи

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Ми явно обчислюємо критичні показники, пов'язані з логарифмічними поправками, виходячи з рівнянь ренормгрупи і середньопольової поведінки для широкого класу моделей як при вищій критичній вимірності (для коротко- і далекосяжних ϕ^2 -теорій), так і нижче від неї. Це дозволяє нам перевірити співвідношення скейлінгу, що пов'язують критичні показники, аналізуючи комплексні сингулярності (нулі Лі-Янга і Фішера) цих моделей. Окрім того, ми запропонували явний метод для обчислення показника $\hat{\nu}$ [означеного як $\xi \sim L(\log L)^{\hat{\nu}}$] і, накінець, ми отримали нове виведення закону скейлінгу, пов'язаного з цим показником.

Ключові слова: ренормгрупа, скейлінг, логарифми, середнє поле