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Gibbs measures of an Ising model with competing interactions on the triangular chandelier-lattice

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In this paper, we consider an Ising model with three competing interactions on a triangular chandelier-lattice (TCL). We describe the existence, uniqueness, and non-uniqueness of translation-invariant Gibbs measures associated with the Ising model. We obtain an explicit formula for Gibbs measures with a memory of length 2 satisfying consistency conditions. It is rigorously proved that the model exhibits phase transitions only for given values of the coupling constants. As a consequence of our approach, the dichotomy between alternative solutions of Hamiltonian models on TCLs is solved. Finally, two numerical examples are given to illustrate the usefulness and effectiveness of the proposed theoretical results.

Key words: *chandelier lattices, Gibbs measures, Ising model, phase transition*

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1. Introduction

As is known, Cayley tree (or Bethe lattice), introduced by Hans Bethe in 1935, is a non-realistic lattice. Since other operations and the calculations on this lattice are easier to understand than the d-dimensional $\mathbf{Z}^d$ lattice, many of the topics in statistical physics have recently been taken into account on the Cayley tree [1]. Thus, the results obtained on the Cayley tree became a source of inspiration for the d-dimensional $\mathbf{Z}^d$ lattice. As a result, many researchers have employed the Ising and Potts models [2, 3] in conjunction with the Cayley tree [2, 4–6]. The Ising model has relevance to physical, chemical, and biological systems [7–9].

We were then able to identify a similar lattice, which we identified as triple, quadruple, quintuple, and so on. So we examined the dynamic behaviours of Ising models on these Cayley-like lattices. Up till now, some studies have been done [10–13]. Although the results are similar to the results for the models on the Cayley tree, we think that we would get many different results in the future. We called this model a triangular, rectangular, pentagonal and similar "chandelier" model. Compared with $\mathbf{Z}^d$ lattice, we think that the chandelier lattice is more realistic than the Cayley tree [10–14]. In this paper, we deal with a Cayley tree-like lattice [12] which we called a **triangular chandelier lattice** (shortly, TCL) from the configuration model.

The theory of probability is one of the basic branches of mathematics lying at the base of the theory of statistical mechanics [8, 15–20]. As is known, one of the fundamental problems of statistical mechanics is to specify the set of all Gibbs measures associated to the given Hamiltonian [21–25]. A Gibbs measure is a probability measure frequently used in many problems of probability theory and statistical mechanics. It is also known that such measures form a convex compact subset that is different from the void in the set of all probability measures. The number of translation-invariant splitting Gibbs measures associated with the Ising model on a Cayley tree can only be one or more than one, depending on temperature [26]. The p-adic counterpart of the Ising-Vanniminus model on the Cayley tree of order two was first studied in [27]. There was proposed a measure-theoretical approach to investigate the model in the p-adic setting.

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In this present paper, we want to investigate translation-invariant Gibbs measures (TIGMs) corresponding to an Ising model on the TCL. It is well known that the comprehension of phase transitions is one of the most interesting, perhaps the central, problems of equilibrium statistical mechanics [7]. By the phase transition we mean the existence of at least two distinct Gibbs measures associated with the given model [1, 8, 18, 28]. We will investigate the existence of translation invariant Gibbs measures on a wide class of the TCL, restricted only to the memory of length 2. We derive specific realizations for the ANNNI model on these structures. We derive the results within the Markov random field framework, making use of the Kolmogorov consistency conditions. We express the solutions of recurrence relations warranting consistency in terms of the fixed points of a function $f(x)$. We provide some diagrams of the behaviour of the function $f(x)$ for different values of the model parameters. We present analytical developments allowing for the identification of Gibbs measures along usual procedures.

The structure of the present article is as follows: in section 2, we give the necessary definition and preliminaries about Ising model with three competing interactions on a TCL. In section 3, we establish the Gibbs measure associated with the model. In section 4, we describe the existence, uniqueness and non-uniqueness of translation-invariant Gibbs measures associated with the Ising model on a TCL. In section 5, it is rigorously proved that the model exhibits phase transitions only for given values of the coupling constants. As a consequence of our approach, the dichotomy between alternative solutions of Hamiltonian models on TCLs is solved. Finally, in section 6, the relevance of the results obtained for systems on the TCL is discussed and the results are compared to ones on the Cayley tree.

2. Preliminary

2.1. Triangular chandelier lattice

Chandelier lattices are simple connected undirected graphs $G = (V, E)$ (V set of vertices, E set of edges). Let $C^k = (V, E, i)$ be order k chandelier lattice with a root vertex $x^{(0)} \in V$, where each vertex has $(k + 3)$ nearest neighbours with V as the set of vertices and the set of edges. It is clear that the root vertex $x^{(0)}$ has k nearest neighbours. The notation i represents the incidence function corresponding to each edge $e \in E$, with end points $x_1, x_2 \in V$. There is a distance $d(x, y)$ on V, the length of the minimal point from x to y, with the assumed length of 1 for any edge (see figure 1).

Let us consider a chandelier with 3 lamps hanging on the ceiling. Suppose that the same three quadrants hanging on each lamp of the first chandelier were added. In this case, we get a weave that resembles a semi-infinite Cayley tree. We assume here that each lamp is connected to the lamps in the nearest neighbours. Thus, we can have the possibility to investigate the titles examined in statistical

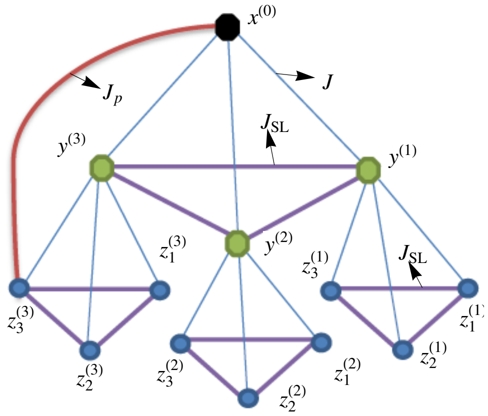

Figure 1. (Colour online) Cayley tree-like lattice: triangular chandelier with 2 level. Three successive generations of TCL (J represents nearest-neighbour interactions; J_p represents prolonged next nearest-neighbour interactions and J_{SL} represents same-level nearest-neighbours interactions.

physics by calculating the internal, external and full energies corresponding to a Hamiltonian on the chandelier lattice that we have defined.

The distance $d(x, y)$, $x, y \in V$, on the chandelier lattice C^k $(k > 2)$, is the number of edges in the shortest path from x to y. The fixed vertex $x^{(0)}$ is called the 0-th level and the vertices in W_n are called the n-th level. For the sake of simplicity we put $|x| = d(x, x^{(0)})$, $x \in V$. We denote the sphere of radius n on V by

$$W_n^{(P)} = \{x \in V : d(x, x^{(0)}) = n\}$$

and the ball of radius n by

$$V_n^{(P)} = \{x \in V : d(x, x^{(0)}) \leqslant n\},$$

where vertex x is prolonged downwards relative to $x^{(0)}$.

$$L_n = \{l = \langle x, y\rangle \in L | x, y \in V_n\}.$$

For example, $W_2^{(P)} = \{z_v^{(u)} : u, v = 1, 2, 3\}$ (see figure 1).

The set of direct prolonged successors of any vertex $x \in W_n$ is denoted by

$$S_k^{(P)}(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

The set of same-level neighbourhoods of any vertex $x \in W_n$ will be denoted by

$$SL_k(x) = \{y \in W_n : d(x, y) = 1\}.$$

It is clear that $|SL_k(x)| = 2$, for all vertexes $x \in W_n$.

Definition 2.1 *Hereafter, we will use the following definitions for neighbourhoods.*

1. *Two vertices x and y, $x, y \in V$ are called* ***nearest-neighbours (NN)*** *if there exists an edge $e \in E$ connecting them, which is denoted by $e = \langle x, y\rangle$.*

2. *The nearest-neighbour vertices $x, y \in V$ that are not prolonged are called* ***same-level nearest-neighbours (SLNN)*** *if $|x| = |y|$ and are denoted by $\widetilde{\rangle x, y\langle}$.*

3. *Two vertices $x, y \in V$ are called* ***the next-nearest-neighbours (NNN)*** *if there exists a vertex $z \in V$ such that x, z and y, z are NN, that is if $d(x, y) = 2$.*

4. *The next-nearest-neighbour vertices $x \in W_n$ and $y \in W_{n+2}$ are called* ***prolonged next-nearest-neighbours (PNNN)*** *if $|x| \neq |y|$ and is denoted by $\rangle x, y\langle$ (see figure 1).*

2.2. Kolmogorov consistency condition

Kolmogorov's extension theorem allows us to construct a variety of measures on infinite-dimensional spaces (see [29] for details).

Now, let us explain this theorem for a one-dimensional situation. Let $S = \{0, 2, \ldots, k-1\}$ be a finite state space. On the infinite product space $\Omega = S^{\mathbf{Z}}$, one can define the product σ-algebra, which is generated by cylinder sets ${}_m[i_1, \ldots, i_N] = \{x \in S^{\mathbf{Z}} : x_m = i_0, \ldots, x_{m+N-1} = i_N\}$ of length N based on the block $(i_1, \ldots, i_N)$ at the place m. Note that a cylinder set is a set of sequences where we fix which symbol can occur in a finite number of places. We denote by $\mathfrak{M}(S^{\mathbf{Z}})$ the set of all measures on $S^{\mathbf{Z}}$. The set of all σ-invariant measures in $S^{\mathbf{Z}}$ is denoted by $\mathfrak{M}_\sigma(S^{\mathbf{Z}})$, where σ is the shift transformation.

Proposition 2.2 *[30, (8.1) Proposition] For $\mu \in \mathfrak{M}_\sigma(S^{\mathbf{Z}})$, the following properties are valid:*

1. $\sum\limits_{i \in S} \mu({}_0[i]) = 1$;

2. $\mu({}_n[i_0, \ldots, i_k]) \geqslant 0$ *for any block* $(i_0, i_1, \ldots, i_k) \in S^{k+1}$ *and any* $n \in \mathbf{Z}$;

3. $\mu({}_n[i_0, \ldots, i_k]) = \sum\limits_{i_{k+1} \in S} \mu({}_n[i_0, \ldots, i_k, i_{k+1}])$;

4. $\mu({}_n[i_0,\ldots,i_k]) = \sum_{i_{-1}\in S} \mu({}_n[i_{-1},i_0,\ldots,i_k])$.

By a special case of Kolmogorov's consistency theorem (see [30]), these properties are sufficient to define a measure. It is well known that a Gibbs measure is a generalization of a Markov measure to any graph. Therefore, any Gibbs measure should satisfy the conditions in the proposition 2.2.

We shall give two examples satisfying the conditions in proposition 2.2 and to illustrate the consistency conditions.

Example 2.1 *Let $\pi = (p_i)_{i\in S}$ be any probability vector on the state set S. For each $n \geqslant 0$, define*

$$\mu_\pi({}_m[i_0,\ldots,i_n]) = p_n(i_0,i_1,\ldots,i_n) = p_{i_0}p_{i_1}\ldots p_{i_n}, \tag{2.1}$$

*where $i_0,i_1,\ldots,i_n \in S$. It is clear that $\{p_n\}_{n\geqslant 0}$ satisfies the consistency conditions (1)–(4) in proposition 2.2 (see [30]). Such a measure μ_π is called a **Bernoulli measure**. One of motivation examples is the Bernoulli measure, which also satisfies the compatible property.*

Example 2.2 *Let $\pi = (p_i)_{i\in S}$ be any probability vector on the state set S and let $P = (p_{ij})_{i,j\in S}$ be any stochastic matrix, i.e., $0 \leqslant p_{ij} \leqslant 1$ and $\sum_{k\in S} p_{ik} = 1$ for each $i,j \in S$. Thus, π is defined as a probability vector such that $\pi P = \pi$. If P is irreducible, π is uniquely defined.*

For each $n \geqslant 0$, the function defined by

$$\mu_{\pi P}({}_m[i_0,\ldots,i_n]) = p_n(i_0,i_1,\ldots,i_n) = p_{i_0}p_{i_0i_1}\ldots p_{i_{n-1}i_n}, \quad \text{where } i_0,i_1,\ldots,i_n \in S$$

*satisfies the consistency conditions (1)–(4) in proposition 2.2 (see [30]). Such a measure $\mu_{\pi P}$ is called a **Markov measure**.*

The proof of the proposition 2.2 can clearly be checked for both the Bernoulli and the Markov measures on σ-algebra [30]. For any cylinder set ${}_m[i_0,\ldots,i_n] = \{x \in S^{\mathbf{Z}} : x_m = i_0,\ldots,x_{m+n-1} = i_n\}$ and any $k \geqslant 1$, we have

$${}_m[i_0,\ldots,i_n] = \bigcup_{i_{n+1}\in S}\cdots\bigcup_{i_{n+k}\in S} ({}_m[i_0,\ldots,i_n,i_{n+1},\ldots,i_{n+k}])$$

and

$$p_n({}_m[i_0,\ldots,i_n]) = \sum_{i_{n+1}\in S}\cdots\sum_{i_{n+k}\in S} p_{n+k}({}_m[i_0,\ldots,i_n,i_{n+1},\ldots,i_{n+k}]).$$

The following is the Kolmogorov extension theorem.

Theorem 2.3 *[29, 4.18 Theorem] Let $S = \{0,1,\ldots,r-1\}$, for some $r \geqslant 2$. Let $\{p_n\}_{n\geqslant 0}$ be a sequence of functions satisfying the consistency conditions, where p_n has domain S^{n+1}. Then, there exists a unique probability measure μ on the measurable space $(\Omega, B(\Omega))$ such that*

$$\mu({}_m[i_0,\ldots,i_n]) = p_n(i_0,i_1,\ldots,i_n)$$

for all $i_0,i_1,\ldots,i_n \in S$ and all $n \geqslant 0$.

3. Gibbs measures

Let us consider the Ising model with competing nearest-neighbour interactions defined by the Hamiltonian

$$H(\sigma) = -J \sum_{\langle x,y\rangle\in L_n} \sigma(x)\sigma(y), \tag{3.1}$$

where the sum runs over nearest-neighbour vertices $\langle x, y\rangle$ and the spins $\sigma(x)$ and $\sigma(y)$ take values in the set $\Phi = \{-1,+1\}$.

Let h_x be a real-valued function of $x \in V$. A finite-dimensional Gibbs distributions on Φ^{V_n} are defined by formula

$$\mu_n(\sigma_n) = \frac{1}{Z_n} \exp\left[-\frac{1}{T} H_n(\sigma_n) + \sum_{x \in W_n} \sigma(x) h_x\right] \tag{3.2}$$

with the associated partition function defined as

$$Z_n = \sum_{\sigma_n \in \Phi^{V_n}} \exp\left[-\frac{1}{T} H_n(\sigma) + \sum_{x \in W_n} \sigma(x) h_x\right],$$

where the spin configurations σ_n belong to Φ^{V_n} and $h = \{h_x \in \mathbf{R}, x \in V\}$ is a collection of real numbers that define boundary condition (see [31]). This distribution is a measure [32, 33].

Bleher and Zalys [32] studied the existence of limit distributions for the ferromagnetic Ising model on infinite diamond-shaped hierarchical lattice (DHL). They have proved that for low temperatures and zero external field, there exist exactly two extreme Gibbs limit distributions, and in other cases the Gibbs distribution is unique.

We say that the probability distributions (3.2) are compatible if for all $n \geqslant 1$ and $\sigma_{n-1} \in \Phi^{V_{n-1}}$:

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}). \tag{3.3}$$

Here, $\sigma_{n-1} \vee \omega_n$ is the concatenation of the configurations. It is clear that the equation (3.3) is the same as condition (4) in proposition 2.2.

In this case, according to theorem 2.3 (the Kolmogorov theorem), there exists a unique measure μ on Φ^V such that, for all n and $\sigma_n \in \Phi^{V_n}$

$$\mu\left(\left\{\sigma|_{V_n} = \sigma_n\right\}\right) = \mu_n(\sigma_n).$$

Such a measure is called a **splitting Gibbs measure** with memory of length 1 corresponding to the Hamiltonian (3.1) and function h_x, $x \in V$ (see [26] for details).

Previously, researchers frequently used memory of length 1 over a Cayley tree to study Gibbs measures [31]. In [1, 21, 22, 34], the authors have studied Gibbs measures with a memory of length 2 for generalized ANNNI models on a Cayley tree of order 2 by means of a vector valued function (see [35] for details). In [1, 21], the next generalizations are considered. These authors have defined Gibbs measures or Gibbs states with a memory of length 2 (on spin-configurations σ) for generalized ANNNI models on Cayley trees of order 2. In [34], the authors have obtained some rigorous results to propose a measure-theoretical approach for the Ising-Vannimenus model.

3.1. New Gibbs measures

In this subsection we are going to construct new Gibbs measures associated with the Ising model on the TCL. We consider the following Hamiltonian

$$H(\sigma) = -J \sum_{\langle x,y \rangle} \sigma(x)\sigma(y) - J_{\mathrm{p}} \sum_{\rangle x,y \langle} \sigma(x)\sigma(y) - J_{\mathrm{SL}} \sum_{\widetilde{\langle x,y \rangle}} \sigma(x)\sigma(y), \tag{3.4}$$

where the first sum ranges all nearest neighbours, the sum in the second term ranges all prolonged next-nearest-neighbours, the third sum ranges all same-level nearest-neighbours and $J, J_{\mathrm{p}}, J_{\mathrm{SL}} \in \mathbf{R}$ are coupling constants (see figure 1). Note that if $J_{\mathrm{SL}} = 0$, then the Hamiltonian (3.4) coincides with the Vannimenus's Hamiltonian [36].

Let us introduce a class of Markov chains on the TCL C^3. For $\Lambda \subset V$ denote $\Phi^{\Lambda} = \{-1, +1\}^{\Lambda}$, the configurational space of the set Λ. For a finite subset V_n of TCL, we define the finite-dimensional Gibbs probability distributions on the configuration space

$$\Omega^{V_n} = \{\sigma_n = \{\sigma(x) = \pm 1, x \in V_n\}\}$$

at inverse temperature $\beta = \frac{1}{kT}$ by formula

$$\mu_{\mathbf{h}}^{(n)}(\sigma_n) = \frac{1}{Z_n}\exp\Bigg[-\beta H_n(\sigma_n) + \sum_{x\in W_{n-1}} \sigma(x)\sigma(y)\sigma(z)\sigma(w)h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)}\Bigg] \tag{3.5}$$

with the corresponding partition function defined by

$$Z_n = \sum_{\sigma_n\in\Omega^{V_n}} \exp\Bigg[-\beta H_n(\sigma_n) + \sum_{x\in W_{n-1}} \sigma(x)\sigma(y)\sigma(z)\sigma(w)h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)}\Bigg],$$

where $y, z, w \in S^{(P)}(x)$.

Let us give the construction of a special class of limiting Gibbs measures associated with the Ising model corresponding to the Hamiltonian (3.4) on the TCL. Firstly, we show that the Gibbs measures associated with the Ising model satisfy the conditions in the proposition 2.2.

The following statement describes conditions on $h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)}$ ensuring compatibility of $\mu_{\mathbf{h}}^{(n)}$.

Theorem 3.1 *Probability distributions $\mu_{\mathbf{h}}^{(n)}$, $n = 1, 2, \ldots,$ in (3.5) are compatible if for any $x \in V$ the following equation holds:*

$$\begin{aligned}\exp\Bigg[\sigma(x)\prod_{v=1}^{3}\sigma(y^{(v)})h_{B_1(x),S^i_{3-i}(\sigma(x))}\Bigg] &= L_2 \sum_{\substack{y^{(v)}\in S(x)\\ z_u^{(v)}\in S(y^{(v)})}} \Bigg\{\exp\Bigg[\beta J\sigma(y^{(v)})\sum_{u=1}^{3}\eta(z_u^{(v)})\Bigg]\\ &\times\exp\Bigg[\beta J_{\mathrm{p}}\sigma(x)\sum_{u,v=1}^{3}\eta(z_u^{(v)})\Bigg]\exp\Bigg[\beta J_{\mathrm{SL}}\Big\{\sigma(z_1^{(v)})\sigma(z_2^{(v)}) + \sigma(z_3^{(v)})\big[\sigma(z_1^{(v)}) + \sigma(z_2^{(v)})\big]\Big\}\Bigg]\\ &\times\prod_{v=1}^{3}\exp\Bigg[\sigma(y^{(v)})\prod_{u=1}^{3}\eta(z_u^{(v)})h_{B_1(y^{(v)}),S^i_{3-i}(\sigma(y^{(v)}))}\Bigg]\Bigg\}.\end{aligned} \tag{3.6}$$

Here, $S(x)$ is the set of direct prolonged successors of x on TCL, and $S(y^{(v)})$ is the set of direct prolonged successors of $y^{(v)}$ on TCL and $L_2 = \frac{Z_1}{Z_2}$.

The proof can be done similarly to [28].

3.2. Basic equations

Denote $B_1(x) = \{x, y, z, w\}$ a unit semi-ball with a center x, where $S(x) = \{y, z, w\}$. For the sake of simplicity, from figure 1, we assume that

$$B_1(x^{(0)}) = \{x^{(0)}, y^{(1)}, y^{(2)}, y^{(3)}\}, \quad B_1(y^{(i)}) = \{y^{(i)}, z_1^{(i)}, z_2^{(i)}, z_3^{(i)}\} \quad \text{for } i = 1, 2, 3.$$

Let $x \in W_n^{(P)}$ for some n and $S(x) = \{y, z, w\}$, where $y, z, w \in W_{n+1}^{(P)}$ are the direct successors of x. We denote the set of all spin configurations on $V_n^{(P)}$ by $\Phi^{V_n^{(P)}}$ and the set of all configurations on unit semi-ball $B_1(x)$ by $\Phi^{B_1(x)}$ (see figure 2). One can get that the set $\Phi^{B_1(x)}$ consists of sixteen configurations

$$\Phi^{B_1(x)} = \left\{\begin{pmatrix} & i & \\ l & & j \\ & k & \end{pmatrix} = (ijkl) : i, j, k, l \in \{-1, +1\}\right\}. \tag{3.7}$$

For example, we have

$$S_3^0(+) = \sigma_1^{(1)} = \begin{pmatrix} & + & \\ + & & + \\ & + & \end{pmatrix} = (++++).$$

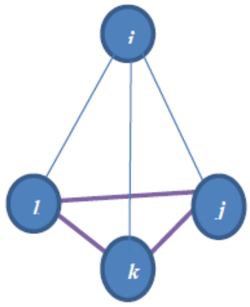

Figure 2. (Colour online) Possible configurations on unit semi-ball $B_1(x)$ on the TCL of order three $(i, j, k, l \in \{-, +\})$.

For the sake of simplicity, let us consider the following abbreviations:

$$
\begin{aligned}
h_1 &= h_{B_1(x),\sigma_1^{(1)}} = h_{B_1(x),S_3^0(+)} = h_{B_1(x),++++},\\
h_2 &= h_{B_1(x),\sigma_2^{(1)}} = h_{B_1(x),\sigma_3^{(1)}} = h_{B_1(x),\sigma_4^{(1)}} = h_{B_1(x),S_2^1(+)} = h_{B_1(x),+-++},\\
h_3 &= h_{B_1(x),\sigma_5^{(1)}} = h_{B_1(x),\sigma_6^{(1)}} = h_{B_1(x),\sigma_7^{(1)}} = h_{B_1(x),S_1^2(+)} = h_{B_1(x),+--+},\\
h_4 &= h_{B_1(x),\sigma_8^{(1)}} = h_{B_1(x),S_0^3(+)} = h_{B_1(x),+---},\\
h_5 &= h_{B_1(x),\sigma_9^{(1)}} = h_{B_1(x),S_3^0(-)} = h_{B_1(x),-+++},\\
h_6 &= h_{B_1(x),\sigma_{10}^{(1)}} = h_{B_1(x),\sigma_{11}^{(1)}} = h_{B_1(x),\sigma_{12}^{(1)}} = h_{B_1(x),S_2^1(-)} = h_{B_1(x),--++},\\
h_7 &= h_{B_1(x),\sigma_{13}^{(1)}} = h_{B_1(x),\sigma_{14}^{(1)}} = h_{B_1(x),\sigma_{15}^{(1)}} = h_{B_1(x),S_1^2(-)} = h_{B_1(x),---+},\\
h_8 &= h_{B_1(x),\sigma_{16}^{(1)}} = h_{B_1(x),S_3^0(-)} = h_{B_1(x),----}.
\end{aligned}
$$

Therefore, we can define the vector-valued function $\mathbf{h} : V \to \mathbf{R}^8$ as follows:

$$\mathbf{h}(x) = (h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8). \tag{3.8}$$

By considering possible configurations as in figure 3 and from (3.6), we can obtain the following equation:

$$
\begin{aligned}
\exp\left[ijkl\, h'_{B_1(x^{(0)}),ijkl}\right] = L_2 \sum_{\substack{a,b,c,d,f,e,\\ g,m,n\in\{-1,+1\}}} \Big(&\exp\left\{J[j(a+b+c)+k(d+e+f)+l(g+m+n)]\right\}\\
&\times \exp\left\{J_{\mathrm{SL}}[bc+a(b+c)+ef+d(e+f)+mn+g(m+n)]\right\}\\
&\times \exp\left[J_{\mathrm{p}}i(a+b+c+d+f+g+m+n+e)\right]\\
&\times \exp\left[abcj\, h_{B_1(y^{(1)}),jabc} + defk\, h_{B_1(y^{(2)}),kdef} + glmn\, h_{B_1(y^{(3)}),lgmn}\right]\Big).
\end{aligned} \tag{3.9}
$$

Here, $i, j, k, l \in \{-1, +1\}$ and $y^{(1)}, y^{(2)}, y^{(3)} \in S(x^{(0)})$ (see figure 3). Therefore, we have obtained an explicit formula for Gibbs measures with memory of length 2 satisfying consistency conditions by means of equation (3.9).

From (3.9), after long and complicated calculations we get the following 8 equations:

$$
\begin{aligned}
\mathrm{e}^{h_1'} &= L_2\left(\frac{\mathrm{e}^{6J+6J_{\mathrm{p}}+4J_{\mathrm{SL}}+h_1}+3\mathrm{e}^{4J+4J_{\mathrm{p}}-h_2}+3\mathrm{e}^{2J+2J_{\mathrm{p}}+h_3}+\mathrm{e}^{4J_{\mathrm{SL}}-h_4}}{\mathrm{e}^{3J+3J_{\mathrm{p}}+J_{\mathrm{SL}}}}\right)^3, \\
\mathrm{e}^{-h_2'} &= L_2\left(\frac{\mathrm{e}^{6J+6J_{\mathrm{p}}+4J_{\mathrm{SL}}+h_1}+3\mathrm{e}^{4J+4J_{\mathrm{p}}-h_2}+3\mathrm{e}^{2J+2J_{\mathrm{p}}+h_3}+\mathrm{e}^{4J_{\mathrm{SL}}-h_4}}{\mathrm{e}^{3J+3J_{\mathrm{p}}+J_{\mathrm{SL}}}}\right)^2
\end{aligned} \tag{3.10}
$$

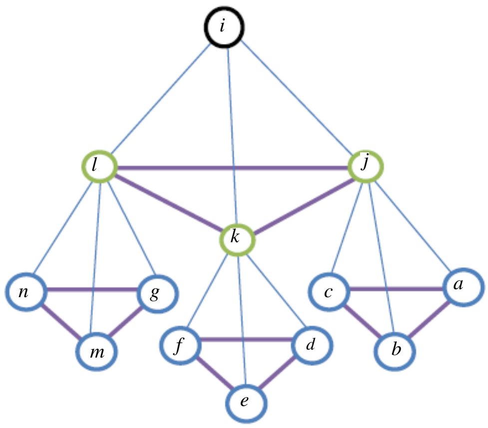

Figure 3. (Colour online) Possible configurations of TCL with level 2 ($\Phi^{V_2^{(P)}}$). Schematic diagram to illustrate the summation used in equation (3.9).

$$\times\left(\frac{e^{6J_p+4J_{SL}-h_5}+3e^{2J+4J_p+h_6}+3e^{4J+2J_p-h_7}+e^{6J+4J_{SL}+h_8}}{e^{3J+3J_p+J_{SL}}}\right), \tag{3.11}$$

$$e^{h_3'} = L_2\left(\frac{e^{6J+6J_p+4J_{SL}+h_1}+3e^{4J+4J_p-h_2}+3e^{2J+2J_p+h_3}+e^{4J_{SL}-h_4}}{e^{3J+3J_p+J_{SL}}}\right)$$
$$\times\left(e^{6J_p+4J_{SL}-h_5}+3e^{2J+4J_p+h_6}+3e^{4J+2J_p-h_7}+e^{6J+4J_{SL}+h_8}\right)^2, \tag{3.12}$$

$$e^{-h_4'} = L_2\left(\frac{e^{6J_p+4J_{SL}-h_5}+3e^{2J+4J_p+h_6}+3e^{4J+2J_p-h_7}+e^{6J+4J_{SL}+h_8}}{e^{3J+3J_p+J_{SL}}}\right)^3, \tag{3.13}$$

$$e^{-h_5'} = L_2\left(\frac{e^{6J+4J_{SL}+h_1}+3e^{4J+2J_p-h_2}+3e^{2J+4J_p+h_3}+e^{6J_p+4J_{SL}-h_4}}{e^{3J+3J_p+J_{SL}}}\right)^3, \tag{3.14}$$

$$e^{h_6'} = L_2\left(\frac{e^{6J+4J_{SL}+h_1}+3e^{4J+2J_p-h_2}+3e^{2J+4J_p+h_3}+e^{6J_p+4J_{SL}-h_4}}{e^{3J+3J_p+J_{SL}}}\right)^2$$
$$\times\left(\frac{e^{4J_{SL}-h_5}+3e^{2J+2J_p+h_6}+3e^{4J+4J_p-h_7}+e^{6J+6J_p+4J_{SL}+h_8}}{e^{3J+3J_p+J_{SL}}}\right), \tag{3.15}$$

$$e^{-h_7'} = L_2\left(\frac{e^{6J+4J_{SL}+h_1}+3e^{4J+2J_p-h_2}+3e^{2J+4J_p+h_3}+e^{6J_p+4J_{SL}-h_4}}{e^{3J+3J_p+J_{SL}}}\right)$$
$$\times\left(\frac{e^{4J_{SL}-h_5}+3e^{2J+2J_p+h_6}+3e^{4J+4J_p-h_7}+e^{6J+6J_p+4J_{SL}+h_8}}{e^{3J+3J_p+J_{SL}}}\right)^2, \tag{3.16}$$

$$e^{h_8'} = L_2\left(\frac{e^{4J_{SL}-h_5}+3e^{2J+2J_p+h_6}+3e^{4J+4J_p-h_7}+e^{6J+6J_p+4J_{SL}+h_8}}{e^{3J+3J_p+J_{SL}}}\right)^3. \tag{3.17}$$

Remark 3.1 *From equations* (3.10)–(3.17)*, one can easily show that*

$$e^{-3h_2'} = e^{2h_1'-h_4'}, \quad e^{3h_3'} = e^{h_1'-2h_4'}, \quad e^{3h_6'} = e^{-2h_5'+h_8'}, \quad e^{-3h_7'} = e^{-h_5'+2h_8'}. \tag{3.18}$$

Remark 3.2 *If the vector-valued function* $\mathbf{h}(x)$ *given in* (3.8) *has the following form:*

$$\mathbf{h}(x) = \left(p, \frac{q-2p}{3}, \frac{p-2q}{3}, q, r, \frac{s-2r}{3}, \frac{r-2s}{3}, s\right),$$

then, the ***consistency*** *condition* (3.3) *is satisfied, where* $p, q, r, s \in \mathbf{R}$.

4. Translation-invariant Gibbs measures (TIGMs) on a TCL

In this section, we describe a set of translation-invariant Gibbs measures (TIGMs) associated with the model (3.4). Assume that $a = \mathrm{e}^{J}$, $b = \mathrm{e}^{J_{\mathrm{p}}}$ and $c = \mathrm{e}^{J_{\mathrm{SL}}}$. By using the equations (3.10)–(3.17), we can take new variables $u_i' = \exp[h_{B_1(x),S^j_{3-j}(\sigma(x))}]$ for $x \in W_{n-1}$ and $u_i = \exp[h_{B_1(y),S^j_{3-j}(\sigma(y))}]$ for $y \in S(x)$. Therefore, selecting the variables u_1', u_4' u_5' and u_8', we will obtain only the following equations:

$$u_1' = L_2(ab)^{-9}c^{-3}\left(a^6b^6c^4u_1 + 3a^4b^4\frac{1}{u_2} + 3a^2b^2u_3 + \frac{c^4}{u_4}\right)^3, \tag{4.1}$$

$$(u_4')^{-1} = L_2(ab)^{-9}c^{-3}\left(\frac{b^6c^4}{u_5} + 3a^2b^4u_6 + \frac{3a^4b^2}{u_7} + a^6c^4u_8\right)^3, \tag{4.2}$$

$$(u_5')^{-1} = L_2(ab)^{-9}c^{-3}\left(a^6c^4u_1 + \frac{3a^4b^2}{u_2} + 3a^2b^4u_3 + \frac{a^6c^4}{u_4}\right)^3, \tag{4.3}$$

$$u_8' = L_2(ab)^{-9}c^{-3}\left[\frac{c^4}{u_5} + 3(ab)^2u_6 + \frac{3(ab)^4}{u_7} + (ab)^6c^4u_8\right]^3. \tag{4.4}$$

Substitution $u_i \rightarrow v_i^3$. From (3.18), it is clear that $v_2 = (v_1^2/v_4)^{1/3}$ and $v_3 = (v_1/v_4^2)^{1/3}$. Similarly, $v_6 = (v_8/v_5^2)^{1/3}$ and $v_7 = (v_8^2/v_5)^{1/3}$.

Thus, we can derive the following recursion relation

$$v_1' = \sqrt[3]{L_2}\frac{(ab)^6c^4(v_1v_4)^3 + 3(ab)^4(v_1v_4)^2 + 3(ab)^2(v_1v_4) + c^4}{(ab)^3cv_4^3}, \tag{4.5}$$

$$(v_4')^{-1} = \sqrt[3]{L_2}\frac{b^6c^4 + 3a^2b^4(v_5v_8) + 3a^4b^2(v_5v_8)^2 + c^4a^6(v_5v_8)^3}{(ab)^3cv_5^3}, \tag{4.6}$$

$$(v_5')^{-1} = \sqrt[3]{L_2}\frac{a^6c^4(v_1v_4)^3 + 3a^4b^2(v_1v_4)^2 + 3a^2b^4(v_1v_4) + b^6c^4}{(ab)^3cv_4^3}, \tag{4.7}$$

$$v_8' = \sqrt[3]{L_2}\frac{c^4 + 3(ab)^2(v_5v_8) + 3(ab)^4(v_5v_8)^2 + (ab)^6c^4(v_5v_8)^3}{(ab)^3cv_5^3}. \tag{4.8}$$

Now we are going to investigate the derived system (4.5)–(4.8). To do this, let us consider the following operator

$$F : (v_1, v_4, v_5, v_8) \in \mathbf{R}_+^4 \rightarrow (v_1', v_4', v_5', v_8') \in \mathbf{R}_+^4. \tag{4.9}$$

It is generally a difficult problem to determine all invariant sets according to operator F. Now to explain this situation, we can write one of the invariant sets of the operator F as

$$\Upsilon := \{(v_1, v_4, v_5, v_8) \in \mathbf{R}_+^4 : v_1 = v_5, v_4 = v_8\}.$$

Note that one can show that the set Υ is invariant with respect to the operator F, i.e., $F(\Upsilon) \subseteq \Upsilon$. We can determine the invariant subsets of this operator, which are used to describe the Gibbs distributions. The equations corresponding to the restrictions of the operator F to all invariant sets are very cumbersome. Their solutions can be determined using a computer, but this lies outside our circle of interests. We note that the relation between the solutions of the equations and the Gibbs distributions is determined by relations (3.3) and (3.6). The restriction of the operator F to the set Υ gives some known of Gibbs distributions. The restriction to Υ leads to new Gibbs distributions.

Divide equation (4.5) by (4.7) and similarly divide equation (4.8) by (4.6). We have

$$v_1'v_5' = \frac{(ab)^6c^4(v_1v_4)^3 + 3(ab)^4(v_1v_4)^2 + 3(ab)^2(v_1v_4) + c^4}{a^6c^4(v_1v_4)^3 + 3a^4b^2(v_1v_4)^2 + 3a^2b^4(v_1v_4) + b^6c^4}, \tag{4.10}$$

$$v_4'v_8' = \frac{c^4 + 3(ab)^2(v_5v_8) + 3(ab)^4(v_5v_8)^2 + (ab)^6c^4(v_5v_8)^3}{b^6c^4 + 3a^2b^4(v_5v_8) + 3a^4b^2(v_5v_8)^2 + c^4a^6(v_5v_8)^3}. \tag{4.11}$$

5. Phase translations

Note that if there is more than one positive fixed point of the operator (4.9), then there is more than one Gibbs measure corresponding to these positive fixed points. One says that a phase transition occurs for the Ising model, if the system of equations (4.5)–(4.8) has more than one solution [8, 18, 24]. The number of the solutions of equations (4.10) and (4.11) depends on the coupling constants and the parameter $\beta = 1/T$. If it is possible to find an exact value of temperature T_{cr} such that a phase transition occurs for all $T < T_{\text{cr}}$, then T_{cr} is called a critical value of temperature.

In this paper, we will only examine the following situation.

From (4.10) and (4.11), if we suppose $v_1 = v_5 = v_4 = v_8 = \sqrt{x}$, then we have

$$x' = f(x) := \frac{c^4 + 3a^2b^2x + 3a^4b^4x^2 + a^6b^6c^4x^3}{c^4b^6 + 3a^2b^4x + 3a^4b^2x^2 + a^6c^4x^3}. \tag{5.1}$$

Therefore, the set $\wp := \{(v_1, v_4, v_5, v_8) \in \mathbf{R}_+^4 : v_1 = v_5 = v_4 = v_8 = \sqrt{x}\}$ is invariant with respect to the operator F. The restriction of the operator F to the set $\wp$ is denoted by the respective symbol $F|_\wp = f$. Where we assume that $F|_\wp = f$.

In order to investigate the phase transition of the model, we will analyze the positive fixed points of the rational function f with real coefficients as a dynamical system defined in (5.1).

The zeros of equation $x = f(x)$ are the zeros of equation

$$p_4(x) = a^6x^4 + Bx^3 - Cx^2 + Dx - 1 = 0, \tag{5.2}$$

where $B = c^{-4}a^4b^2(a^2b^4c^4 - 3)$, $C = 3c^{-4}a^2b^4(1 - a^2)$, $D = c^{-4}b^2(3a^2 - b^4c^4)$ and $c^4 < 3$.

By using Descartes' rule of signs to find the zeroes of a polynomial, we can determine the number of real solutions to equation (5.2). We can count the number of sign changes. For example, if $B > 0$, $C > 0$ and $D > 0$, then there are three sign changes in the "positive" case. This number "three" is the maximum number of possible positive zeroes (x-intercepts) for the polynomial $p_4(x)$. In this case, there are either 3 or 1 positive roots.

Now let us look at $p_4(-x)$ for the case if $B > 0$, $C > 0$ and $D > 0$ (that is, having changed the sign on x, so this is the "negative" case). There is only one sign change in this "negative" case, so there is exactly one negative root. For the other cases, similarly, we can estimate the possible number of positive and negative roots of the polynomial $p_4(x)$ (table 1).

Thus, we have two critical temperatures $T_{\text{cr}}^* = \frac{-J+2(J_{\text{p}}+J_{\text{LS}})}{\ln\sqrt{3}}$ and $T_{\text{cr}}^{**} = \frac{J+2(J_{\text{p}}+J_{\text{LS}})}{\ln\sqrt{3}}$.

For the antiferromagnetic Ising model ($J < 0$) if we assume

$$\frac{-J + 2(J_{\text{p}} + J_{\text{LS}})}{\ln\sqrt{3}} < T < \frac{J + 2(J_{\text{p}} + J_{\text{LS}})}{\ln\sqrt{3}}$$

then, there exist three positive fixed points of f.

Table 1. The table of possible positive or negative roots of the polynomial $p_4(x) = c^4 - Bx + Cx^2 - Dx^3 - a^6c^4x^4$.

B	C	D	Positive roots	Negative roots
+	+	+	3,1	1
+	+	-	3,1	1
+	-	+	1	3,1
+	-	-	3,1	1
-	+	+	1	3,1
-	+	-	1	3,1
-	-	+	1	3,1
-	-	-	3,1	1

Then, taking the first and the second derivatives of the function g, we have

$$f'(x) = \frac{3a^2\left(b^4-1\right)A(x,a,b,c)}{\left(b^2+a^2x\right)^2\left(b^4c^4+3a^2b^2x-a^2b^2c^4x+a^4c^4x^2\right)^2},$$

where

$$A(x,a,b,c) = b^4c^4 + 2a^2b^2c^4(1+b^4)x + a^4(3b^4+c^8+b^4c^8+b^8c^8)x^2 + 2a^6b^2c^4(1+b^4)x^3 + a^8b^4c^4x^4).$$

If $b^4 < 1$ (with $x \geqslant 0$), then f is decreasing and there can only be one solution of $f(x) = x$, where it is obvious that $A(x,a,b,c) > 0$. Thus, we can restrict ourselves to the case in which $b^4 > 1$. That is, we will assume that $\frac{J_\text{p}}{T} > 0$.

Theorem 5.1 *For the Ising model on the TCL of order 3, the following statements are true:*

1. *If $T < T^*_\text{cr}$, $T > T^{**}_\text{cr}$ and $\frac{J_\text{p}}{T} < 0$, then there is a unique translation-invariant Gibbs measure μ_0.*
2. *If $\frac{-J+2(J_\text{p}+J_\text{LS})}{\ln\sqrt{3}} < T < \frac{J+2(J_\text{p}+J_\text{LS})}{\ln\sqrt{3}}$, then there are 3 translation-invariant Gibbs measures μ_-, μ_0, μ_+ indicating a phase transition (μ_0 is called disordered Gibbs measure). Moreover, μ_-, μ_+ are extreme.*

5.1. An example indicating a phase transition

By using an elementary analysis, we can obtain the fixed points of the function f given in (5.1) by finding real roots of equation (5.2). Thus, we need to identify all the roots of the polynomial (5.2) of degree 4. Previously, a documented analysis has solved these equations, which we will not show here due to the complicated nature of formulas and coefficients [37]. Nonetheless, we have manipulated the polynomial equation via Mathematica [37]. Here, we will only deal with positive fixed points, because of the positivity of exponential functions.

Recall that the set of the fixed points of the function f is defined by

$$\text{Fix}(f) = \{x \in \mathbf{R} : f(x) = x\}.$$

According to the size of the derivative, the fixed points are classified as

- unstable if $|f'| > 1$,
- neutral if $|f'| = 1$,
- stable if $|f'| < 1$,
- superstable if $|f'| = 0$.

We have obtained 3 positive real roots for some parameters J, J_p and J_SL (coupling constants) and temperature T. For example, in figure 4, we have manipulated that there are 3 positive fixed points of the function (5.1) for $J = -1$, $J_\text{p} = 29$, $J_\text{SL} = 5.3$, $T = 68$. As a result, there are three translation-invariant Gibbs measures associated with the positive fixed points. Therefore, for $J = -1$, $J_\text{p} = 29$, $J_\text{SL} = 5.3$, $T = 68$, the phase transitions occur.

For $J = -1$, $J_\text{p} = 29$, $J_\text{SL} = 5.3$, $T = 68$, there exist four fixed points of the function f obtained in (5.1) as follows

$$\text{Fix}(f) = \{-1.0376, 0.127421, 1.11525, 7.40762\}.$$

It is clear that $|f'(0.127421)| = 0.470903 < 1$ and $|f'(7.40762)| = 0.520525 < 1$. Therefore, $x_- = 0.127421$ and $x_+ = 7.40762$ are stable fixed points of the function f. The corresponding Gibbs measures are extreme ones. Also, $|f'(1.11525)| = 1.36756 > 1$, thus, $x_1 = 1.11525$ is unstable fixed point of the function f (see figure 4).

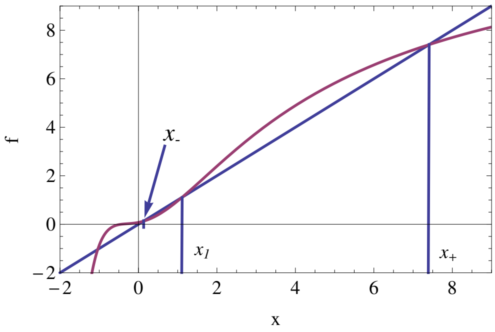

Figure 4. (Colour online) There exist three positive fixed points of the function f for $J = -1$, $J_{\mathrm{p}} = 29$, $J_{\mathrm{SL}} = 5.3$, $T = 68$.

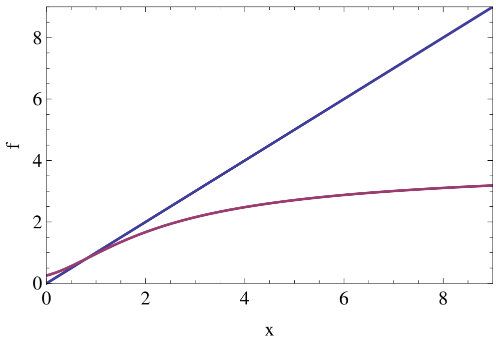

Figure 5. (Colour online) There exists a unique positive fixed point of the function f for $J = -1$, $J_{\mathrm{p}} = 10$, $J_{\mathrm{SL}} = 5.3$, $T = 44$.

In figure 5, there exists only a single positive fixed point of the function (5.1) for $J = -1$, $J_{\mathrm{p}} = 10$, $J_{\mathrm{SL}} = 5.3$, $T = 44$. The set of the other fixed points are $\{-1.05633, 0.554978 - 1.02241\mathrm{i}, 0.554978 + 1.02241\mathrm{i}, 0.801718\}$. Therefore, there is a unique Gibbs measure corresponding to the fixed point $x = 0.801718$ on the CL with parameters $J = -1$, $J_{\mathrm{p}} = 10$, $J_{\mathrm{SL}} = 5.3$, $T = 44$. Therefore, the phase transition does not occur for $J = -1$, $J_{\mathrm{p}} = 10$, $J_{\mathrm{SL}} = 5.3$, $T = 44$.

Note that an attractive fixed point of a function f is a fixed point x_+ of f such that for any value of x in a domain that is close enough to x_0, the iterated function sequence $x, f(x), f(f(x)), f(f(f(x))), \ldots$ converges to x_+. An attractive fixed point is said to be a stable fixed point if it is also Lyapunov stable (see [3] for details).

Remark 5.1 *From theorem 5.1, one can say that the stable roots describe extreme Gibbs distributions. Therefore, from the figure 4, we can conclude that the Gibbs measures μ_- and μ_+ corresponding to the stable fixed points x_- and x_+ are extreme Gibbs distributions [9, 21, 22, 38].*

Remark 5.2 *We conclude that there are at most 3 translation-invariant Gibbs measures corresponding to the positive real roots of the equation* (5.1). *Also, one can show that translation-invariant Gibbs measures corresponding stable solutions are extreme.*

6. Conclusions

When our model is compared with the model given in [28], we can see the role of the competing coupling J_{SL} which represents the same level nearest neighbour on the phase transition phenomenon. If we take $J_{\mathrm{SL}} = 0$, then the equation is the same as the dynamical system in [28]. The recurrence equations obtained in the present paper totally differ from [1, 28, 34, 39]. Note that for the Ising model associated with the Hamiltonian (3.4) on the chandelier lattices of order k, in contrast to the symmetry of arbitrary order Cayley tree [6, 39], if $k > 3$, then the chandelier lattice of order k is not symmetry. Therefore, in order to construct the recurrence equations associated with the given Hamiltonian (3.4) for $k > 3$ is much more difficult.

To describe the set of all the corresponding Gibbs distributions is one of the main problems for the given Hamiltonian [40]. However, the attempts to completely describe this set have not been accomplished until now, even for rather simple Hamiltonians. An exact description of all positive fixed points of the operator F given in (4.9) is rather tricky. Therefore, under some assumptions, description of the translation-invariant Gibbs measures for the model has been given. We have also shown that for some parameter values of the model there is a phase transition. We state some unsolved problems that turned out to be rather complicated and require further consideration:

1. Do any other invariant sets of the operator F exist?

2. Do positive fixed points of the operator F exist outside the invariant sets?

Ganikhodjaev and Rozikov [41] have given a complete description of periodic Gibbs measures for the Ising model, i.e., a characterization of such measures with respect to any normal subgroup of finite index in G_k. Akın et al. [42] have studied the periodic extreme Gibbs measures with memory length 2 of Vannimenus model. Description of the periodic (non transition-invariant) Gibbs measures with a memory of length 2 on the chandelier lattice remains an open problem.

In [34, 43], the authors have presented, for the Ising model on the Cayley tree, some explicit formulae of the free energies (and entropies) according to boundary conditions (b.c.). By applying the general formulae to various known boundary conditions on arbitrary order chandelier-lattices, we plan to obtain some explicit formula of free energy and relative entropy corresponding to the boundary conditions in our future work.

We think that the present paper is of certain interest for the statistical physics community due to an interesting application of the **chandelier-lattices** model to real problems.

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Ґібсові міри моделі Ізінга з конкурентними взаємодіями на трикутній люстровій гратці

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В цій статті ми розглядаємо модель Ізінга з трьома конкурентними взаємодіями на трикутній люстровій гратці. Описано існування, єдиність і неєдиність трансляційно інваріантних Ґібсових мір, пов'язаних з моделлю Ізінга. Отримано явну формулу для Ґібсових мір з пам'яттю довжиною 2, що задовільняють умови консистентності. Строго доведено, що дана модель проявляє фазові переходи лише для даних констант зв'язку. В результаті застосування даного підходу вирішено проблему дихотомії між альтернативними розв'язками Гамільтонових моделей на трикутних люстрових гратках. Нарешті, показано два числових приклади, що ілюструють корисність і ефективність запропонованих теоретичних результатів.

Ключові слова: *люстрові гратки, Ґібсові міри, модель Ізінга, фазовий перехід*