ENTROPIC EQUATION OF STATE AND SCALING FUNCTIONS FOR SPIN MODELS ON SCALE-FREE NETWORKS

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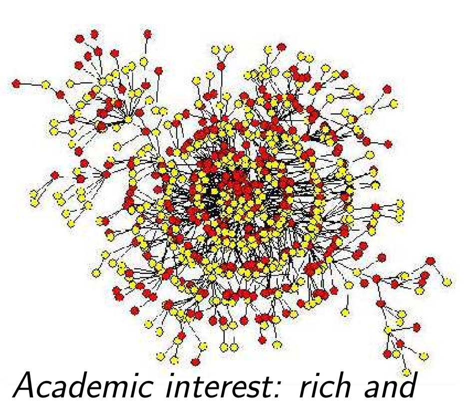
Abstract

Many real world processes may be analyzed applying the methods and tools of statistical mechanics to the corresponding systems. A prototype example is that of the standard Ising model. However, instead of spins on a regular lattice one considers spins associated with the individuals (nodes) of a social network. The latter model may be used to describe opinion formation processes in society. In this way fundamental equations and laws of statistical mechanics interpreted appropriately may quantitatively describe aspects of social behaviour. In the present work a model with a single order parameter (Ising model) on an uncorrelated scale-free network is analyzed by determining the entropic equation of state and the relative scaling functions. Both are of fundamental importance within the theory of phase transitions and critical phenomena. Here we obtain general scaling functions for the entropy, the constant-field heat capacity, and the isothermal magnetocaloric coefficient near the critical point in scale-free networks. The essential influence of the basic network structure as expressed e.g. by the node degree distribution on the behaviour of the above mentioned quantities is confirmed, extending the principle of universality to systems on scale-free networks.

Motivation



Academic interest: rich and non-trivial critical behavior

Social networks as models for opinion formation (see e.g. [1])

The node degree distribution of many real-world networks obeys a power law decay (see e.g. [2]), scale-free networks:

$$P(k) \sim k^{-\lambda}, \quad k \to \infty. \tag{1}$$

The properties of critical phenomena of spin models on scale-free networks strongly depend on the network topology (see e.g. [3]):

- if $\langle k^4 \rangle$ is finite ($\lambda > 5$), the critical behavior is governed by the usual mean-field critical exponents;
- if $\langle k^4 \rangle \to \infty$ and $\langle k^2 \rangle < \infty$, logarithmic corrections to scaling appear ($\lambda = 5$) or the critical exponents acquire λ dependence ($3 < \lambda < 5$);
- if $\langle k^2 \rangle \to \infty$ ($2 < \lambda \leq 3$), $T_c \to \infty$ and the system is ordered at any finite temperature.

Our tasks

Using the free energy expressions for the single order parameter systems on uncorrelated scale-free networks [4]

$$f(m,\tau) = \pm\frac{\tau}{2}m^2 + \frac{1}{4}m^4, \qquad \lambda > 5, \tag{2}$$

$$f(m,\tau) = \pm\frac{\tau}{2}m^2 + \frac{1}{4}m^{\lambda-1}, \quad 3 < \lambda < 5, \tag{3}$$

complete an analysis of the critical behavior of the systems by calculating entropic equation of state and the corresponding scaling functions (entropy, heat capacity and isothermal magnetocaloric coefficient scaling functions).

General thermodynamics

The scaling hypothesis states that the singular part of a thermodynamic potential near the critical point has the form of a generalized homogeneous function. For the Helmholtz potential the latter in dimensionless variables can be written as

$$f(\tau, m) = \tau^{2-\alpha} f_{\pm}(m/\tau^{\beta}). \tag{4}$$

Here $\tau = |T - T_c|/T_c$, and the sign $\pm$ corresponds to $T > T_c$ or $T < T_c$ respectively. Starting from general definition

$$\mathrm{d}F = -S\mathrm{d}T + H\mathrm{d}M \tag{5}$$

one arrives at both magnetic and entropic forms of the equation of states

$$H = \frac{\partial F}{\partial M}\Big|_T, \qquad S = \frac{\partial F}{\partial T}\Big|_M. \tag{6}$$

While scaling properties of the magnetic equation of state were previously analyzed [7], scaling analysis of the entropic equation of state and corresponding characteristics (entropy, heat capacity and isothermal magnetocaloric coefficient) is the point of the current investigation.

In dimensionless variables both equations of state given by Eq. (6) follow

$$h(m,\tau) = \frac{\partial f(m,\tau)}{\partial m}\Big|_\tau, \quad s(m,\tau) = \mp\frac{\partial f(m,\tau)}{\partial \tau}\Big|_m. \tag{7}$$

For the Landau-like free energy, constant-magnetic-field heat capacity $c_H = T\frac{\mathrm{d}S}{\mathrm{d}T}\Big|_H$ may be written in dimensionless variables as follows

$$c_h = (\tau \pm 1)\frac{\partial s}{\partial m}\Big|_\tau \frac{\partial m}{\partial \tau}\Big|_h. \tag{8}$$

For both free energy expressions (2) and (3), c_h may be presented as

$$c_h = \mp(1 \pm \tau)m\frac{\partial m(\tau,h)}{\partial \tau}\Big|_h. \tag{9}$$

Isothermal magnetocaloric coefficient serves as a direct measure of the heat released by the system due to the magnetocaloric effect upon an isothermal increase of the magnetic field. Initially defined as $M_T = -T(\partial M/\partial T)_H$ (see e.g. [5]) the coefficient in dimensionless variables may be presented as

$$m_T = -(1 \pm \tau)\frac{\partial s(m,\tau)}{\partial m}\Big|_\tau \frac{\partial m(\tau,h)}{\partial h}\Big|_\tau. \tag{10}$$

Scaling functions

For the free energy (4) the entropy can be presented in a scaling form

$$s(\tau, m) = \tau^{1-\alpha}\mathcal{S}(x). \tag{11}$$

Here and below the scaling variable $x = m/\tau^{\beta}$. For both free energies given by expressions (2) and (3), the entropy scaling function $\mathcal{S}(x)$ follows

$$\mathcal{S}(x) = -\frac{x^2}{2}. \tag{12}$$

On the contrary to $\mathcal{S}(x)$, other scaling functions essentially depend on the value of exponent λ. Namely, there are two ranges for λ where different functional dependencies for the scaling functions are found. The heat capacity scaling function $\mathcal{C}_{\pm}(x)$ defined by equation

$$c_h = (1 \pm \tau)\tau^{-\alpha}\mathcal{C}_{\pm}(x) \tag{13}$$

follows

$$\mathcal{C}_{\pm}(x) = \frac{x^2}{3x^2 \pm 1}, \qquad \lambda > 5, \tag{14}$$

$$\mathcal{C}_{\pm}(x) = \frac{x^2}{\frac{(\lambda-1)(\lambda-2)}{4}x^{\lambda-3} \pm 1}, \qquad 3 < \lambda < 5, \tag{15}$$

with the critical exponent α given in Table I.

	α	β	γ	δ	ω	α_c	γ_c	ω_c
$\lambda \geq 5$	0	1/2	1	3	1/2	0	2/3	1/3
$3 < \lambda < 5$	$\frac{\lambda-5}{\lambda-3}$	$\frac{1}{\lambda-3}$	1	$\lambda - 2$	$\frac{\lambda-4}{\lambda-3}$	$\frac{\lambda-5}{\lambda-2}$	$\frac{\lambda-3}{\lambda-2}$	$\frac{\lambda-4}{\lambda-2}$

Table I. *Critical exponents governing temperature and field dependencies of the thermodynamic quantities for different values of λ.*

Scaling representation for the dimensionless isothermal magnetocaloric coefficient m_T follows

$$m_T = (1 \pm \tau)\tau^{-\omega}\mathcal{M}_{\pm}(x), \tag{16}$$

with the critical exponent ω:

$$\omega = 1 - \beta\,. \tag{17}$$

For the systems with the entropy scaling function given by Eq. (12) a simple presentation of the magnetocaloric coefficient scaling function $\mathcal{M}_{\pm}(x)$ holds

$$\mathcal{M}_{\pm}(x) = x\chi_{\pm}(x). \tag{18}$$

Here $\chi_{\pm}(x)$ is the isothermal magnetic susceptibility scaling function ($\chi_T(m,\tau) = \tau^{-\gamma}\chi_{\pm}(x)$). Having in hand expression for χ_T [7] we arrive at the corresponding dependencies for $\mathcal{M}_{\pm}(x)$. Resulting expressions for the scaling functions obtained in the current work as well as in the previous one [7] are summarized in Table II.

	$3 < \lambda < 5$	$\lambda > 5$
$f_{\pm}$	$\pm x^2/2 + x^{\lambda-1}/4$	$\pm x^2/2 + x^4/4$
$H_{\pm}$	$\frac{\lambda-1}{4}x^{\lambda-2} \pm x$	$x^3 \pm x$
$\chi_{\pm}$	$\frac{1}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4 \pm 1}$	$\frac{1}{3x^2 \pm 1}$
$\mathcal{S}$	$-x^2/2$	$-x^2/2$
$\mathcal{C}_{\pm}$	$\frac{x^2}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4 \pm 1}$	$\frac{x^2}{3x^2 \pm 1}$
$\mathcal{M}_{\pm}$	$\frac{x}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4 \pm 1}$	$\frac{x}{3x^2 \pm 1}$

Table II. *Scaling functions for a single order parameter system near the critical point in scale-free networks. The scaling variable is $x = m/\tau^{\beta}$* [7, 8].

Finally, the scaling relation for the critical exponent ω_c that governs the field dependence of $m_T(\tau = 0, h)$ [6]:

$$\omega_c = \frac{1-\beta}{\beta\delta}. \tag{19}$$

The value of the latter exponent as well as other ones are given in Table I.

In figures Fig. I and Fig. II constant-magnetic-field heat capacity and isothermal magnetocaloric coefficient scaling functions for different values of parameter λ are presented as a functions of scaling variables $x = m/\tau^{\beta}$ and $y = h/\tau^{\beta\delta}$. While the asymptotic value of $\mathcal{C}_{\pm}(x)$ and $\mathcal{C}_{\pm}(y)$ for $\lambda > 5$ converges to a constant $\mathcal{C}_{\pm}(x \to \infty) = \mathcal{C}_{\pm}(y \to \infty) = 1/3$, the asymptotic behavior changes for $3 < \lambda < 5$. Namely, $\mathcal{C}_{\pm}(x \to \infty) \sim x^{5-\lambda}$ and $\mathcal{C}_{\pm}(y \to \infty) \sim y^{\frac{5-\lambda}{\lambda-2}}$.

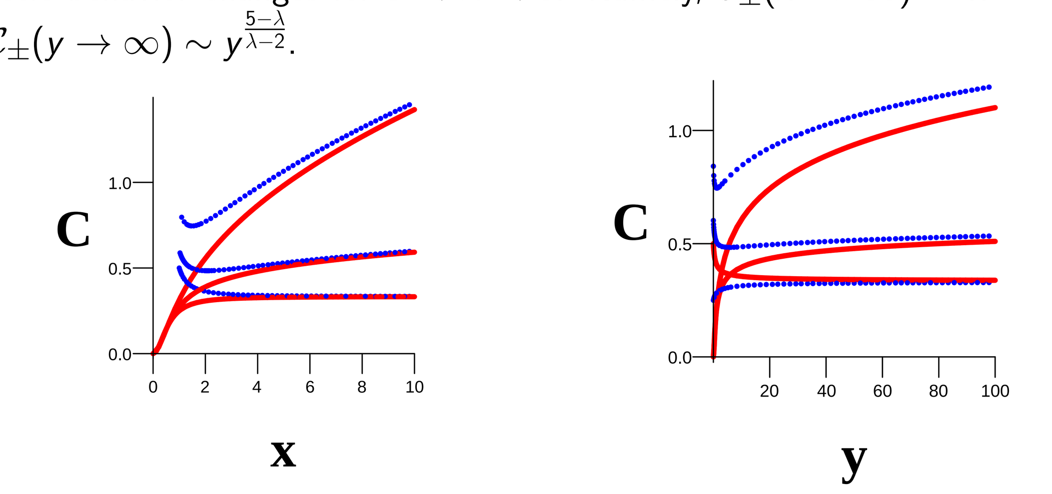


Fig.1. *Heat capacity scaling functions $\mathcal{C}_-$ (blue curves) and $\mathcal{C}_+$ (red curves) as functions of the scaling variables $x = m/\tau^{\beta}$ (left plot) and $y = h/\tau^{\beta\delta}$ (right plot) at $\lambda > 5$, $\lambda = 4.8$, and $\lambda = 4.5$ (lower, middle and upper pairs of curves, respectively).*

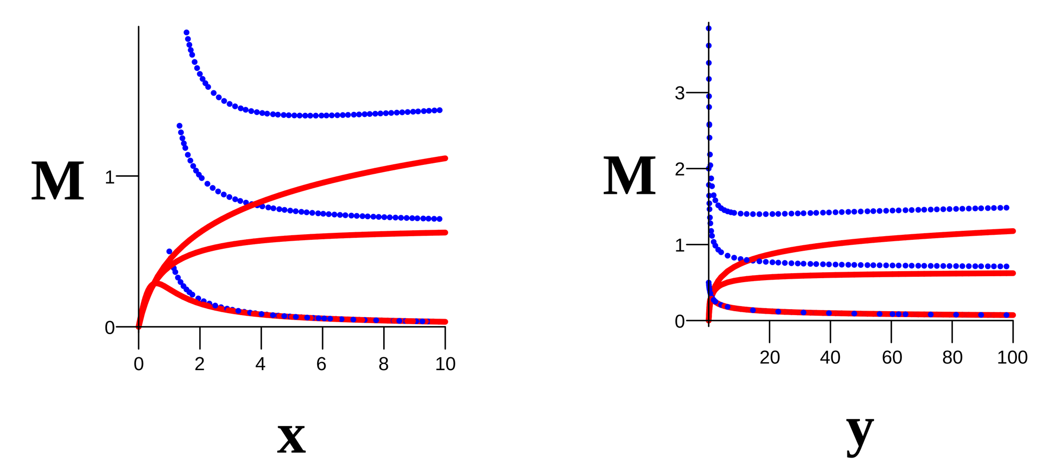


Fig.2. *Isothermal magnetocaloric coefficientscaling functions $\mathcal{M}_-$ (blue curves) and $\mathcal{M}_+$ (red curves) as functions of the scaling variables $x = m/\tau^{\beta}$ (left plot) and $y = h/\tau^{\beta\delta}$ (right plot) at $\lambda > 5$, $\lambda = 4$, and $\lambda = 3.8$ (lower, middle and upper pairs of curves, respectively).*

Conclusions

- The comprehensive list of observables that describe the scaling and characterize the criticality in uncorrelated scale-free networks is completed [7, 8].
- The node degree distribution exponent λ plays the role of a global variable as far as the critical behavior is considered. The universal quantities that govern criticality become λ-dependent for small enough λ.
- The difference between the global variable λ and the dimension d is considerable with respect to the inuence of the uctuations on the critical behavior of the physical quantities: the fluctuation in network structure only enters via the magnetization and since the entropy S is measured at constant magnetization, it is given by a usual Landau-like mean field value for any $\lambda > 3$.

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