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Confinement interaction in nonlinear generalizations of the Wick–Cutkosky model

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Abstract

We consider nonlinear-mediating-field generalizations of the Wick–Cutkosky model. Using an iterative approach and eliminating the mediating field by means of the covariant Green’s function we arrive at a Lagrangian density containing many-point time-nonlocal interaction terms. In low-order approximations of $\varphi^3 + \varphi^4$ theory we obtain the usual two-current interaction as well as a three-current interaction of a confining type. The same result is obtained without approximation for a version of the dipole model. The transition to the Hamiltonian formalism and subsequent canonical quantization is performed with time non-locality taken into account approximately. A relativistic three-particle wave equation is derived variationally by using a three-particle Fock space trial state. The non-relativistic limit of this equation is obtained and its properties are analyzed and discussed.

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1. Introduction

Confinement is evidently related to the nonlinearity of chromodynamics. Since confining solutions of classical non-Abelian field equations are not known at present [1], it is believed that confinement is an essential quantum effect. This is supported by numerical computations of QCD on the lattice [2, 3]. However, the analytical study of confinement, particularly in gauge field theory like QCD, remains a challenging task [3]. Thus, the study of simpler field theoretical models that simulate the characteristic features of confinement remains relevant.

In this regard, earlier models are worth mentioning such as the dipole model [4] and the related higher derivative model [5] with its subsequent non-Abelian generalization [6]. They indicate a $1/k^4$ infrared behavior of the ‘gluon’ propagator, and thus a linear interaction potential, even at the classical level. In spite of some quantization inconsistencies, these phenomenological models treat the confinement interaction as an elementary process, i.e. a

two-particle interaction arising from the lowest-order approximation of perturbative dynamics of the models.

More realistic models are the Dual Abelian Higgs model [3] and non-Abelian versions [3, 7] in which the spontaneous symmetry breaking mechanism is used to generate a vacuum condensate with confining properties. In this approach the confinement interaction is a kind of collective effect similar to that in condensed matter physics.

The two classes of models mentioned above represent quite different points of view on the confinement mechanism. The purpose of this study is to shed light on the question: Is an intermediate mechanism possible, in which confinement cannot be reduced to elementary processes but is governed by cluster interactions involving finite numbers of particles?

To investigate such a possibility, we utilize the variational method, in a reformulated Hamiltonian formalism of quantum field theory (QFT), which has been demonstrated to be a promising and powerful approach to the relativistic bound-state problem [8–13]. In particular, this approach has been used to derive (and solve approximately) relativistic equations for two and three fermion systems, such as Positronium (Ps) and Muonium (Mu) [14], and also Ps^- and Mu^- [15], and it was shown that the derived bound-state energies agree with conventional perturbation theory and with experimental results (where available).

The use of many-particle Fock-space components in the variational trial states leads to wave equations with systematically improvable bound-state energy levels, as has been shown, for example, in the simple scalar Yukawa model [12, 13].

In this paper, we analyze the interactions that arise from the nonlinear terms in the mediating-field sector of the QFT Lagrangian. In particular, we consider the $(\varphi^3 + \varphi^4)$ -generalization of the Wick–Cutkosky (i.e. massless scalar Yukawa) model [16] as well as a version of the dipole model [4, 11].

We note that the models being considered are not of a non-Abelian gauge-field type. The only two features which are common to the models of this paper and QCD are the massless and nonlinear nature of the mediating field. Both features are important in the generating confinement but the mechanism of this effect here is different from that in gauge models [3, 7].

2. Partially-reduced Wick–Cutkosky model

The Wick–Cutkosky model [16] is based on the classical action integral:

$$I = \int d^4x \mathcal{L}(x), \quad (2.1)$$

with the Lagrangian density ($\hbar = c = 1$)

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - g \phi^* \phi \chi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi, \quad (2.2)$$

where $\phi(x)$ is a complex scalar ‘matter’ field with the rest mass m , and $\chi(x)$ is a real massless scalar field interacting with ϕ via the Yukawa term $g \phi^* \phi \chi$ (here g is an interaction constant).

The stationary property of the action (2.1)–(2.2), i.e. $\delta I = 0$, leads to the coupled set of the Euler–Lagrange equations:

$$(\square + m^2)\phi = -g\phi\chi, \quad (2.3)$$

$$(\square + m^2)\phi^* = -g\phi^*\chi, \quad (2.4)$$

$$\square\chi = \rho, \quad (2.5)$$

which determine the field dynamics; here $\rho \equiv -g\phi^*\phi$.

Equation (2.5) can be solved exactly:

$$\chi = D * \rho + \chi_0, \tag{2.6}$$

where ‘*’ denotes the convolution $[D * \rho](x) \equiv \int d^4x' D(x - x')\rho(x')$ and $D(x) = \frac{1}{4\pi}\delta(x^2)$ is the symmetric Green’s function of the d’Alembert equation. If the free χ field plays no role in the investigation the arbitrary solution χ_0 of the homogeneous d’Alembert equation can be omitted. Then the use of the formal solution (2.6) (with $\chi_0 = 0$) in the Lagrangian (2.2) leads to a self-contained variational principle for the interacting fields $\phi(x)$ and $\phi^*(x)$. The modified Lagrangian $\tilde{\mathcal{L}}$ which we shall refer to as the partially-reduced Lagrangian, is an important basis for the quantization of the model; cf [9, 11].

We demonstrate here how to derive the partially-reduced Lagrangian for the Wick–Cutkosky model without the use of the condition $\chi_0 = 0$. For this purpose we consider equality (2.6) as a change of variable $\chi \rightarrow \chi_0$ where the new field χ_0 is not *a priori* subjected to any field equation. The substitution of (2.6) directly in the Lagrangian (2.2) gives

$$\begin{aligned} \mathcal{L} &= \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \rho(D * \rho + \chi_0) + \frac{1}{2}[\partial_\mu(D * \rho + \chi_0)\partial^\mu(D * \rho + \chi_0)] \\ &\simeq \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \rho(D * \rho + \chi_0) - \frac{1}{2}(D * \rho + \chi_0)\square(D * \rho + \chi_0) \\ &\simeq \underbrace{\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \frac{1}{2}\rho D * \rho}_{\tilde{\mathcal{L}}} + \underbrace{\frac{1}{2}\partial_\mu \chi_0 \partial^\mu \chi_0}_{\Delta\mathcal{L}_{\text{free}}}, \end{aligned} \tag{2.7}$$

where \simeq denotes equality modulo surface terms. In this form the system is effectively split into two independent subsystems: the interacting ϕ matter field and the free χ_0 field. From this point on the physically trivial χ_0 -dependent $\Delta\mathcal{L}_{\text{free}}$ term can be ignored (as indicated above)³.

The partially-reduced Lagrangian $\tilde{\mathcal{L}}$ is non-local in spacetime coordinates. The treatment of non-local theories of this type is a conceptually intricate, but practically realizable procedure. In particular, partially-reduced versions of Yukawa-like models are worked out in [11]. In the next section we consider a nonlinear generalization of Wick–Cutkosky model within the partially-reduced formulation.

3. Nonlocal Lagrangian from a nonlinear Wick–Cutkosky model

We proceed from the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - g \phi^* \phi \chi - \frac{1}{4}\lambda(\phi^* \phi)^2 + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - \mathcal{V}(\chi), \tag{3.1}$$

where $\lambda > 0$ is a self-interaction coupling constant and $\mathcal{V}(\chi)$ is an arbitrary potential (all other quantities are the same as in (2.2)).

The new terms, $\lambda(\phi^* \phi)^2$ and $\mathcal{V}(\chi)$, modify the Euler–Lagrange equations (2.3)–(2.5). In particular, equation (2.5) becomes the nonlinear inhomogeneous d’Alembert equation

$$\square\chi = \rho - \mathcal{V}'(\chi), \tag{3.2}$$

where $\mathcal{V}'(\chi) \equiv d\mathcal{V}(\chi)/d\chi$. It can be formally solved by iteration (cf [17]). In the first-order approximation we have

$$\chi = D * [\rho - \mathcal{V}'(D * \rho)] + \chi_0, \tag{3.3}$$

where χ_0 includes an arbitrary solution of the homogeneous equation.

³ It is noteworthy that, within the variational problem based on (2.7), the primary meaning of χ_0 in (2.6) as general solution of the homogeneous d’Alembert equation is restored.

Similarly to the case of the linear Wick–Cutkosky model, we use the replacement (3.3) (where χ_0 is a new field variable) in the Lagrangian (3.1). In first order this gives

$$\begin{aligned} \mathcal{L} \simeq & \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \frac{1}{2} \rho D * \rho - \frac{1}{4} \lambda (\phi^* \phi)^2 \\ & + \frac{1}{2} \partial_\mu \chi_0 \partial^\mu \chi_0 - \mathcal{V}(D * \rho + \chi_0) + \chi_0 \mathcal{V}'(D * \rho). \end{aligned} \quad (3.4)$$

Unlike the Lagrangian (2.7), this functional is not completely split in the ϕ and χ_0 variables. The Euler–Lagrange equation for χ_0 ,

$$\square \chi_0 = -\mathcal{V}'(D * \rho + \chi_0) + \mathcal{V}'(D * \rho), \quad (3.5)$$

is a free-field one only in the zero-order approximation. Nevertheless, it possesses the solution $\chi_0 = 0$ which, upon substitution into (3.4), gives the reduced Lagrangian:

$$\begin{aligned} \bar{\mathcal{L}} \simeq & \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \frac{1}{2} \rho D * \rho - \frac{1}{4} \lambda (\phi^* \phi)^2 - \mathcal{V}(D * \rho) \\ \equiv & \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}^{(2)} + \mathcal{L}_{\text{int}}^{(>2)}. \end{aligned} \quad (3.6)$$

It is non-local, and the action (2.1) and (3.6) includes 1-, 2- and >2-fold integrations over Minkowsky space.

The difference $\Delta \mathcal{L} = \mathcal{L} - \bar{\mathcal{L}}$, i.e. the χ_0 -dependent part of the total Lagrangian (3.4), is at least quadratic in the χ_0 variable:

$$\begin{aligned} \Delta \mathcal{L} &= \Delta \mathcal{L}_{\text{free}} + \Delta \mathcal{L}_{\text{int}} \quad \text{where} \quad \Delta \mathcal{L}_{\text{free}} = \frac{1}{2} \partial_\mu \chi_0 \partial^\mu \chi_0, \\ \Delta \mathcal{L}_{\text{int}} &= \mathcal{V}(D * \rho) + \chi_0 \mathcal{V}'(D * \rho) - \mathcal{V}(D * \rho + \chi_0) \\ &= -\frac{1}{2!} \chi_0^2 \mathcal{V}''(D * \rho) - \frac{1}{3!} \chi_0^3 \mathcal{V}'''(D * \rho) - \dots \end{aligned} \quad (3.7)$$

This structure shows that the term $\Delta \mathcal{L}$ is not important in this work, as will be explained in more detail in section 7.

The non-local Lagrangian (3.6) is the first-order approximate result of the reduction procedure applied to nonlinear generalizations of the Wick–Cutkosky model. In the appendix we construct another local model, a kind of dipole model (with a pair of mediating fields), that can be reduced to the Lagrangian (3.6) exactly.

4. Quantization

In order to proceed further we need to specify the interaction potential $\mathcal{V}(\chi)$. We choose

$$\mathcal{V}(\chi) = \frac{1}{3} \kappa \chi^3 + \frac{1}{4} \varkappa \chi^4, \quad (4.1)$$

where κ and $\varkappa > 0$ are the coupling constants. In this case the nonlinear Wick–Cutkosky model (3.1) possesses a stable perturbative vacuum and is renormalizable.

We proceed from the partially-reduced Lagrangian (3.6), construct the Hamiltonian of the model and perform the canonical quantization. Due to the non-locality of the Lagrangian (3.6), the Hamiltonization is a rather complicated procedure. It can be performed perturbatively, following [11–13]. In the leading-order approximation the Hamiltonization proceeds as follows. We work out the Hamiltonian density:

$$\mathcal{H} = \mathcal{H}_{\text{free}} + \mathcal{H}_{\text{int}}^{(2)} + \mathcal{H}_{\text{int}}^{(3)} + \mathcal{H}_{\text{int}}^{(4)}, \quad (4.2)$$

where

$$\begin{aligned} \mathcal{H}_{\text{int}}^{(2)}(x) &= -\frac{1}{2} \int d^4 x' \rho(x) D(x - x') \rho(x') + \frac{1}{4} \lambda (\phi^*(x) \phi(x))^2 \\ &\equiv -\frac{1}{2} \int d^4 x' \rho(x) \left[D(x - x') - \frac{\lambda}{2g^2} \delta(x - x') \right] \rho(x'), \end{aligned} \quad (4.3)$$

$$\mathcal{H}_{\text{int}}^{(3)}(x) = \frac{1}{3}\kappa \iiint d^4x' d^4x'' d^4z D(z-x)D(z-x')D(z-x'')\rho(x)\rho(x')\rho(x''), \quad (4.4)$$

$$\mathcal{H}_{\text{int}}^{(4)}(x) = \frac{1}{4} \iiint d^4x' d^4x'' d^4x''' d^4z D(z-x)D(z-x')D(z-x'')D(z-x''') \times \rho(x)\rho(x')\rho(x'')\rho(x'''). \quad (4.5)$$

The total interaction Hamiltonian density (4.2) is then expressed in terms of the Fourier amplitudes A_k , B_k and A_k^\dagger , B_k^\dagger , of the field $\phi(x)$ (see equation (2.14) in [13]; actually, the procedure is somewhat more intricate [11] but the result is the same). Upon quantization these amplitudes satisfy the standard commutation relations and become the creation and annihilation operators. Then the canonical Hamiltonian operator is given by

$$H = \int d^3x : \mathcal{H}(t=0, \mathbf{x}) :, \quad (4.6)$$

where ‘:’ denotes the normal ordering of operators. Other canonical generators, such as linear and angular momentum, can be easily obtained.

The term H_{free} is the standard Hamiltonian of the free complex scalar field. The explicit form of the pair interaction term $H_{\text{int}}^{(2)}$ is known (see [9, 11]) and so we shall concentrate on the $H_{\text{int}}^{(3)}$ term. It has the following somewhat cumbersome form:

$$H_{\text{int}}^{(3)} = -\frac{\kappa g^3}{24(2\pi)^6} \int \frac{d^3k_1 \dots d^3k_6}{\sqrt{k_{10} \dots k_{60}}} \sum_{\substack{\eta_1=\pm \\ \dots \\ \eta_6=\pm}} \tilde{D}(\eta_1 k_1 + \eta_2 k_2) \tilde{D}(\eta_3 k_3 + \eta_4 k_4) \tilde{D}(\eta_5 k_5 + \eta_6 k_6) \times \delta(\eta_1 \mathbf{k}_1 + \dots + \eta_6 \mathbf{k}_6) : B_{k_1} A_{k_2} B_{k_3} A_{k_4} B_{k_5} A_{k_6} :, \quad (4.7)$$

where $B^+ = B$, $B^- = A^\dagger$, $A^+ = A$, $A^- = B^\dagger$ and the Fourier transform, $\tilde{D}(k) = -\mathcal{P}/k^2$, of the symmetric Green’s function of the d’Alembert equation depends on the on-shell 4-momentum $k = \{k_0, \mathbf{k}\}$, where $k_0 = \sqrt{m^2 + \mathbf{k}^2}$. Expression (4.7) includes $2^6 = 64$ terms. The term $H_{\text{int}}^{(4)}$ is of similar but more cumbersome form. We do not exhibit it explicitly, since, as will be seen below, it makes no contribution to the three-body equation derived in this work.

5. Variational three-particle wave equations

In the variational approach to QFT the trial state of the system is built of few particle channel components [12, 13] such as the two-particle state vector $|2\rangle = \frac{1}{\sqrt{2}} \int d^3p_1 d^3p_2 F_2(\mathbf{p}_1, \mathbf{p}_3) A_{p_1}^\dagger A_{p_2}^\dagger |0\rangle$, the particle-antiparticle one $|1+\bar{1}\rangle = \int d^3p_1 d^3p_2 G(\mathbf{p}_1, \mathbf{p}_3) A_{p_1}^\dagger B_{p_2}^\dagger |0\rangle$, and so on. The three-particle component has the form

$$|3\rangle = \frac{1}{\sqrt{3!}} \int d^3p_1 d^3p_2 d^3p_3 F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) A_{p_1}^\dagger A_{p_2}^\dagger A_{p_3}^\dagger |0\rangle, \quad (5.1)$$

where the channel wavefunction F , which is to be determined variationally, is completely symmetric under the permutation of the variables $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. In the variational method the channel components, $|\psi_N\rangle$, are used to determine the matrix elements of the Hamiltonian, namely $\langle \psi_N | H | \psi_{N'} \rangle$, where N, N' stand for $1, \bar{1}, 2, 1+\bar{1}, \bar{2}, 3, 2+\bar{1}, 2+\bar{2}, \dots$.

We are interested here in the matrix element of the interaction $H_{\text{int}} = H_{\text{int}}^{(2)} + H_{\text{int}}^{(3)} + H_{\text{int}}^{(4)}$ of the Hamiltonian. We note that $\langle 1+\bar{1} | H_{\text{int}}^{(3)} | 1+\bar{1} \rangle = 0$, $\langle 2 | H_{\text{int}}^{(3)} | 2 \rangle = 0$. In other words, purely two-particle trial states, and so the resulting variational wave equations, do not sample the

term $H_{\text{int}}^{(3)}$. Thus, we first consider the three-particle case and calculate the matrix element

$$\langle 3|H_{\text{int}}|3\rangle = \int d^3 p'_1 \dots d^3 p'_3 d^3 p_1 \dots d^3 p_3 F^*(\mathbf{p}'_1 \dots \mathbf{p}'_3) F(\mathbf{p}_1 \dots \mathbf{p}_3) \mathcal{K}_{33}(\mathbf{p}'_1 \dots \mathbf{p}'_3, \mathbf{p}_1 \dots \mathbf{p}_3), \quad (5.2)$$

where the kernel $\mathcal{K}_{33} = \mathcal{K}_{33}^{(2)} + \mathcal{K}_{33}^{(3)}$ consists of the following components:

$$\begin{aligned} \mathcal{K}_{33}^{(2)}(\mathbf{p}'_1 \dots \mathbf{p}'_3, \mathbf{p}_1 \dots \mathbf{p}_3) &= -\frac{3}{4(2\pi)^3} \delta(\mathbf{p}'_1 + \mathbf{p}'_2 + \mathbf{p}'_3 - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ &\times \frac{\delta(\mathbf{p}'_3 - \mathbf{p}_3)}{\sqrt{p'_{10} p'_{20} p_{10} p_{20}}} [g^2 \tilde{D}(p'_2 - p_2) - \lambda/2], \end{aligned} \quad (5.3)$$

$$\begin{aligned} \mathcal{K}_{33}^{(3)}(\mathbf{p}'_1 \dots \mathbf{p}'_3, \mathbf{p}_1 \dots \mathbf{p}_3) &= -\frac{\kappa g^3}{4(2\pi)^6} \delta(\mathbf{p}'_1 + \mathbf{p}'_2 + \mathbf{p}'_3 - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \\ &\times \frac{\tilde{D}(p'_1 - p_1) \tilde{D}(p'_2 - p_2) \tilde{D}(p'_3 - p_3)}{\sqrt{p'_{10} \dots p'_{30} p_{10} \dots p_{30}}}, \end{aligned} \quad (5.4)$$

and $p_{i0} = \sqrt{m^2 + \mathbf{p}_i^2}$ and similarly for p'_{j0} ($i, j = 1, 2, 3$). The term $H_{\text{int}}^{(4)}$ does not contribute in \mathcal{K}_{33} , i.e. $\mathcal{K}_{33}^{(4)} = 0$.

The kernel \mathcal{K}_{33} determines the interaction in the relativistic three-particle wave equation that follows from the variational principle $\delta \langle 3|H - E|3\rangle = 0$, namely

$$\begin{aligned} \{p_{10} + p_{20} + p_{30} - E\} F(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \\ + \int d^3 p'_1 d^3 p'_2 d^3 p'_3 \mathcal{K}_{33}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3) F(\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3) = 0, \end{aligned} \quad (5.5)$$

where the kernel is understood to be the completely symmetrized expression (with respect to the variables $\mathbf{p}'_1, \mathbf{p}'_2, \mathbf{p}'_3$ and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$) of (5.3) and (5.4).

The term $\mathcal{K}_{33}^{(2)}$ of the kernel corresponds to the attractive interaction via massless boson exchange and repulsive contact interaction between each pair of particles while $\mathcal{K}_{33}^{(3)}$ describes a cluster three-particle interaction.

From the mathematical viewpoint the three-body wave equation (5.5) is an integral equation with a singular kernel. Even in simpler (say, two-particle) cases such equations are usually solved approximately (variationally, numerically, perturbatively), and it is not easy to get a general qualitative characteristic of the solutions, or to estimate the role of different terms of the kernel.

In order to have some understanding of the properties of the cluster interaction we consider the non-relativistic limit of equation (5.5), in which case the kernels simplify considerably, and then perform the Fourier transformation into coordinate space. In this representation the equation is simply a Schrödinger equation for the three-particle eigenfunction $\Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ (see [12]) and eigenenergy $\epsilon = E - 3m$:

$$\left\{ \frac{1}{2m} (\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{p}_3^2) + V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) - \epsilon \right\} \Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = 0, \quad (5.6)$$

where $\mathbf{p}_i = -i\nabla_i$ ($i = 1, 2, 3$), and the potential $V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, like the relativistic kernel \mathcal{K}_{33} , consists of two parts, $V = V_{33}^{(2)} + V_{33}^{(3)}$:

$$\begin{aligned} V_{33}^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= -\frac{g^2}{16\pi m^2} \left\{ \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} + \frac{1}{|\mathbf{x}_2 - \mathbf{x}_3|} + \frac{1}{|\mathbf{x}_3 - \mathbf{x}_1|} \right\} \\ &+ \frac{\lambda}{8m^2} \{ \delta(\mathbf{x}_1 - \mathbf{x}_2) + \delta(\mathbf{x}_2 - \mathbf{x}_3) + \delta(\mathbf{x}_3 - \mathbf{x}_1) \}, \end{aligned} \quad (5.7)$$

$$V_{33}^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{2\kappa g^3}{(8\pi m)^3} U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \quad (5.8)$$

where

$$U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \equiv - \int \frac{d^3 z}{|z - \mathbf{x}_1| |z - \mathbf{x}_2| |z - \mathbf{x}_3|}. \quad (5.9)$$

The integral on the r.h.s. of (5.9) is a divergent quantity and thus equation (5.6) may seem to be meaningless. However, the gradients $\partial U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)/\partial \mathbf{x}_i$ ($i = 1, 2, 3$) which determine the forces in the classical background of this problem, are well defined and finite. Thus, ‘function’ (5.9) can be presented in the form

$$U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \tilde{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + U_0, \quad (5.10)$$

where $\tilde{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is a regular (finite) function and U_0 is an infinite negative constant (independent of the variables $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$). This constant can be absorbed by the eigenenergy ϵ so that the wave equation (5.6) gets reformulated as follows:

$$\begin{aligned} V_{33}^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &\rightarrow \tilde{V}_{33}^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{2\kappa g^3}{(8\pi m)^3} \{U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) - U_0\} \\ &\equiv \frac{2\kappa g^3}{(8\pi m)^3} \tilde{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \end{aligned} \quad (5.11)$$

$$\epsilon \rightarrow \tilde{\epsilon} = E - 3m - \frac{2\kappa g^3}{(8\pi m)^3} U_0, \quad (5.12)$$

where the eigenenergy $\tilde{\epsilon}$ is finite (as is the potential $\tilde{V}_{33}^{(3)}$).

In order to perform this reformulation explicitly, we need to resort to regularization of integral (5.9) which we consider in the next section.

The problem of divergences is expected in the relativistic case too. But the analysis of the integral equation (5.5) is a more subtle problem which shall not be undertaken in this work.

6. Properties and evaluation of the three-point potential

Various regularization procedures are possible. In essence, one introduces some cut-off parameter which finally is put to 0 (or ∞). We enumerate some possibilities:

- (1) We could consider the case where the mediating χ field is massive, whereupon there would be a mass term $-\frac{1}{2}\mu^2 \chi^2$ in the Lagrangian (3.1). In that case the gravity-like $\frac{1}{r}$ factors would be replaced by the Yukawa forms $\frac{e^{-\mu r}}{r}$. Thus, we could regard U of equation (5.9) as the massless-mediating-field limit of the massive-mediating-field case:

$$U_\mu(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = - \int d^3 z \frac{e^{-\mu|z-\mathbf{x}_1|}}{|z-\mathbf{x}_1|} \frac{e^{-\mu|z-\mathbf{x}_2|}}{|z-\mathbf{x}_2|} \frac{e^{-\mu|z-\mathbf{x}_3|}}{|z-\mathbf{x}_3|}, \quad (6.1)$$

which is well defined and finite for any $\mu > 0$.

We note that by changing the variable of integration from z to $\mathbf{v} = z - \mathbf{x}_1$ in equation (6.1), we can write U_μ as

$$U_\mu(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = - \int d^3 v \frac{e^{-\mu v}}{v} \frac{e^{-\mu|\mathbf{v}+\mathbf{x}_{12}|}}{|\mathbf{v}+\mathbf{x}_{12}|} \frac{e^{-\mu|\mathbf{v}+\mathbf{x}_{13}|}}{|\mathbf{v}+\mathbf{x}_{13}|} = \tilde{U}_\mu(\mathbf{x}_{12}, \mathbf{x}_{13}), \quad (6.2)$$

where $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ and $v = |\mathbf{v}|$.

(2) Another way would be to regard U of equation (5.9) as a limiting case, as $R \rightarrow \infty$, of

$$\bar{U}_R(\mathbf{x}_{12}, \mathbf{x}_{13}) = - \int_0^R dv v \int d\hat{v} \frac{1}{|v + \mathbf{x}_{12}||v + \mathbf{x}_{13}|}, \quad (6.3)$$

where $\hat{v} = v/v$ and R is an arbitrarily large, but finite, ‘radius of space’.

(3) We could, also, regard U of equation (5.9) as a limiting case, as $\Lambda \rightarrow 0_+$, of

$$\bar{U}_\Lambda(\mathbf{x}_{12}, \mathbf{x}_{13}) = - \int_0^\infty dv v e^{-\Lambda v} \int d\hat{v} \frac{1}{|v + \mathbf{x}_{12}||v + \mathbf{x}_{13}|}. \quad (6.4)$$

Evidently, any other suitable and convenient cut-off function can be used in place of $e^{-\Lambda v}$.

Of course, physical results would be meaningful to the extent that they were independent of the choice of the regularization procedure.

Below we establish some general properties of the regularized \tilde{U} function. We discuss in detail a convenient method of its evaluation and show that it possesses a logarithmic confining property when $\mu \rightarrow 0$.

Let us consider the regularization $U_\mu(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ (6.1). It obviously obeys the following symmetry properties:

- (i) translational invariance: $U_\mu(\mathbf{x}_1 + \boldsymbol{\lambda}, \mathbf{x}_2 + \boldsymbol{\lambda}, \mathbf{x}_3 + \boldsymbol{\lambda}) = U_\mu(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, where $\boldsymbol{\lambda} \in \mathbb{R}^3$;
- (ii) rotational invariance: $U_\mu(\mathbf{R}\mathbf{x}_1, \mathbf{R}\mathbf{x}_2, \mathbf{R}\mathbf{x}_3) = U_\mu(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, where $\mathbf{R} \in \text{SO}(3)$;
- (iii) permutational invariance: $U_\mu(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3) = U_\mu(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_2) = U_\mu(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$;
- (iv) scaling transformation: $U_\mu(\lambda\mathbf{x}_1, \lambda\mathbf{x}_2, \lambda\mathbf{x}_3) = U_{\lambda\mu}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, where $\lambda \in \mathbb{R}_+$.

These properties have implications for the structure of the regularized potential.

The properties (i)–(iii) hold for arbitrary values of the cut-off parameter μ , including the formal limiting case $\mu \rightarrow 0$. Moreover, these are fundamental symmetries inherent to any interaction potential of a closed (nonrelativistic) system of three identical particles. Thus, the regularized potential must possess the properties (i)–(iii) of necessity.

The scaling property (iv) has specific implication for regularization (6.1). In the formal limit $\mu \rightarrow 0$ the ‘function’ $U \equiv U_{\mu=0}$ is scale invariant:

$$(iv)' \text{ scale invariance: } U(\lambda\mathbf{x}_1, \lambda\mathbf{x}_2, \lambda\mathbf{x}_3) = U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \text{ where } \lambda \in \mathbb{R}_+.$$

However, as is shown below, the scaling property of the regularized potential $\tilde{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is different.

We note that an important property of the potential $U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, with any of the regularizations (6.1)–(6.4), follows from the symmetries 1–3, namely that it actually depends only on the three inter-point distances x_{12}, x_{13}, x_{23} , where $x_{ij} = |\mathbf{x}_{ij}|$. Explicitly, this is readily seen if the factors $\frac{e^{-\mu|v+\mathbf{x}_{ij}|}}{|v+\mathbf{x}_{ij}|}$ in equations (6.1)–(6.4) are expanded in spherical harmonics ($\mu \equiv 0$ in (6.3) and (6.4)), the angular integrations $\int d\hat{v} \dots$ are carried out, and the orthogonality properties of the spherical harmonics are used, then (after the remaining integration over dv), the result is seen to depend only on the lengths of the two vectors $\mathbf{x}_{12}, \mathbf{x}_{13}$ and on the angle between them (or, equivalently, on x_{12}, x_{13}, x_{23}).

The direct calculation of the regularized potential, with any of the regularizations (6.1)–(6.4), is complicated. Instead, we propose a representation for function (5.9) in which its dependence on scalar arguments is manifest. This greatly simplifies the regularization and evaluation of U . Let us apply the well-known formula:

$$\frac{1}{r} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dk e^{-k^2 r^2}$$

to each factor of the integrand of expression (5.9) (which shall be treated formally). Then changing the order of integration we have

$$\begin{aligned} U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= -\frac{1}{\pi^{3/2}} \int d^3k \int d^3z e^{-k_1^2(z-x_1)^2 - k_2^2(z-x_2)^2 - k_3^2(z-x_3)^2} \\ &= -\int \frac{d^3k}{k^3} e^{-(k_1^2 x_{12}^2 + k_2^2 x_{23}^2 + k_3^2 x_{13}^2)/k^2} \\ &= -\int d\hat{\mathbf{k}} \int_0^\infty \frac{dk}{k} e^{-(\hat{k}_1^2 x_{12}^2 + \hat{k}_2^2 x_{23}^2 + \hat{k}_3^2 x_{13}^2)k^2}, \end{aligned} \quad (6.5)$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$. It is obvious in this form that $U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = U(x_{12}, x_{23}, x_{13})$ and, in addition, that the internal integral in the last line of (6.5) is divergent at its lower boundary $k = 0$.

The potential difference:

$$U(x_{12}, x_{23}, x_{13}) - U(y_{12}, y_{23}, y_{13}) = -\int d\hat{\mathbf{k}} \int_0^\infty \frac{dk}{k} [e^{-X^2 k^2} - e^{-Y^2 k^2}], \quad (6.6)$$

where $X^2 = \hat{k}_1^2 \hat{k}_2^2 x_{12}^2 + \hat{k}_2^2 \hat{k}_3^2 x_{23}^2 + \hat{k}_1^2 \hat{k}_3^2 x_{13}^2$, $Y^2 = \hat{k}_1^2 \hat{k}_2^2 y_{12}^2 + \hat{k}_2^2 \hat{k}_3^2 y_{23}^2 + \hat{k}_1^2 \hat{k}_3^2 y_{13}^2$, will be finite since infinite constants U_0 (see (5.10)) from the first and second terms of (6.6) mutually cancel. Indeed, using the cut-off parameter ε in the internal integral on the r.h.s. of (6.6) yields

$$\int_\varepsilon^\infty \frac{dk}{k} [e^{-Y^2 k^2} - e^{-X^2 k^2}] = \left[\int_{Y\varepsilon}^\infty - \int_{X\varepsilon}^\infty \right] \frac{dt}{t} e^{-t^2} = \int_{Y\varepsilon}^{X\varepsilon} \frac{dt}{t} e^{-t^2} \xrightarrow{\varepsilon \rightarrow 0} \ln \frac{X}{Y},$$

i.e. the integral is convergent.

Next, we introduce angular variables $\{\vartheta, \varphi\}$ on the unit sphere in k -space, so that $\hat{k}_1 = \sin \vartheta \cos \varphi$, $\hat{k}_2 = \sin \vartheta \sin \varphi$, $\hat{k}_3 = \cos \vartheta$. Then

$$U(x_{12}, x_{23}, x_{13}) - U(y_{12}, y_{23}, y_{13}) = W(\bar{x}_{12}, \bar{x}_{23}, \bar{x}_{13}) - W(\bar{y}_{12}, \bar{y}_{23}, \bar{y}_{13}), \quad (6.7)$$

where

$$\begin{aligned} W(\bar{x}_{12}, \bar{x}_{23}, \bar{x}_{13}) &= \frac{1}{2} \int_0^{2\pi} d\varphi \int_0^\pi \sin \vartheta d\vartheta \ln[(\bar{x}_{12} \sin \vartheta \cos \varphi \sin \varphi)^2 \\ &\quad + (\bar{x}_{23} \cos \vartheta \sin \varphi)^2 + (\bar{x}_{13} \cos \vartheta \cos \varphi)^2] \end{aligned} \quad (6.8)$$

and $\bar{x}_{ij} = x_{ij}/a$. The arbitrary constant a (with dimension of length) is introduced so that the argument of the logarithm will be dimensionless. Actually, the potential difference (6.7) does not depend on a while function (6.8) itself does. Since this function is well defined and finite, it can be considered, up to some additive constant, as the regularized potential:

$$\tilde{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = W(\bar{x}_{12}, \bar{x}_{23}, \bar{x}_{13}) - W_0. \quad (6.9)$$

The choice of the constant W_0 is a matter of taste; it can be canceled by an appropriate rescaling of the constant a : $W(x_{12}/a, \dots) = W(x_{12}/b, \dots) + 4\pi \ln(b/a)$. Thus, an arbitrariness of the regularized potential arises due to the scale constant a .

We note that the regularized function (6.9) obeys the following scaling property:

$$(\tilde{\text{iv}}) \text{ scale invariance: } \tilde{U}(\lambda \mathbf{x}_1, \lambda \mathbf{x}_2, \lambda \mathbf{x}_3) = \tilde{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) + 4\pi \ln \lambda, \text{ where } \lambda \in \mathbb{R}_+.$$

The inner integration (over ϑ) in (6.8) can be performed explicitly. Then the change of variable $\varphi \rightarrow s = \cos \varphi$ yields

$$\tilde{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = 4\pi \ln \frac{x_{13} + x_{23}}{4a} + I(\xi, \eta), \quad (6.10)$$

where

$$I(\xi, \eta) = 4 \int_{-1}^1 \frac{ds}{\sqrt{(s + \xi)^2 + \eta^2}} \arctan \sqrt{\frac{(s + \xi)^2 + \eta^2}{1 - s^2}}, \quad (6.11)$$

$$\xi = \frac{x_{13}^2 - x_{23}^2}{x_{12}^2}, \quad \eta^2 = \frac{[(x_{13} + x_{23})^2 - x_{12}^2][x_{12}^2 - (x_{13} - x_{23})^2]}{x_{12}^4}, \quad (6.12)$$

and we have chosen for convenience: $W_0 = 4\pi(\ln 2 - 1)$. We note that the interparticle distances must satisfy the triangle inequalities: $x_{13} + x_{23} \geq x_{12}$, $x_{23} + x_{12} \geq x_{13}$ and $x_{12} + x_{13} \geq x_{23}$.

The regularized potential (6.10)–(6.12) possesses the permutational invariance (iii) implicitly. This is evident from the fact that any particle permutation is equivalent to some renumbering of k -variables in integrals (6.5) and (6.6) and, finally, to another choice of angular variables in integral (6.8).

In particular cases where the points x_1 , x_2 and x_3 lie on a straight line integral (6.11) can be calculated analytically:

$$\tilde{U}(x_1, x_2, x_3) = 4\pi \ln \frac{x_{>}}{2a}, \quad \text{where } x_{>} = \max(x_{12}, x_{13}, x_{23}). \quad (6.13)$$

Another analytically solvable case is that of equidistant points, $x_{12} = x_{13} = x_{23} = r$, whereupon in (6.12), $\xi = 0$ and $\eta^2 = 3$, so that I of (6.11) is a finite constant independent of r . Thus, $\tilde{U}(r, r, r) = 4\pi \ln(r/a) + c_1$, where c_1 is a finite constant, which we can ignore (it does not affect energy differences). For convenience we shall use ‘atomic units’, that is, energies will be in units of $m\alpha^2$, and lengths in units of $a = m\alpha$, where $\alpha = \frac{g^2}{16\pi m^2}$ is the dimensionless ‘fine structure constant’. The total potential $V = V_{33}^{(2)} + V_{33}^{(3)}$ (cf equation (5.7) and (5.8)) is (with $\lambda = 0$), in atomic units,

$$V(r) = -\frac{3}{r} + \gamma \ln r, \quad (r \text{ is } r\alpha, \text{ and } V \text{ is } V/m\alpha^2), \quad (6.14)$$

where $\gamma = 4\kappa/g$. We see that $V(r)$, in this equidistant-points subspace, is a uniformly increasing, logarithmically confining potential (for $\gamma > 0$). Note that $V(r) \simeq -\frac{3}{r}$ for small r ($r \rightarrow 0_+$) but $V(r) \simeq \gamma \ln r$ for large r . Recall that if $\kappa = \gamma = 0$, the bound-state eigenvalue spectrum (in atomic units) is the Rydberg spectrum $\epsilon_n = -\frac{3}{2n^2}$, where $n = 1, 2, 3, \dots$, and there are no bound states for $\epsilon > 0$. However, for $\gamma > 0$, the logarithmic confining potential stretches out this Rydberg spectrum, so that there is a purely bound-state spectrum for $\epsilon > 0$. Using various approximations [18, 19] one can estimate $\epsilon_n \simeq \gamma \ln n$ for $n \gg 1$. (The repulsive contact (delta-function) potentials, which we have ignored by taking $\lambda = 0$, are of little consequence, since such repulsive contact potentials have an insignificant effect on the energy spectrum.)

Other regularization methods lead, basically, to the same results. For example, if we use the cut-off regularization of (6.3), then for the case $x_{13} = 0$, we obtain $\tilde{U}_R = -4\pi[1 + \ln(R/x_{12})] = 4\pi \ln(x_{12}/a) - c_2$, where a is an arbitrary length parameter (length unit), and $c_2 = 4\pi[1 + \ln(R/a)]$ is a very large constant, which has to be absorbed into a redefined (shifted) energy, as in (5.12). This result is the same as equation (6.13).

In the general case, a numerical integration of (6.11) is required. We illustrate the behavior of the potential in figure 1 for the particular case $x_1 = a$, $x_2 = -a$ as a function of $x_3 = r$. The value of potential for arbitrary configuration can be obtained from it using the symmetry properties (i)–(iii) and (iv).

In the case where one of the points is far from the others, equality (6.13) is valid asymptotically. Thus, the regularized potential reveals logarithmic confinement properties.

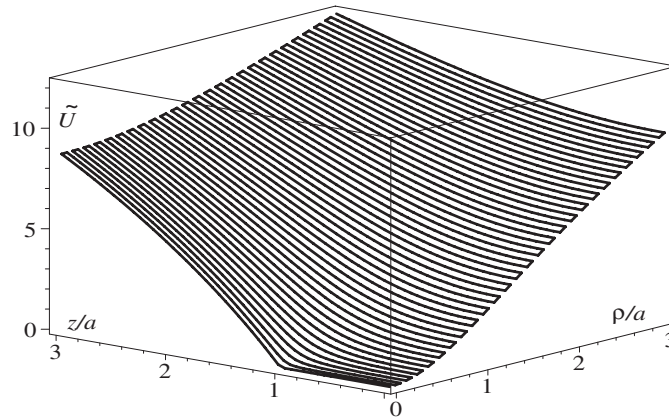


Figure 1. The potential $\tilde{U}(a, -a, \mathbf{r})$ as a function of $\mathbf{r} = \{x, y, z\}$; $\rho = \sqrt{x^2 + y^2}$; $a = |a|$. The function is symmetric under the inversion $z \rightarrow -z$ and rotation around $0z$. In particular, $\tilde{U} = 4\pi\theta(|z| - a) \ln \frac{1}{2}(|z|/a + 1)$ if $\rho = 0$.

A detailed analysis of the (non-relativistic) bound-state spectrum for the general case requires the solution of the three body equation (5.6). This is a quite challenging task in itself. However, from the confining nature of the three-point potential, we can see that the spectrum will reflect confinement, much like for the equidistant-points subspace of (6.14).

7. Concluding remarks

We have considered generalizations of the Wick–Cutkosky (massless scalar Yukawa) model that include nonlinear mediating fields. Covariant Green’s functions were used to eliminate the mediating field, thus arriving at a Lagrangian that contains nonlocal interaction terms.

In the case of a massless mediating field χ , with a $\frac{1}{3}\kappa\chi^3 + \frac{1}{4}\varkappa\chi^4$ nonlinearity, we evaluate the corresponding interaction term explicitly and show that the kernel has the form of a three- and four-point ‘cluster potential’, cf (4.4) and (4.5).

We consider the quantized version of this model in the Hamiltonian formalism, and use the variational method, with trial states built from Fock-space components, to derive a relativistic integral wave equation for the three-particle system. The kernels (relativistic potentials) are shown to contain one-quantum exchange terms and a three-point cluster term. In the non-relativistic limit we evaluate the explicit coordinate-space form of the interaction potentials and show that they consist of attractive pairwise Coulombic potentials and a cluster three-point confining potential. The three-point potential, which arises from the $\frac{1}{3}\kappa\chi^3$ term in the Hamiltonian, is divergent (and so needs regularization), but the potential differences are finite. The regularized three-point potential is shown to be logarithmically confining, and dependent only on the three inter-point distances. Its evaluation, for arbitrary values of its arguments, is shown to be reducible to a single quadrature.

The three-body wave equation derived in this paper is quite complicated and must be solved using approximation methods. This will be the subject of forthcoming work.

The three-particle trial state (5.1) is found to be the simplest variational ansatz which manifests the confinement properties of the model. However, other sectors of the Fock space in the variational problem are also of interest. For example, an open problem is the role of the three-point interaction in the particle-antiparticle problem. It was pointed out in the section 5

that the simple variational particle-antiparticle trial state $|1+\bar{1}\rangle$ does not sample the $H_{\text{int}}^{(3)}$ term (4.7) of the Hamiltonian. Thus, this term does not influence the variational wave equation derived only by using $|1+\bar{1}\rangle$ (see [9, 11, 12]), in which case the only Coulomb-like interaction arise. But the inclusion of both the $|1+\bar{1}\rangle$ and $|2+\bar{2}\rangle$ sectors leads to a coupled set of two many-body wave equations [13] in which the effects of $H_{\text{int}}^{(3)}$ and $H_{\text{int}}^{(4)}$ are present. Whether these effects are confining is a question that needs to be investigated.

Lastly, we comment on the role of ‘chion’ Fock-space sector in the variational bound-state problem within the reduced Hamiltonian formalism of QFT used in this work. This role can be examined by taking into account the χ_0 -dependent extra terms $\Delta\mathcal{L}$ of the total non-local Lagrangian (3.4). They are at least quadratic in χ_0 including the free-field term $\frac{1}{2}\partial_\mu\chi_0\partial^\mu\chi_0$ and interaction terms; see equation (3.7). Thus, the additional Hamiltonian corresponding to the extra terms, ΔH , has no effect on variational states $|\Psi\rangle$ without free ‘chions’ (i.e. quanta of the field χ_0), since $\langle\Psi|\Delta H|\Psi\rangle = 0$ for such states. A non-trivial contribution to a variational bound-state problem may arise from states with two or more virtual ‘chions’ but this is a higher order effect in the coupling constants (κ , \varkappa or others) of the potential \mathcal{V} .

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Appendix. Nonlocal Lagrangian from a nonlinear dipole model

In this section we consider a model which is built in analogy to the linear ‘dipole model’ [4, 11] that simulates the confinement interaction of quarks in mesons. This model is nonlinear and gives Yukawa + cluster interactions. It is specified by the Lagrangian

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \frac{1}{4}\lambda(\phi^*\phi)^2 + \rho(\chi + \frac{1}{2}\varphi) + \partial_\mu\chi\partial^\mu\varphi - \mathcal{V}(\varphi), \quad (\text{A.1})$$

where both the $\chi(x)$ and $\varphi(x)$ are real massless scalar fields and $\rho = -g\phi^*\phi$ as in (3.1).

The variation of the action (2.1) and (A.1) leads to the coupled set of the Euler–Lagrange equations

$$(\square + m^2)\phi = -g\phi(\chi + \frac{1}{2}\varphi) - \lambda\phi(\phi^*\phi), \quad (\text{A.2})$$

$$(\square + m^2)\phi^* = -g\phi^*(\chi + \frac{1}{2}\varphi) - \lambda\phi^*(\phi^*\phi), \quad (\text{A.3})$$

$$\square\varphi = \rho, \quad (\text{A.4})$$

$$\square\chi = \frac{1}{2}\rho - \mathcal{V}'(\varphi), \quad (\text{A.5})$$

which determine the field dynamics.

Equations (A.4) and (A.5) possess the exact formal solution:

$$\varphi = D * \rho, \quad (\text{A.6})$$

$$\chi = D * \left\{ \frac{1}{2}\rho - \mathcal{V}'(\varphi) \right\} = D * \left\{ \frac{1}{2}\rho - \mathcal{V}'(D * \rho) \right\}, \quad (\text{A.7})$$

which can immediately be used on the r.h.s. of equation (A.2) and (A.3):

$$(\square + m^2)\phi = -g\phi D * \left\{ \rho - \mathcal{V}'(D * \rho) \right\} - \lambda\phi(\phi^*\phi), \quad (\text{A.8})$$

and similarly for ϕ^* . These equations can be derived from $\delta I = 0$, with a Lagrangian identical to (3.6) (but note that no iterative expansion, like that in equation (3.6), needs to be made in this case).

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