Національна академія наук України

ІНСТИТУТ
ФІЗИКИ
КОНДЕНСОВАНИХ
СИСТЕМ

ICMP–03–08E

Yuri Kozitsky*

MATHEMATICAL THEORY OF THE ISING MODEL AND ITS GENERALIZATIONS: AN INTRODUCTION

*Institute of Mathematics, Maria Curie-Skłodowska University, 20-031 Lublin, Poland,E-mail: jkozi@golem.umcs.lublin.pl

ЛЬВІВ

УДК: 530.145

PACS: 05.50.+q, 64.60.Cn, 75.10.Hk

Математична теорія моделі Ізінга та її узагальнення: вступ

Юрій Козицький

Анотація. Стаття є вступом в строгу теорію рівноважного стану множини граткових моделей класичної та квантової статистичної фізики. Розглядаються узагальнені моделі Ізінга з дискретними, неперервними, обмеженими і необмеженими спінами, трансляційно іварінтні та з ієрархічною структурою; квантові спінові моделі, моделі квантових ангармонічних осциляторів. Для класичних моделей обговорюються деякі властивості гібсових станів, такі як теорема Лі-Янга, кореляційні нерівності, фазові переходи, самоподібність. Для квантових моделей на елементарному рівні описано підхід, побудований на функціональному інтегруванні.

Mathematical theory of the Ising model and its generalizations: an introduction

Yuri Kozitsky

Abstract. An introduction into the rigorous theory of equilibrium states of a number of lattice models of classical and quantum statistical physics is given. Generalized Ising models with discrete, continuous, bounded and unbounded spins, translation invariant and with a hierarchical structure; quantum spin models, models of interacting quantum anharmonic oscillators are considered. For the classical models, certain properties of local Gibbs states, such as the Lee-Yang theorem, correlation inequalities, phase transitions, self-similarity, are discussed. For the quantum models, an approach based on functional integration is presented on an introductory level.

Подається в "Order, Disorder, and Criticality: Advanced Problems of Phase Transitions Theory", edited by Yu. Holovatch (World Scientific, Singapore, 2003)
Submitted to "Order, Disorder, and Criticality: Advanced Problems of Phase Transitions Theory", edited by Yu. Holovatch (World Scientific, Singapore, 2003)

1. Introduction

This article is addressed to those physicists working in statistical physics who would want to learn modern mathematical methods and concepts used by mathematicians also working in this area. Although both communities study the same object, there exists a serious gap between the ways of getting and expressing knowledge, which quite often impedes substantially the exchange of this knowledge between them. The intention of the article is to help to make just first steps towards treating equilibrium states, phase transitions, critical points, etc., as mathematical objects. As a continuation, a serious work on such classical sources as Refs. [1]– [14] is recommended. The article is more or less self-contained, nevertheless the reader is supposed to possess certain knowledge in functional analysis (linear operators on Banach and Hilbert spaces, see Ref. [15]), analytic functions (holomorphic functions of one and several complex variables, entire functions, see Refs. [16], [17]), measure theory, see Refs. [18]– [20], probability and stochastic processes, see Ref. [21]. The article is mainly a review, although certain results and approaches are new. Among them – a new approach to the description of the critical point in one-dimensional models (classical and quantum) with long-range interactions.

The Ising model was introduced in 1925. Ising solved the model in the one-dimensional case [22] (see also Refs. [23], [24]) and came to the conclusion that it has no phase transitions in all dimensions. Later, due to Onsager's solution [25], it had become clear that the two-dimensional version of the model does have a phase transition and a critical point. Since that time, the Ising model has become one of the most popular models of statistical physics. A very important conclusion, which one can come up to by analyzing Onsager's solution, is that the phase transition singularities of thermodynamic functions, such as the free energy density, magnetization etc., occur only in the infinite-volume (thermodynamic) limit. Another important peculiarity of Onsager's solution is that it cannot be extended to the three-dimensional case[1]. This fact stimulated a more serious mathematical approach to the description of lattice models of this kind. The state of the art account in this area may be found in the monographs Refs. [13], [14].

Originally the Ising model was considered as a quantum model described in terms of spin operators. Later, it was understood [29] that

[1]It is believed [26] that the three-dimensional and two-dimensional Ising models have different types of time complexity. The $3D$-model has a non-polynomial time complexity, whereas the $2D$-model – polynomial. More about complexity – a very popular conception of modern science – see Refs. [27], [28].

there exists a deep connection between the Ising model, the φ^4-models of the Euclidean quantum field theory and classical lattice models. Moreover, due to its diagonality, the Ising model may be considered as a classical model as well. In accordance with this duality the main body of the article consists of two parts dedicated to classical and quantum models respectively. In the first part (section 2) we consider a number of generalizations of the Ising model, which may be described in terms of systems of depending random variables (spins) indexed by the elements of a d-dimensional simple cubic lattice of unit spacing $\mathbb{Z}^d$. In this context the Ising model describes a system of interacting spins taking values ± 1. In its generalizations the spins take values: (a) from a finite sets $s_1, \ldots, s_n$ (discrete spins), (b) from intervals like $[a, b]$ (bounded continuous spins); (c) from the whole real line (unbounded spins). These values are taken with certain probability (in the Ising model both ± 1 are taken with probability 1/2). Different types of probability laws, which prescribe these probabilities are discussed. Local Gibbs states are introduced as probability measures, which are constructed by means of local Hamiltonians and the probability laws discussed above. Here and below *local* means related to a finite subset of the lattice $\mathbb{Z}^d$. The central notion of this part is the infinite-volume Gibbs state, which is defined by means of local Gibbs states as a probability measure. As it has been pointed out above, the only possibility to describe phase transitions in such models is to construct these infinite-volume states, or at least to get information about their properties. Such information may be obtained by studying local Gibbs states, in particular analytic properties of local partition functions. Valuable information may be obtains with the help of correlation inequalities, which we discuss in subsection 2.3 . In subsection 2.6 we show how to prove that the infinite-volume Gibbs state of the Ising model with a nonzero external field is unique at all temperatures. This uniqueness means that only one phase may exist hence no phase transitions are possible. The proof is based on the correlation inequalities and analytic properties of the model partition function as a function of the external field. Among the main problems of statistical physics a special place belongs to the problem of criticality. At a critical point the infinite-volume Gibbs state possesses unusual properties. In particular, it is characterized by large fluctuations due to which the usual central limit theorem fails to hold whereas the law of large numbers is still valid. Such a phenomenon is interesting not only for physicists - the appearance of the strong dependence between random patterns is studied in population genetics, mathematical finance, etc. In subsection 2.7 we consider some new aspects of the theory of critical points in a number of models discussed in this section.

In section 2 we have restricted ourselves to real-valued spin models. Therefore, we do not consider classical models with vector spins, taking values in $\mathbb{R}^n$ with $n > 1$. We also leave without consideration models like the Potts model, the clock model, etc. Finally, we do not consider a very interesting class of spin models on graphs (like the Bethe lattices), which now are getting more and more popular (see e.g., Refs. [30], [31] and pp.170-173 in the book Ref. [13]).

In the second part (section 3) we discuss how to construct local Gibbs states of quantum lattice models, which can be considered as generalizations of the Ising model. We consider two types of such models: (a) non-diagonal spin models (like the Heisenberg spin model), which may be described by means of finite complex matrices; (b) models of interacting localized quantum particles, described by unbounded momentum and position operators. A typical example of the latter models is the model of quantum anharmonic oscillators, which now is extensively employed in the theory of structural phase transitions [32]. The local Gibbs states of quantum models are constructed as positive linear normalized functionals on non-commutative algebras of observables. Such functionals are defined by means of density matrices, which in turn are defined in terms of local Hamiltonians. All these objects - local Hamiltonians, density matrices, observables, may be realized as operators acting on certain Hilbert spaces. In subsection 3.1 we give a brief introduction and some examples on this matter, including a number of facts from the theory of such operators. In subsection 3.2 we discuss the main technical tool in quantum statistical physics which gives a possibility to describe local Gibbs states by means of Matsubara functions constructed for observables taken from a commutative subalgebra of the algebra of all observables. In the approach to the description of the models of quantum anharmonic oscillators initiated in Ref. [33], the Matsubara functions are written as integrals on function spaces, which makes this description similar to the description of models of classical statistical physics, the models considered in section 2 in particular. The only difference is that now the spins are infinite-dimensional. This approach is called *Euclidean* because of its similarity to the corresponding approach in quantum field theory. Within this approach it is possible to construct infinite-volume Gibbs states on the same base as in the case of classical models. We present here certain aspects of this approach, a full description of which may be found in Ref. [34]. In subsection 3.3 we give some statements regarding phase transitions and critical phenomena in the models of quantum anharmonic oscillators obtained in the Euclidean approach.

2. Classical Models

2.1. Local Hamiltonians and Gibbs states

We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ the sets of positive integer, nonnegative integer, integer, real and complex numbers respectively. For simplicity reasons, we consider a simple cubic lattice of unit spacing, i.e., our lattice is $\mathbb{Z}^d$, $d \in \mathbb{N}$. Let Δ be a finite subset of the lattice $\mathbb{Z}^d$. Among such subsets we will distinguish boxes

$$\Lambda = (-L, L]^d \cap \mathbb{Z}^d, \quad L \in \mathbb{N}. \tag{2.1}$$

Below, otherwise explicitly stated, Δ, respectively Λ, will always stand for an arbitrary finite subset of $\mathbb{Z}^d$, respectively a box (2.1). The number of lattice points in Δ, Λ will be denoted by $|\Delta|$, $|\Lambda|$. Since Λ is a particular case of Δ, everything stated for subsets Δ will be valid also for boxes Λ.

We start the description of the models we consider by introducing the measure ϱ on $\mathbb{R}$ which describes a priori probability distribution of a random variable corresponding to a "particle". The measure which would describe a system of non-interacting such particles, each of which is labelled by an element of a subset Δ, should be the product of this ϱ taken over Δ. If the particles interact with each other, the measure which describes this system is obtained [6] as a "Gibbsian reconstruction" of the product measure performed by means of the energy functional. In the simplest case this is a quadratic form on the Euclidean space $\mathbb{R}^{|\Delta|}$ consisting of vectors $\sigma_\Delta = (\sigma_{\mathbf{l}})_{\mathbf{l}\in\Delta}$ with the components $\sigma_{\mathbf{l}}$, $\mathbf{l} \in \Delta$, which reads

$$H_\Delta = -\frac{1}{2}\sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'} - \sum_{\mathbf{l}\in\Delta} h_{\mathbf{l}}\sigma_{\mathbf{l}}, \tag{2.2}$$

where $J_{\mathbf{l}\mathbf{l}'} = J_{\mathbf{l}'\mathbf{l}}$, $h_{\mathbf{l}}$ are real parameters of the model defined for all $\mathbf{l}, \mathbf{l}' \in \mathbb{Z}^d$. By means of these objects, we introduce the following probability measure on the space $\mathbb{R}^{|\Delta|}$

$$\begin{aligned} \mathrm{d}\nu_\Delta(\sigma_\Delta) &= Z_{\beta,\Delta}^{-1}\exp\left(-\beta H_\Delta\right)\prod_{\mathbf{l}\in\Delta}\mathrm{d}\varrho(\sigma_{\mathbf{l}}), \\ Z_{\beta,\Delta} &= \int_{\mathbb{R}^{|\Delta|}}\exp\left(-\beta H_\Delta\right)\prod_{\mathbf{l}\in\Delta}\mathrm{d}\varrho(\sigma_{\mathbf{l}}), \end{aligned} \tag{2.3}$$

where β is the inverse temperature measured in energy units. This measure is called *the local Gibbs measure*, or equivalently *the local Gibbs state*, corresponding to the zero condition on the boundary (i.e., outside) of Δ. The mentioned reconstruction was performed by multiplying the product measure in (2.3) by the corresponding factor, which turns to be one for $\beta = 0$ when the particles become non-interacting. The normalization constant $Z_{\beta,\Delta}$, which provides that $\int d\nu_\Delta = 1$, is called *the partition function* in the subset Δ and

$$F_{\beta,\Delta} = -\frac{1}{\beta|\Delta|} \ln Z_{\beta,\Delta}, \tag{2.4}$$

is called *the free energy density*.

A particular case of the above model, where the reference measure ϱ is

$$d\varrho(\sigma_{\mathbf{l}}) = \delta(\sigma_{\mathbf{l}}^2 - 1)d\sigma_{\mathbf{l}} = \frac{1}{2}\left[\delta(\sigma_{\mathbf{l}} - 1) + \delta(\sigma_{\mathbf{l}} + 1)\right] d\sigma_{\mathbf{l}}, \tag{2.5}$$

is nothing else but the Ising model with the interaction potential $J_{\mathbf{l}\mathbf{l}'}$ in the external field $h_{\mathbf{l}}$. This field is called *homogeneous* if $h_{\mathbf{l}} = h$ for all $\mathbf{l} \in \mathbb{Z}^d$. In (2.5) δ is the Dirac δ-function, thus the above measure is symmetric and concentrated at ± 1.

Due to the fact that the Ising model is a particular case of the model described by (2), the random variables in a general situation are called "spins", the energy functional (2.2) is called "Hamiltonian", the measure ϱ is called "single-spin measure". The models for which, like for the Ising model, the measure ϱ is concentrated at points $s_1, s_2, \dots s_n \in \mathbb{R}$ are called models with discrete spin. Such a model with $s_1 = -1, s_2 = 0, s_3 = 1$ and with the single-spin measure

$$d\varrho(\sigma_{\mathbf{l}}) = c[\delta(\sigma_{\mathbf{l}} + 1) + \delta(\sigma_{\mathbf{l}} - 1)] + (1 - 2c)\delta(\sigma_{\mathbf{l}}), \quad c \in (0, 1/2),$$

was studied in Ref. [35]. The models for which the measure ϱ is not concentrated at any points are called models with continuous spin. As an example here one may take the model with the single-spin measure

$$d\varrho(\sigma_{\mathbf{l}}) = \frac{1}{2}\varpi_{[-1,1]}(\sigma_{\mathbf{l}})d\sigma_{\mathbf{l}}, \tag{2.6}$$

where $\varpi_{[-1,1]}(t) = 1$ if $t \in [-1, 1]$ and $\varpi_{[-1,1]}(t) = 0$ otherwise. The models for which there exists $a > 0$ such that

$$\int_{[-a,a]} d\varrho \stackrel{\text{def}}{=} \int_{-a}^{a} d\varrho = 1,$$

are called models with bounded (compact) spins. Thus, discrete spins are always bounded. The models for which the measure ϱ is not concentrated on a bounded interval are called models with unbounded spins. Among

such models, a significant role belongs to the so called polynomial models, for which

$$d\varrho(\sigma_{\mathbf{l}}) = C^{-1} \exp\left(-P(\sigma_{\mathbf{l}})\right) d\sigma_{\mathbf{l}}, \quad C = \int_{\mathbb{R}} \exp\left(-P(\sigma_{\mathbf{l}})\right) d\sigma_{\mathbf{l}}, \tag{2.7}$$

where P is a polynomial, i.e.,

$$P(\sigma_{\mathbf{l}}) = b_1\sigma_{\mathbf{l}} + \dots b_{2r}\sigma_{\mathbf{l}}^{2r}, \quad b_{2r} > 0. \tag{2.8}$$

Such a polynomial is semi-bounded, which means that for all its arguments $P(\sigma_{\mathbf{l}}) \geq b_0$ for some real b_0. A measure ϱ on $\mathbb{R}$ is called symmetric if for every $0 \leq a < b$,

$$\int_a^b d\varrho = \int_{-b}^{-a} d\varrho.$$

The measure (2.7) is symmetric if P is even, i.e., only even powers appear in (2.8). A typical example of such a measure is the symmetric Gaussian measure, for which $P(\sigma_{\mathbf{l}}) = (b/2)\sigma_{\mathbf{l}}^2$. Another typical example one obtains by setting $r = 2$,

$$\begin{aligned} d\varrho(\sigma_{\mathbf{l}}) &= C^{-1} \exp\left(-a\sigma_{\mathbf{l}}^2 - b\sigma_{\mathbf{l}}^4\right) d\sigma_{\mathbf{l}}, \quad a \in \mathbb{R}, \quad b > 0; \\ C &= \int_{\mathbb{R}} \exp\left(-P(\sigma_{\mathbf{l}})\right) d\sigma_{\mathbf{l}}, \end{aligned} \tag{2.9}$$

which is known as the φ^4 measure.

Such polynomial models have another interpretation. For the above P and the Hamiltonian (2.1), set

$$E_\Delta = -\frac{1}{2}\sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'} - \sum_{\mathbf{l}\in\Delta} h_{\mathbf{l}}\sigma_{\mathbf{l}} + \sum_{\mathbf{l}\in\Delta} P(\sigma_{\mathbf{l}}). \tag{2.10}$$

This functional may be considered as the potential energy of a system of interacting classical (non-quantum) oscillators, in which the first term is responsible for the inter-particle interaction whereas the second and the third ones represent the single-particle potential energy. In case $P(\sigma_{\mathbf{l}}) = (b/2)\sigma_{\mathbf{l}}^2$, $b > 0$, for all $\mathbf{l} \in \mathbb{Z}^d$, these oscillators are harmonic. A generalization of (2.7) and (2.10) may be made by replacing the polynomial P by a differentiable semi-bounded function. With the help of the potential energy (2.10) the measure (2.3) may be written in the form

$$d\nu_\Delta(\sigma_\Delta) = Z_{\beta,\Delta}^{-1} \exp\left(-\beta E_\Delta\right) \prod_{\mathbf{l}\in\Delta} d\sigma_{\mathbf{l}}, \tag{2.11}$$

which is the Gibbs measure of a system of classical oscillators, it is Gaussian if they are harmonic. Let us describe the latter case in more details. For $P(\sigma_{\mathbf{l}}) = (b/2)\sigma_{\mathbf{l}}^2$, we set

$$S_{\mathbf{ll}'} = b\delta_{\mathbf{ll}'} - J_{\mathbf{ll}'}, \quad \mathbf{l}, \mathbf{l}' \in \Delta, \tag{2.12}$$

where $\delta_{\mathbf{ll}'}$ is the Kronecker delta. Let S be the $|\Delta|\times|\Delta|$ symmetric matrix which elements are given by (2.12). It may be diagonalized and all its eigenvalues have to be real. The measure (2.11) will exist for all $d \in \mathbb{N}$ if all these eigenvalues are strictly positive. In this case the inverse matrix S^{-1} exists and the partition function may be written explicitly

$$Z_{\beta,\Delta} = (2\pi)^{|\Delta|/2} \left[\det S\right]^{-1/2} \exp\left\{ \frac{1}{2} \sum_{\mathbf{l},\mathbf{l}' \in \Delta} (S^{-1})_{\mathbf{ll}'} h_{\mathbf{l}} h_{\mathbf{l}'} \right\}. \tag{2.13}$$

If $J_{\mathbf{ll}'} \geq 0$ for all $\mathbf{l}, \mathbf{l}' \in \mathbb{Z}^d$, the necessary and sufficient condition for the mentioned eigenvalues to be positive is

$$b > \max_{\mathbf{l} \in \Delta} \sum_{\mathbf{l}' \in \Delta} J_{\mathbf{ll}'}. \tag{2.14}$$

The thermodynamic properties of the model make sense to consider only if the following

$$\sum_{\mathbf{l}' \in \mathbb{Z}^d} J_{\mathbf{ll}'} < \infty, \tag{2.15}$$

holds for all $\mathbf{l} \in \mathbb{Z}^d$. In this case the condition (2.14) will be satisfied for any Δ if

$$b > \sup_{\mathbf{l} \in \mathbb{Z}^d} \sum_{\mathbf{l}' \in \mathbb{Z}^d} J_{\mathbf{ll}'}. \tag{2.16}$$

If the latter condition fails to hold, the same will be with (2.14) for sufficiently large subsets Δ. In this case the infinite-volume Gibbs measure does not exist.

By means of the local Gibbs measure (2.3), one obtains physical quantities as the integrals

$$\int_{\Omega_\Delta} f(\sigma_\Delta) \mathrm{d}\nu_\Delta(\sigma_\Delta) \stackrel{\mathrm{def}}{=} \langle f \rangle_{\nu_\Delta}, \tag{2.17}$$

where we have set $\Omega_\Delta = \mathbb{R}^{|\Delta|}$. Such integrals are called expectation values of the functions f with respect to the measure ν_Δ. In particular, the mean magnetization in the set Δ, the two-point correlation function and the susceptibility of the model read respectively

$$M_\Delta = \frac{1}{|\Delta|} \sum_{\mathbf{l} \in \Delta} \langle \sigma_{\mathbf{l}} \rangle_{\nu_\Delta}, \quad K^\Delta_{\mathbf{ll}'} = \langle \sigma_{\mathbf{l}} \sigma_{\mathbf{l}'} \rangle_{\nu_\Delta} - \langle \sigma_{\mathbf{l}} \rangle_{\nu_\Delta} \langle \sigma_{\mathbf{l}'} \rangle_{\nu_\Delta}, \tag{2.18}$$

$$\chi_\Delta = \frac{1}{|\Delta|} \sum_{\mathbf{l},\mathbf{l}' \in \Delta} K^\Delta_{\mathbf{ll}'}. \tag{2.19}$$

The infinite-volume limit of such quantities, if it exists, will describe thermodynamic properties of the model. In general, the integrals (2.17) exist not for all functions $f : \Omega_\Delta \to \mathbb{R}$. If such a function is continuous and polynomially bounded, its expectation value $\langle f \rangle_{\nu_\Delta}$ exists for all measures ϱ of the type of (2.7). A polynomially bounded function by definition is a function $f : \Omega_\Delta \to \mathbb{R}$, which satisfies the condition

$$|f(\sigma_\Delta)| \leq f_0 + \left[\sum_{\mathbf{l} \in \Delta} \sigma_{\mathbf{l}}^2 \right]^n, \tag{2.20}$$

with certain $f_0 > 0$ and $n \in \mathbb{N}$. In the case of compact spins, the integrals (2.17) exist for all continuous functions. It should be pointed out here that the integrals (2.17) exist not only for continuous function, but their extension to wider classes of functions will complicate mathematics, which we are going to avoid in this article. Moreover, all functions, for which the expectations $\langle f \rangle_{\nu_\Delta}$ have a physical reason, are continuous, hence we may restrict ourselves to considering such functions only. Thereby, by $\mathcal{F}_\Delta$ we denote the set of all polynomially bounded continuous functions $f : \Omega_\Delta \to \mathbb{R}$. All the single-spin measures we consider in this article are supposed to satisfy the condition

$$\int_{\mathbb{R}} \exp\left(a s^2\right) \mathrm{d}\varrho(s) < \infty, \tag{2.21}$$

with a certain $a > 0$. In this case all function from $\mathcal{F}_\Delta$ will be integrable with respect to the local Gibbs measures (2.3).

2.2. Analytic properties of local Gibbs states

In this subsection we study the dependence of the partition function $Z_{\beta,\Delta}$, and hence of the free energy density $F_{\beta,\Delta}$, on the parameters $J_{\mathbf{ll}'}$ and $h_{\mathbf{l}}$. Here we extensively use notions and fact from the theory of entire functions, which may be found in the books Refs. [16], [17].

For discrete spins, the partition function $Z_{\beta,\Delta}$, as a sum of exponents, may be extended to an exponential type entire function of $h_\Delta = (h_\mathbf{l})_{\mathbf{l}\in\Delta} \in \mathbb{C}^{|\Delta|}$. The same remains true for all types of bounded spins – a fact which follows from the Paley-Wiener theorem (see. e.g., Ref. [17]). For the model of classical harmonic oscillators, i.e., for Gaussian ν_Δ, it may be extended to an entire function of h_Δ of order two (see (2.13)). For unbounded spins with the measure ϱ in the form (2.7), $Z_{\beta,\Delta}$ may be extended to an entire function of order between one and two, depending on the degree of the polynomial P (see below). Since for real $h_\mathbf{l}$, the function $\exp(-\beta H_\Delta)$ takes positive values only, the partition function $Z_{\beta,\Delta}$ is also positive, which means that the free energy density $F_{\beta,\Delta}$ is an analytic function on a domain in $\mathbb{C}^{|\Delta|}$, which contains $\mathbb{R}^{|\Delta|}$. This shows one more time that no phase transitions can arise until the volume (i.e., the subset Δ) remains finite since these phenomena are connected with the singularities of the free energy density (see Refs. [8], [10], [14]). On the other hand, by (2.4), the singularities of the free energy density may be connected with the zero points of the partition function. A classical result in this domain, known as the Lee-Yang theorem [36], see also Refs. [10], [11], states that for the Ising model, the only point of the real line which such zeros may reach in the infinite-volume limit is the origin. Here we present a generalization of this statement proved in the article Ref. [37]. To this end we introduce the following notion.

Definition 1: A symmetric probability measure ϱ on the real line $\mathbb{R}$ is said to possess the Lee-Yang property if

$$\varphi_\varrho(z) = \int_\mathbb{R} e^{zt} \mathrm{d}\varrho(t), \quad z \in \mathbb{C}, \tag{2.22}$$

is an entire function which has imaginary zeros only or has them none.

Then the Lee-Yang theorem in the version of E.H. Lieb and A.D. Sokal may be formulated as follows.

Proposition 2: *Let the spin model defined by the Hamiltonian (2.2) and the single-spin measure ϱ possess the properties: (a) $J_{\mathbf{l}\mathbf{l}'} \geq 0$ for all $\mathbf{l}, \mathbf{l}' \in \mathbb{Z}^d$; (b) the measure ϱ has the Lee-Yang property . Then, for every finite subset $\Delta \subset \mathbb{Z}^d$ and every $\beta > 0$, the partition function (2.3), as a function of h_Δ, can be extended to an entire function, which has nonzero values whenever $\Re(h_\mathbf{l}) > 0$ for all $\mathbf{l} \in \Delta$.*

Here $\Re(z)$ stands for the real part of $z \in \mathbb{C}$. The spin model for which $J_{\mathbf{l}\mathbf{l}'} \geq 0$ is called *ferromagnetic* . A corollary of the above statement, a particular case of which is equivalent to the original theorem proved by T.D. Lee and C.N. Yang (see below), is formulated as follows.

Proposition 3: *Let the conditions of the above proposition be satisfied and let $h_\mathbf{l} = z$ for all $\mathbf{l} \in \Delta$. Then, for every $\beta > 0$, the partition function (2.3), as a function of $z \in \mathbb{C}$, can be extended to an entire function, which has nonzero values whenever $\Re(z) \neq 0$.*

Let us analyze this statement in more details. For $h_\mathbf{l} = z$, one has from (2.2) and (2.3)

$$Z_{\beta,\Delta}(z) = \int_{\Omega_\Delta} \exp\left(\beta z \sum_{\mathbf{l}\in\Delta} \sigma_\mathbf{l} + \frac{\beta}{2} \sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}\sigma_\mathbf{l}\sigma_{\mathbf{l}'}\right) \prod_{\mathbf{l}\in\Delta} \mathrm{d}\varrho(\sigma_\mathbf{l}). \tag{2.23}$$

As it has been mentioned, the order ρ of $Z_{\beta,\Delta}$, as an entire function of z, belongs to the interval $\rho \in [1, 2]$. Since the measure ϱ is supposed to be symmetric, this function ought to be even. Thus, it can be written as the following infinite product (see e.g., Ref. [17])

$$Z_{\beta,\Delta}(z) = Z_{\beta,\Delta}(0) \exp\left(\gamma_0(\beta,\Delta)z^2\right) \prod_{j=1}^{\infty} \left(1 + \gamma_j(\beta,\Delta)z^2\right), \tag{2.24}$$

where $Z_{\beta,\Delta}(0) > 0$, $\gamma_j(\beta,\Delta) \geq 0$, for all $j = 0, 1, 2, \ldots$. The case of all $\gamma_j(\beta,\Delta) = 0$ is degenerate, it holds if and only if the single-spin measure is concentrated at zero, i.e., $\mathrm{d}\varrho(s) = \delta(s)\mathrm{d}s$. The case $\gamma_0(\beta,\Delta) > 0$ and $\gamma_j(\beta,\Delta) = 0$ for $j \in \mathbb{N}$, corresponds to the model of harmonic oscillators, for which the single-spin measure is Gaussian. In this case the function (2.24) may be written (see (2.13))

$$Z_{\beta,\Delta}(z) = Z_{\beta,\Delta}(0) \exp\left\{\left(\frac{\beta^2}{2} \sum_{\mathbf{l},\mathbf{l}'\in\Delta} (S^{-1})_{\mathbf{l}\mathbf{l}'}\right) z^2\right\}.$$

It has no zeros at all and the corresponding infinite-volume free energy density exists and has no singularities. For polynomial models with even polynomials, for which the single-spin measure has the form (2.7), (2.8) with $r \geq 2$, one has $\gamma_0(\beta,\Delta) = 0$ and $\gamma_j(\beta,\Delta) > 0$ for all $j \in \mathbb{N}$. In this case the order of grows of the function (2.24) is $\rho = 2r/(2r-1)$. This function has imaginary zeros at the points $z = \pm i/\sqrt{\gamma_j(\beta,\Delta)}$, $j \in \mathbb{N}$. An immediate consequence of the above analysis is that the only value of the homogeneous external field $h_\mathbf{l} = z$ at which the infinite-volume free energy density may have a singularity is $z = 0$. It occurs, when the zeros of $Z_{\beta,\Delta}(z)$ reach the origin as $\Delta \to \mathbb{Z}^d$. Further analysis of measures with the Lee-Yang property may be found in Refs. [38], [39]. So far, we have had no examples of measures ϱ possessing the Lee-Yang property.

Two such measures are well-known. These are the measure (2.5) and the symmetric Gaussian measure, i.e., a polynomial measure (2.7) with $P(\sigma_{\mathbf{l}}) = (b/2)\sigma_{\mathbf{l}}^2$

$$\mathrm{d}\varrho(s) = [b/2\pi]^{1/2} \exp\left(-(b/2)s^2\right) \mathrm{d}s, \quad b > 0. \tag{2.25}$$

In fact, for the former measure, one has $\varphi_\varrho(z) = \cosh(z)$, whereas for the latter one, $\varphi_\varrho(z) = \exp\left(z^2/2b\right)$. The original Lee-Yang theorem is equivalent to Proposition 3 with the single-spin measure (2.5) and with the nearest-neighbor interaction potential $J_{\mathbf{l}\mathbf{l}'} = J\delta_{1,|\mathbf{l}-\mathbf{l}'|}$, $J > 0$.

Another examples of measures possessing the Lee-Yang property are described by the following statement, which was proved in Ref. [40] (see also Refs. [41], [42]). Let $\mathbb{L}$ stand for the class of entire functions of a single complex variable, which are the polynomials possessing real non-positive zeros only or the limits of sequences of such polynomials, taken in the topology of uniform convergence on compact subsets of $\mathbb{C}$. Such functions are called *Laguerre entire functions*, their theory may be found in Ref. [16].

Proposition 4: *Given an entire function $g : \mathbb{C} \to \mathbb{C}$ let: (a) for every real $z \in [0, +\infty)$, this function have real values; (b) there exist $b > 0$ such that the function $\phi(z) = b + g'(z)$, where $g' = \mathrm{d}g/\mathrm{d}z$, belongs to the class $\mathbb{L}$. Then the measure*

$$\mathrm{d}\varrho(s) = C \exp\left(-g(s^2)\right) \mathrm{d}s, \quad C = \int_{\mathbb{R}} \exp\left(-g(s^2)\right) \mathrm{d}s, \tag{2.26}$$

possesses the Lee-Yang property. For this measure, φ_ϱ is of order $\rho = 2r/(2r-1)$ if g is a polynomial of degree $r \in \mathbb{N}$, and $\rho = 1$, if g is a transcendental function.

An immediate consequence of this statement is that the φ^4 measure (2.9) possesses this property (see also Theorem IX.15 in Ref. [11]).

2.3. Correlation inequalities

As it was mentioned in Introduction, correlation inequalities constitute the base of a number of powerful methods in the theory of models we consider. Here we describe the most important of them, a more detailed description of such inequalities and their applications may be found in the book Ref. [3]. Mostly these inequalities hold for ferromagnetic spin models only, i.e., for the models with $J_{\mathbf{l}\mathbf{l}'} \geq 0$, though some of them may be extended to more general interaction potentials. Therefore, in the statements presented below all inequalities hold for ferromagnetic spin models described by the Hamiltonian (2.2) and a single-spin measure ϱ, which satisfies (2.21). In some cases we impose more specific conditions on the measure ϱ. The model with $h_{\mathbf{l}} \geq 0$ (respectively $h_{\mathbf{l}} = 0$) will be called a model with a nonnegative (respectively with the zero) external field.

The first example is known as the Fortuin-Kastelyn-Ginibre (FKG) inequality [43]. To formulate it we will need the following notion. For $\sigma_\Delta = (\sigma_{\mathbf{l}})_{\mathbf{l}\in\Delta}$ and $\sigma'_\Delta = (\sigma'_{\mathbf{l}})_{\mathbf{l}\in\Delta}$, we write $\sigma_\Delta \leq \sigma'_\Delta$ if $\sigma_{\mathbf{l}} \leq \sigma_{\mathbf{l}'}$ for all $\mathbf{l} \in \Delta$. A function $f \in \mathcal{F}_\Delta$ is said to be monotone on Ω_Δ if $f(\sigma_\Delta) \leq f(\sigma'_\Delta)$, whenever $\sigma_\Delta \leq \sigma'_\Delta$.

Proposition 5: *For any two monotone functions $f, g \in \mathcal{F}_\Delta$*

$$\langle fg \rangle_{\nu_\Delta} \geq \langle f \rangle_{\nu_\Delta} \langle g \rangle_{\nu_\Delta}. \tag{2.27}$$

The next correlation inequalities are known mostly due to R.B. Griffiths [44], they are called Griffiths-Kelly-Sherman (GKS) inequalities.

Proposition 6: *Let the functions $f_1, \ldots, f_n, g_1, \ldots, g_n \in \mathcal{F}_\Delta$ be given. Suppose that each of them satisfies the following conditions: (a) it depends on one component $\sigma_{\mathbf{l}}$ of the vector σ_Δ only; (b) as a function of this $\sigma_{\mathbf{l}}$, it is either odd and monotone or even and monotone as a function of $|\sigma_{\mathbf{l}}|$. Then for any ferromagnetic spin model with the zero external field,*

$$\langle f_1 \ldots f_n \rangle_{\nu_\Delta} \geq 0; \tag{2.28}$$

$$\langle f_1 \ldots f_n g_1 \ldots g_n \rangle_{\nu_\Delta} \geq \langle f_1 \ldots f_n \rangle_{\nu_\Delta} \langle g_1 \ldots g_n \rangle_{\nu_\Delta}. \tag{2.29}$$

Definition 7: A probability measure ϱ on the real line $\mathbb{R}$ is said to be a Bridges-Fröhlich-Spencer (BFS) measure if it is of the form

$$\mathrm{d}\varrho(s) = C^{-1} \exp\left(-v(s^2)\right) \mathrm{d}s, \quad C = \int_{\mathbb{R}} \exp\left(-v(s^2)\right) \mathrm{d}s, \tag{2.30}$$

where the function $v : [0, +\infty) \to \mathbb{R}$ has the following properties: (a) there exist $v_0 \in \mathbb{R}$, $v_1 > 0$ such that $v(s^2) \geq v_0 + v_1 s^2$ for all $s \in \mathbb{R}$; (b) it is convex on $[0, +\infty)$, i.e., for any $\tau_1, \tau_2 \geq 0$ and $\theta \in [0, 1]$, it obeys $v(\theta\tau_1 + (1-\theta)\tau_2) \leq \theta v(\tau_1) + (1-\theta)v(\tau_2)$.

Definition 8: A probability measure ϱ on the real line $\mathbb{R}$ is said to be a Ellis-Monroe (EM) measure if it has the form (2.7) with an even polynomial (2.8), in which $b_2 \in \mathbb{R}$, $b_4, \ldots, b_{2r-2} \geq 0$, $b_{2r} > 0$.

The Gaussian measure (2.25), the φ^4 -measure (2.9), the measures (2.26) are both BFS- and EM-measures. Moreover, every EM-measure is a BFS-measure.

Now we introduce the Griffiths-Hurst-Sherman (GHS) inequality, its proof may be found in the article Ref. [45].

Proposition 9: *For any ferromagnetic model with a nonnegative external field and a EM single-spin measure ϱ, the following inequality*

$$\begin{aligned}\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta} \leq \langle\sigma_{\mathbf{l}_1}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta} + \langle\sigma_{\mathbf{l}_2}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta} + \\ +\langle\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_2}\rangle_{\nu_\Delta} - 2\langle\sigma_{\mathbf{l}_1}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta},\end{aligned} \tag{2.31}$$

holds for any $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3 \in \Delta$.

The proof of the Lebowitz inequality, which we formulate below, may be found in the book Ref. [3].

Proposition 10: *For any ferromagnetic model with the zero external field and a BFS single-spin measure ϱ, the following inequality*

$$\begin{aligned}\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_3}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta} \leq \langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_2}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_3}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta} + \langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta} + \\ +\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta},\end{aligned} \tag{2.32}$$

holds for any $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3, \mathbf{l}_4 \in \Delta$.

Remark 11: All the above correlation inequalities hold for the ferromagnetic Ising model with the corresponding external field.

The proof of this statement follows from the fact that the Ising model can be approximated by the model with the single-spin measure of the form (2.9), for which all these inequalities hold. Let us provide some more details. The measure

$$\mathrm{d}\varrho_\lambda(\sigma_\mathbf{l}) = C_\lambda^{-1}\exp\left(-\lambda(\sigma_\mathbf{l}^2-1)^2\right)\mathrm{d}\sigma_\mathbf{l}, \quad C_\lambda = \int_\mathbb{R}\exp\left(-\lambda(\sigma_\mathbf{l}^2-1)^2\right)\mathrm{d}\sigma_\mathbf{l} \tag{2.33}$$

with $\lambda > 0$ is evidently of the type of (2.9). By means of the Laplace method [46], one may prove the following statement. For any continuous polynomially bounded function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{\lambda\to+\infty}\int_\mathbb{R} f(\sigma_\mathbf{l})\mathrm{d}\varrho_\lambda(\sigma_\mathbf{l}) = \int_\mathbb{R} f(\sigma_\mathbf{l})\mathrm{d}\varrho^\mathrm{I}(\sigma_\mathbf{l}) = \frac{1}{2}\left[f(1)+f(-1)\right], \tag{2.34}$$

where ϱ^I is the measure (2.5). This statement has the following important corollary. Let ν_Δ^I (respectively ν_Δ^λ) denote the local Gibbs measure (2.3) corresponding to the Ising mode, i.e., the model with the single-spin measure (2.5) (respectively to the model with the single-spin measure (2.33)). Then for any continuous polynomially bounded function $f : \mathbb{R}^{|\Delta|} \to \mathbb{R}$, the following holds

$$\lim_{\lambda\to+\infty}\int_{\Omega_\Delta} f(\sigma_\Delta)\mathrm{d}\nu_\Delta^\lambda(\sigma_\Delta) = \int_{\Omega_\Delta} f(\sigma_\Delta)\mathrm{d}\nu_\Delta^\mathrm{I}(\sigma_\Delta), \tag{2.35}$$

which proves Remark 2.3. On the other hand, for any $\lambda > 0$ the measure ν_Δ^λ may be approximated by ν_Δ^I [29].

For the measure ν_Δ^I with the zero external field, the function

$$\psi_\Delta(z) = \int_{\Omega_\Delta}\exp\left(z\sum_{\mathbf{l}\in\Delta}\sigma_\mathbf{l}\right)\mathrm{d}\nu_\Delta^\mathrm{I}(\sigma_\Delta), \tag{2.36}$$

may be written also as $\psi_\Delta(z) = Z_{\beta,\Delta}(\beta^{-1}z)/Z_{\beta,\Delta}(0)$, where $Z_{\beta,\Delta}(\beta^{-1}z)$ is defined by (2.23) with the single-spin measure ϱ^I. By the Bochner theorem [18], there exists a unique probability measure ϱ_Δ on $\mathbb{R}$, such that this ψ_Δ may be written (c.f., the equation (2.22))

$$\psi_\Delta(z) = \int_\mathbb{R}\exp(zt)\mathrm{d}\varrho_\Delta(t).$$

The measure ϱ_Δ defines the probability distribution of the total spin

$$S_\Delta = \sum_{\mathbf{l}\in\Delta} s_\mathbf{l}.$$

By (2.24) and Definition 1, this measure has the Lee-Yang property.

Let $\mathcal{D}$ be a sequence of subsets $\Delta \subset \mathbb{Z}^d$, such that for any two its elements Δ, Δ', one of them is contained in the other one. We also suppose that this sequence exhausts the lattice $\mathbb{Z}^d$, which means that any finite subset $A \subset \mathbb{Z}^d$ is contained in a certain $\Delta \in \mathcal{D}$. As a countable set, the sequence $\mathcal{D}$ can be enumerated in such a way that for any two its elements Δ_n, Δ_m, their numbers satisfy $n < m$ if and only if $\Delta_n \subset \Delta_m$. Then we may write $\mathcal{D} = \{\Delta_n\}_{n\in\mathbb{N}}$. For a sequence $\{c_\Delta\}_{\Delta\in\mathcal{D}}$ indexed by the elements of such $\mathcal{D}$, we write $\lim_\mathcal{D} c_\Delta$ or, if the sequence $\mathcal{D}$ has been specified, $\lim_{\Delta\to\mathbb{Z}^d} c_\Delta$ meaning $\lim_{n\to+\infty} c_{\Delta_n}$.

Definition 12: A sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ is said to converge weakly to the measure μ if for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to+\infty}\int_\mathbb{R} f(t)\mathrm{d}\mu_n(t) = \int_\mathbb{R} f(t)\mathrm{d}\mu(t). \tag{2.37}$$

Definition 13: A probability measure ϱ on $\mathbb{R}$ is called a Griffiths-Simon (GS) measure if it may be written as the defined above ϱ_Δ with corresponding Δ and nonnegative $J_{\mathbf{l}\mathbf{l}'}$ or it is the weak limit of a sequence $\{\varrho_\Delta\}_{\Delta\in\mathcal{D}}$ of such measures.

Thus by Ref. [29], the φ^4-measure is a GS-measure. Another example is

$$\mathrm{d}\varrho(t) = \frac{1}{3}[\delta(t-1)+\delta(t)+\delta(t+1)]\mathrm{d}t.$$

One may show that the sequence of φ_{μ_n}, defined by (2.22) for the above measures μ_n converges uniformly on bounded closed (i.e., compact) subsets of the complex plane $\mathbb{C}$ to the function φ_μ, which means that φ_μ has the representation (2.24) and hence the measure μ possesses the Lee-Yang property. It should be pointed out that not all of the EM- and BFS-measures possess this property (c.f., Proposition 4).

The Lebowitz inequality (2.32) may be generalized to the case of nonzero external field, but for another type of single-spin measures. The result presented below was proven in Ref. [45].

Proposition 14: *For any ferromagnetic model with a nonnegative external field and a single-spin measure ϱ, which is either of EM or of GS type, the following inequality*

$$\begin{aligned}\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_3}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta} \leq \langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_2}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_3}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta} + \langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta} + \\ +\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta} - 2\langle\sigma_{\mathbf{l}_1}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta},\end{aligned} \quad (2.38)$$

holds for any $\mathbf{l}_1,\mathbf{l}_2,\mathbf{l}_3,\mathbf{l}_4 \in \Delta$.

Yet another result connected with the Lebowitz inequality (2.32) was proven in Ref. [47], it is called the Aizenman-Fröhlich inequality (see also Ref. [3]).

Proposition 15: *For any ferromagnetic model with the zero external field and a GS single-spin measure ϱ, the following inequality*

$$\begin{aligned}\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_3}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta} - \langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_2}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_3}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta} - \\ -\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_4}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}_3}\rangle_{\nu_\Delta} \geq -2\sum_{\mathbf{l}\in\Delta}\langle\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_2}\sigma_{\mathbf{l}}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_3}\sigma_{\mathbf{l}}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}_4}\sigma_{\mathbf{l}}\rangle_{\nu_\Delta},\end{aligned} \quad (2.39)$$

holds for any $\mathbf{l}_1,\mathbf{l}_2,\mathbf{l}_3,\mathbf{l}_4 \in \Delta$.

It is worth noting that the Lebowitz inequality (2.32) gives an upper bound (it is zero) for the left-hand side of (2.39), whereas the latter inequality gives its lower bound.

By Proposition 2, the partition function of the Ising model $Z_{\beta,\Delta}(h_\Delta)$ can be extended to an entire function of complex h_Δ, which does not vanish in the vicinity of the point $h_\Delta = 0$. Therefore, in this vicinity its logarithm will be holomorphic and hence may be written

$$\begin{aligned}\Phi_{\beta,\Delta}(h_\Delta) \stackrel{\mathrm{def}}{=} \ln Z_{\beta,\Delta}(h_\Delta) = \ln Z_{\beta,\Delta}(0) + \\ +\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{\mathbf{l}_1,\ldots,\mathbf{l}_{2n}\in\Delta} U^{(2n)}_{\beta,\Delta}(\mathbf{l}_1,\ldots,\mathbf{l}_{2n})h_{\mathbf{l}_1}\ldots h_{\mathbf{l}_{2n}},\end{aligned} \quad (2.40)$$

where

$$U^{(2n)}_{\beta,\Delta}(\mathbf{l}_1,\ldots,\mathbf{l}_{2n}) = \left(\frac{\partial^{2n}}{\partial h_{\mathbf{l}_1}\ldots\partial h_{\mathbf{l}_{2n}}}\Phi_{\beta,\Delta}\right)(0), \quad n\in\mathbb{N}, \quad (2.41)$$

are called *Ursell functions*. They are also called *cumulants* or *semiinvariants*. By direct calculation,

$$U^{(2)}_{\beta,\Delta}(\mathbf{l}_1,\mathbf{l}_2) = K^{\Delta}_{\mathbf{l}_1\mathbf{l}_2}(0),$$

where the latter is the correlation function for this model with the zero external field (see (2.18)). By Remark 2.3 and Propositions 2.3, 2.3

$$\begin{aligned}- \quad 2\sum_{\mathbf{l}\in\Delta}U^{(2)}_{\beta,\Delta}(\mathbf{l}_1,\mathbf{l})U^{(2)}_{\beta,\Delta}(\mathbf{l}_2,\mathbf{l})U^{(2)}_{\beta,\Delta}(\mathbf{l}_3,\mathbf{l})U^{(2)}_{\beta,\Delta}(\mathbf{l}_4,\mathbf{l}) \leq \\ \leq U^{(4)}_{\beta,\Delta}(\mathbf{l}_1,\ldots,\mathbf{l}_4) \leq 0.\end{aligned} \quad (2.42)$$

For certain models, $U^{(2n)}_{\beta,\Delta}(\mathbf{l}_1,\ldots,\mathbf{l}_{2n})$, $n\in\mathbb{N}$ satisfy the following sign rule [48].

Proposition 16: *For all $\beta > 0$ and any finite subset Δ, the Ursell functions (2.41) for the ferromagnetic model with the zero external field and a GS single-spin measure, satisfy the sign rule*

$$(-1)^{n-1}U^{(2n)}_{\beta,\Delta}(\mathbf{l}_1,\ldots,\mathbf{l}_{2n}) \geq 0, \quad n\in\mathbb{N}. \quad (2.43)$$

Moreover, equality in (2.43) occurs either if the single-spin measure ϱ is Gaussian, or if and only if among the indices $\mathbf{l}_1,\ldots,\mathbf{l}_{2n}$ one finds the pair $\mathbf{l}_i,\mathbf{l}_j$ such that Δ may be divided onto disjoint Δ_1,Δ_2, $\mathbf{l}_i\in\Delta_1$ and $\mathbf{l}_j\in\Delta_2$, such that for any $\mathbf{l}\in\Delta_1$ and $\mathbf{l}'\in\Delta_2$, one has $J_{\mathbf{l}\mathbf{l}'} = 0$.

The following statements are important corollaries of the inequalities presented above. First, we obtain a property of the correlation functions $K^{\Delta}_{\mathbf{l}\mathbf{l}'}(h_\Delta)$ defined by (2.18).

Corollary 17: *For any ferromagnetic model with a nonnegative external field*

$$K^{\Delta}_{\mathbf{l}\mathbf{l}'}(h_\Delta) \geq 0. \tag{2.44}$$

Proof: By (2.18),

$$K^{\Delta}_{\mathbf{l}\mathbf{l}'}(h_\Delta) = \langle fg\rangle_{\nu_\Delta} - \langle f\rangle_{\nu_\Delta}\langle g\rangle_{\nu_\Delta} \geq 0,$$

where we have set $f(\sigma_\Delta) = \sigma_{\mathbf{l}}$, $g(\sigma_\Delta) = \sigma_{\mathbf{l}'}$ and employed the FKG inequality (2.27).

Recall that $h_\Delta \leq h'_\Delta$ means $h_{\mathbf{l}} \leq h'_{\mathbf{l}}$ for all $\mathbf{l} \in \Delta$, moreover, 0_Δ stands for such a vector h_Δ with all $h_{\mathbf{l}} = 0$.

Corollary 18: *For any ferromagnetic model with a EM single-spin measure ϱ, the correlation function, has the property*

$$K^{\Delta}_{\mathbf{l}\mathbf{l}'}(h'_\Delta) \leq K^{\Delta}_{\mathbf{l}\mathbf{l}'}(h_\Delta) \leq K^{\Delta}_{\mathbf{l}\mathbf{l}'}(0_\Delta), \tag{2.45}$$

for any h_Δ, h'_Δ such that $0_\Delta \leq h_\Delta \leq h'_\Delta$.

Proof: By (2.18) and (2.2), (2.3), one has

$$\frac{\partial}{\partial h_{\mathbf{l}''}} K^{\Delta}_{\mathbf{l}\mathbf{l}'}(h_\Delta) = \langle\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'}\sigma_{\mathbf{l}''}\rangle_{\nu_\Delta} - \langle\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}''}\rangle_{\nu_\Delta} - \langle\sigma_{\mathbf{l}}\sigma_{\mathbf{l}''}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}'}\rangle_{\nu_\Delta} -$$
$$-\langle\sigma_{\mathbf{l}'}\sigma_{\mathbf{l}''}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}}\rangle_{\nu_\Delta} + 2\langle\sigma_{\mathbf{l}}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}'}\rangle_{\nu_\Delta}\langle\sigma_{\mathbf{l}''}\rangle_{\nu_\Delta} \leq 0,$$

where we have used the GHS-inequality (2.31). Thus, as a function of h_Δ, $K^{\Delta}_{\mathbf{l}\mathbf{l}'}(h_\Delta)$ is monotone decreasing for all h_Δ, for which (2.31) is valid, i.e., for $h_\Delta \geq 0$.

Corollary 19: *For every monotone function $f \in \mathcal{F}_\Delta$, the expectation value $\langle f\rangle_{\nu_\Delta}$ is a monotone function of the external field h_Δ. This means that for any h_Δ, h'_Δ, such that $h_\Delta \leq h'_\Delta$, the following holds*

$$\langle f\rangle_{\nu_\Delta} \leq \langle f\rangle_{\nu'_\Delta}, \tag{2.46}$$

where ν_Δ (respectively ν'_Δ) is the local Gibbs measure (2.3) corresponding to the local Hamiltonian (2.2) with the external field h_Δ (respectively h'_Δ).

Proof: For $t \in [0, 1]$ and the mentioned h_Δ, h'_Δ, we set $x_\Delta(t) = h_\Delta + t(h'_\Delta - h_\Delta)$ and let $\nu_\Delta(t)$ be the measure (2.3) corresponding to the

external field $x_\Delta(t)$, which obviously satisfies the following "boundary condition" $\nu_\Delta(0) = \nu_\Delta$ and $\nu_\Delta(1) = \nu'_\Delta$. Then

$$\langle f\rangle_{\nu_\Delta(t)} = \tag{2.47}$$
$$= \frac{1}{Z_{\beta,\Delta}(t)} \int_{\Omega_\Delta} f(\sigma_\Delta) \exp\left(\frac{\beta}{2}\sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'} + \beta\sum_{\mathbf{l}\in\Delta} x_{\mathbf{l}}(t)\sigma_{\mathbf{l}}\right)\prod_{\mathbf{l}\in\Delta} \mathrm{d}\varrho(\sigma_{\mathbf{l}}),$$

where

$$Z_{\beta,\Delta}(t) = \int_{\Omega_\Delta} \exp\left(\frac{\beta}{2}\sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'} + \beta\sum_{\mathbf{l}\in\Delta} x_{\mathbf{l}}(t)\sigma_{\mathbf{l}}\right)\prod_{\mathbf{l}\in\Delta} \mathrm{d}\varrho(\sigma_{\mathbf{l}}). \tag{2.48}$$

Differentiating both parts of (2.47) and taking into account (2.48) one obtains

$$\frac{\partial}{\partial t}\langle f\rangle_{\nu_\Delta(t)} = \langle fg\rangle_{\nu_\Delta(t)} - \langle f\rangle_{\nu_\Delta(t)}\langle g\rangle_{\nu_\Delta(t)}, \tag{2.49}$$

where

$$g(\sigma_\Delta) \stackrel{\mathrm{def}}{=} \beta\sum_{\mathbf{l}\in\Delta}(h'_{\mathbf{l}} - h_{\mathbf{l}})\sigma_{\mathbf{l}},$$

which is a monotone function since $h_\Delta \leq h'_\Delta$. Now we apply in (2.49) the FKG-inequality, which obviously holds for the measure $\nu_\Delta(t)$ with any $t \in [0, 1]$, and obtain

$$\frac{\partial}{\partial t}\langle f\rangle_{\nu_\Delta(t)} \geq 0,$$

for all $t \in [0, 1]$. Therefore, $\langle f\rangle_{\nu_\Delta(t)}$ is a monotone function of t, which yields (2.46).

Corollary 20: *For every monotone function $f \in \mathcal{F}_\Delta$ and for any ferromagnetic model with the zero external field, the expectation value $\langle f\rangle_{\nu_\Delta}$ is a monotone function of the interaction parameter $J_{\mathbf{l}\mathbf{l}'}$. This means that for $0 \leq J_{\mathbf{l}\mathbf{l}'} \leq J'_{\mathbf{l}\mathbf{l}'}$ for all $\mathbf{l}, \mathbf{l}' \in \Delta$, the following holds*

$$\langle f\rangle_{\nu_\Delta} \leq \langle f\rangle_{\nu'_\Delta}, \tag{2.50}$$

where ν_Δ (respectively ν'_Δ) is the local Gibbs measure (2.3) corresponding to the local Hamiltonian (2.2) with the interaction potential $J_{\mathbf{l}\mathbf{l}'}$ (respectively $J'_{\mathbf{l}\mathbf{l}'}$).

The proof of this statement is almost the same as the proof of the former one and is based on the GKS-inequalities. Below we give another applications of the above inequalities.

A wide variety of correlation inequalities and examples of their applications are presented in the book Ref. [3], see also Ref. [49].

2.4. Infinite-volume Gibbs states

As has been pointed out above, phase transitions are possible only in the infinite-volume limit $\Delta \to \mathbb{Z}^d$. In order to pass to such a limit, we have to relate with each other functions belonging to $\mathcal{F}_\Delta$ and $\mathcal{F}_{\Delta'}$ with $\Delta \subset \Delta'$. Thus, each $f \in \mathcal{F}_\Delta$ will be considered also as an element of $\mathcal{F}_{\Delta'}$, which is independent of $\sigma_{\mathbf{l}}$ with $\mathbf{l} \in \Delta' \setminus \Delta$. This defines an embedding $\mathcal{F}_\Delta \subset \mathcal{F}_{\Delta'}$ for $\Delta \subset \Delta'$. We recall that a sequence $\mathcal{D}$ of subsets Δ is called increasing if, for every two its elements, one of them is a subset of the other one. It exhausts the lattice $\mathbb{Z}^d$ if every finite subset of the latter is contained in an element of $\mathcal{D}$. For such a sequence $\mathcal{D}$, we set

$$\mathcal{F} = \bigcup_{\Delta \in \mathcal{D}} \mathcal{F}_\Delta. \tag{2.51}$$

Therefore, for every $f \in \mathcal{F}$, one may find Δ_1, such that $f \in \mathcal{F}_{\Delta_1}$, and a sequence $\mathcal{D}$ in which this Δ_1 is the first element. Then the infinite-volume limit $\lim_{\mathcal{D}} \langle f \rangle_{\nu_\Delta}$ will make sense. By $\lim_{\mathcal{D}}$ we mean the limit in which $\Delta \to \mathbb{Z}^d$ along the sequence $\mathcal{D}$. At first glance, this setting is sufficient to describe possible phase transitions in our model. But a more detailed consideration immediately yields that it is not. For example, in the case of symmetric ϱ and all $h_{\mathbf{l}} = 0$ there is no way to break the symmetry $\sigma_{\mathbf{l}} \to -\sigma_{\mathbf{l}}$ of the model in such a limit. This means that certain thermodynamic properties of the model, especially those related to phase transitions, are described by quantities, which cannot be obtained as infinite-volume limits of expectations (averages) with respect to the measures (2.3). To obtain such quantities N.N. Bogolyubov [50] introduced a notion of quasi-averages. They are obtained by adding to the Hamiltonian (2.2) corresponding infinitesimal fields, which are to be removed after passing to the infinite-volume limit. At the same time, an approach to the construction of "all possible" infinite-volume Gibbs measures[2] on the base of conditional probabilities was elaborated by R.L. Dobrushin [51], [52] and by O.E. Lanford and D. Ruelle [53]. In this approach such measures are obtained as solutions of a certain equation, now known as *the Dobrushin-Lanford-Ruelle (DLR) equation*. A detailed description of this approach is given in the book Ref. [4]. Here we present a short introduction into this theory.

First of all we will need the space on which such infinite-volume measures are defined. Set

$$\Omega = \{\sigma = (\sigma_{\mathbf{l}})_{\mathbf{l} \in \mathbb{Z}^d} \mid \sigma_{\mathbf{l}} \in \mathbb{R}\}. \tag{2.52}$$

[2]More details on the relation between infinite-volume Gibbs states and phase transitions are given in the next susection.

This set consists of vectors σ, which have infinitely many real components $\sigma_{\mathbf{l}}$ indexed by the points of the lattice. Such vectors are called *configurations*. This set can be metrized by introducing the following "distance" between any two its elements σ, σ'

$$\mathbf{d}(\sigma, \sigma') = \sum_{\mathbf{l} \in \mathbb{Z}^d} \frac{1}{2^{|\mathbf{l}|}} \cdot \frac{|\sigma_{\mathbf{l}} - \sigma'_{\mathbf{l}}|}{1 + |\sigma_{\mathbf{l}} - \sigma'_{\mathbf{l}}|}, \tag{2.53}$$

where $|\mathbf{l}| = \sqrt{|l_1|^2 + \cdots + |l_d|^2}$. This enables us to introduce the set of all probability measures on Ω , which will be denoted by $\mathcal{P}(\Omega)$. It appears, see Ref. [4] p. 59, that the above introduced set of functions $\mathcal{F}$ separates the points of $\mathcal{P}(\Omega)$. This means that if the measures $\mu_1, \mu_2 \in \mathcal{P}(\Omega)$ have the property $\langle f \rangle_{\mu_1} = \langle f \rangle_{\mu_2}$ for all $f \in \mathcal{F}$, then they coincide. Here we write

$$\langle f \rangle_\mu = \int_\Omega f \mathrm{d}\mu. \tag{2.54}$$

In order to have the things we deal with as much simple as we can we suppose in this subsection that the interaction potential $J_{\mathbf{l}\mathbf{l}'}$ in the Hamiltonian (2.2) has a finite range, which means that this potential takes zero values whenever the distance $|\mathbf{l} - \mathbf{l}'|$ exceeds a certain $R > 0$. Given a subset Δ and a configuration $\xi \in \Omega$, we define the following probability measures

$$\mathrm{d}\nu_\Delta(\sigma_\Delta|\xi) = Z^{-1}_{\beta,\Delta}(\xi) \exp\left(-\beta H_\Delta(\xi)\right) \prod_{\mathbf{l} \in \Delta} \mathrm{d}\varrho(\sigma_{\mathbf{l}}), \tag{2.55}$$

$$\mathrm{d}\pi_\Delta(\sigma|\xi) = \mathrm{d}\nu_\Delta(\sigma_\Delta|\xi) \prod_{\mathbf{l} \in \Delta^c} \left[\delta(\sigma_{\mathbf{l}} - \xi_{\mathbf{l}}) \mathrm{d}\sigma_{\mathbf{l}}\right], \tag{2.56}$$

where

$$H_\Delta(\xi) = -\frac{1}{2} \sum_{\mathbf{l},\mathbf{l}' \in \Delta} J_{\mathbf{l}\mathbf{l}'} \sigma_{\mathbf{l}} \sigma_{\mathbf{l}'} - \sum_{\mathbf{l} \in \Delta} h_{\mathbf{l}} \sigma_{\mathbf{l}} - \sum_{\mathbf{l} \in \Delta, \mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'} \sigma_{\mathbf{l}} \xi_{\mathbf{l}'}, \tag{2.57}$$

$$Z_{\beta,\Delta}(\xi) = \int_{\Omega_\Delta} \exp\left(-\beta H_\Delta(\xi)\right) \prod_{\mathbf{l} \in \Delta} \mathrm{d}\varrho(\sigma_{\mathbf{l}}), \tag{2.58}$$

and $\Delta^c = \mathbb{Z}^d \setminus \Delta$. The first two terms of the latter Hamiltonian describe the energy of the self-interaction of the spins located in Δ whereas the third term corresponds to the interaction of these spins with the fixed configuration ξ outside Δ. Due to our assumption regarding the range of $J_{\mathbf{l}\mathbf{l}'}$, the sum in this term is finite hence no convergence problems appear. The essential difference between the above introduced measures is that $\nu_\Delta(\cdot|\xi)$[3] is defined on the space Ω_Δ consisting of vectors σ_Δ of finitely many components. The measure $\pi_\Delta(\cdot|\xi)$ is defined on the space Ω, but due to the presence of the δ-functions on the right-hand side of (2.56), the components of σ labelled by $\mathbf{l} \in \Delta^c$ are "frozen", i.e., they should coincide with the corresponding components of the configuration ξ. The partition function $Z_{\beta,\Delta}(\xi)$ is defined in the same way as in (2.3). In what follows, the measure (2.3) is a particular case of (2.55), which corresponds to the zero configuration $\xi = 0$. Hence ν_Δ is called the local Gibbs measure with the zero boundary condition, whereas $\nu_\Delta(\cdot|\xi)$ is the local Gibbs measure with the boundary condition defined by the configuration ξ, or simply, with the boundary condition ξ. For a function $f \in \mathcal{F}_\Delta$, one readily has

$$\int_{\Omega_\Delta} f(\sigma_\Delta)\mathrm{d}\nu_{\beta,\Delta}(\sigma_\Delta|\xi) = \int_\Omega f(\sigma)\mathrm{d}\pi_{\beta,\Delta}(\sigma|\xi),$$

where the function under the integral on the right-hand side is the same f but considered as a function defined on the whole Ω, which is independent of the components of σ labelled by $\mathbf{l} \in \Delta^c$. The measure $\pi_{\beta,\Delta}$ has the following property. For every $\mu \in \mathcal{P}(\Omega)$, the integral

$$\int_\Omega \mathrm{d}\pi_{\beta,\Delta}(\sigma|\xi)\mathrm{d}\mu(\xi),$$

is again a probability measure on Ω. We will denote this new measure by $\pi_{\beta,\Delta} \circ \mu$, that is we set

$$\mathrm{d}(\pi_{\beta,\Delta} \circ \mu)(\sigma) = \int_\Omega \mathrm{d}\pi_{\beta,\Delta}(\sigma|\xi)\mathrm{d}\mu(\xi). \tag{2.59}$$

The above integration has the following interpretation. Given a configuration $\xi \in \Omega$, the measure $\pi_{\beta,\Delta}(\cdot|\xi)$ defines a probability distribution of configurations $\sigma \in \Omega$ which ought to coincide with this ξ outside Δ. In the course of integration (2.59) the boundary conditions are averaged, i.e., they are taken with their weights which are prescribed by the measure μ. Suppose now that this new measure (2.59) coincides with the averaging measure μ and this takes place for every finite subset Δ. Then for any $f \in \mathcal{F}$, one finds Δ such that $f \in \mathcal{F}_\Delta$ and then

$$\int_\Omega f\mathrm{d}\mu = \int_\Omega \left(\int_\Omega f(\sigma)\mathrm{d}\pi_{\beta,\Delta}(\sigma|\xi) \right) \mathrm{d}\mu(\xi) = \int_\Omega f\mathrm{d}(\pi_{\beta,\Lambda} \circ \mu),$$

[3]In this way we indicate the dependence of ν_Δ, π_Δ on ξ.

which means that the expectation value of f with respect to the local Gibbs measure with averaged boundary conditions is the same as its expectation taken directly with respect to the measure μ. In other words, a kind of equilibrium between configurations inside and outside each such a Δ holds.

Definition 21: A probability measure $\mu \in \mathcal{P}(\Omega)$ is called an equilibrium (Gibbs) measure if, for any finite $\Delta \subset \mathbb{Z}^d$, the following equilibrium condition holds

$$\pi_{\beta,\Delta} \circ \mu = \mu. \tag{2.60}$$

The set of all Gibbs measures existing at a given β will be denoted by $\mathcal{G}_\beta$. It is defined by the family of all local Hamiltonians (2.2). The equality (2.60) may be considered as an equation, in which the unknown is μ. It is called the Dobrushin-Lanford-Ruelle equation. On can show (see e.g., Ref. [4]) that under the assumptions (2.21) made regarding the single-spin measure ϱ the set $\mathcal{G}_\beta$ in nonempty for all $\beta > 0$. Its elements are also called *phases* . If for a given β, it contains more than one element, the system considered may exists in more than one phases at the same conditions. And alternatively, if this set consists of exactly one element at all temperatures, no phase transitions are possible for this system. Suppose now that μ_1, μ_2 are two different elements of $\mathcal{G}_\beta$. Then, for every $\theta \in [0, 1]$, the combination

$$\mu = \theta\mu_1 + (1 - \theta)\mu_2, \tag{2.61}$$

which is called *a mixture* of the measures μ_1 and μ_2, is a probability measure (it is normalized since $\theta + (1 - \theta) = 1$). It solves the DLR equation (2.60) hence belongs to $\mathcal{G}_\beta$. This means that the latter set may contain either one or infinitely many elements. If an element $\mu \in \mathcal{G}_\beta$ cannot be written as a convex combination (2.61) with $\theta \neq 1, 0$ of any other elements of this set, it is called *a pure state*. In the case of the Ising model such pure states $\mu_\pm$ may be obtained as infinite-volume limits of the measures (2.55) corresponding to the boundary conditions $\xi_\pm$, all elements of which are equal ± 1. For $d \geq 2$ and large enough β, $\mu_+ \neq \mu_-$.

Now let us discuss how, for a given model, one may get its Gibbs states or at least how to get information regarding such states. Another question of this kind is whether one can get such states as infinite volume limits of local states, which would be very natural. The direct construction of Gibbs states may be made only for simple models, for example in the case of the Gaussian single-spin measure ϱ. In more nontrivial situations such states are studied by means of the DLR equation without their explicit construction. As for the second question, the answer is yes. It was obtained recently [54] and even for much more general objects than those we consider. In order to formulate the corresponding statement we have to make precise the notion of convergence. Recall that the set $\mathcal{F}$ consists of local polynomially bounded continuous functions $f : \Omega \to \mathbb{R}$, where "local" means that there exists a finite subset Δ, depending on f, such that f depends on $\sigma_{\mathbf{l}}$, $\mathbf{l} \in \Delta$ only. By $\mathcal{F}^{(0)}$ we denote the subset of $\mathcal{F}$ consisting of bounded functions. Let again $\mathcal{D}$ be an increasing sequence of subsets Δ, which exhausts the lattice $\mathbb{Z}^d$ and for which we write $\lim_{\mathcal{D}}$ meaning the limit $\Delta \to \mathbb{Z}^d$ taken along this sequence.

Definition 22: A sequence of probability measures $\{\mu_\Delta\}_{\Delta\in\mathcal{D}}$, defined on the configuration space Ω is said to converge locally weakly to a probability measure μ if for every $f \in \mathcal{F}^{(0)}$,

$$\lim_{\mathcal{D}} \left(\int_\Omega f \mathrm{d}\mu_\Delta \right) = \int_\Omega f \mathrm{d}\mu. \tag{2.62}$$

Then we have the following result [54].

Proposition 23: *For every $\xi \in \Omega$ and any sequence $\mathcal{D}$, the locally weak limit of the sequence of $\{\pi_{\beta,\Delta}(\cdot|\xi)\}_{\Delta\in\mathcal{D}}$, if it exists, is a Gibbs measure.*

Another result of this kind is taken from the book Ref. [4]. Pure states, i.e., the Gibbs measures which cannot be written as nontrivial convex combinations (2.61), are called extreme elements [4] of $\mathcal{G}_\beta$. Theorem 7.26 of Ref. [4], p.13 has the following corollary.

Proposition 24: *The set of extreme elements $\mathcal{G}_\beta^{\mathrm{ex}} \subset \mathcal{G}_\beta$ is nonempty. If $|\mathcal{G}_\beta^{\mathrm{ex}}| = 1$, then $|\mathcal{G}_\beta| = 1$.*

Another result of this book (Theorem 7.12 on p.122) reads as follows.

Proposition 25: *For every $\mu \in \mathcal{G}_\beta^{\mathrm{ex}}$ and any sequence $\mathcal{D}$, the sequence $\{\pi_{\beta,\Delta}(\cdot|\xi)\}_{\Delta\in\mathcal{D}}$ locally weakly converges to μ for almost all ξ. The latter means that this convergence may fail to hold only for boundary conditions ξ, which belong to a subset $A \subset \Omega$, such that $\mu(A) = 0$.*

Given $\mathbf{l}_0 \in \mathbb{Z}^d$ and a configuration $\sigma = (\sigma_{\mathbf{l}})_{\mathbf{l}\in\mathbb{Z}^d}$, we set $\vartheta_{\mathbf{l}_0}\sigma = (\sigma_{\mathbf{l}+\mathbf{l}_0})_{\mathbf{l}\in\mathbb{Z}^d}$. Now for a function $f \in \mathcal{F}$, we set

$$t_{\mathbf{l}_0}(f)(\sigma) = f(\vartheta_{\mathbf{l}_0}\sigma), \tag{2.63}$$

i.e., $\vartheta_{\mathbf{l}_0}$ and $t_{\mathbf{l}_0}$ are translations defined on the set of configurations and real valued functions of configurations respectively.

[4] The extreme elements of a plane triangle are its vertices.

Definition 26: The model, described by the family of Hamiltonians (2.2) with all possible finite $\Delta \subset \mathbb{Z}^d$, is called translation invariant if, for every $\mathbf{l_0} \in \mathbb{Z}^d$, its parameters satisfy the following conditions

$$\forall \mathbf{l}, \mathbf{l}' \in \mathbb{Z}^d : \quad J_{\mathbf{l}\mathbf{l}'} = J_{(\mathbf{l}+\mathbf{l}_0)(\mathbf{l}'+\mathbf{l}_0)}, \qquad \forall \mathbf{l} \in \mathbb{Z}^d : \quad h_{\mathbf{l}} = h_{\mathbf{l}+\mathbf{l}_0}.$$

The external field $h_{\mathbf{l}}$ which satisfies the above condition is homogeneous, i.e., it is constant $h_{\mathbf{l}} = h$. A particular case of the translation invariant interaction potentials is $J_{\mathbf{l}\mathbf{l}'} = \phi(|\mathbf{l}-\mathbf{l}'|)$, where ϕ is a real valued function. One has to remark here that each Hamiltonian (2.2) is not translation invariant but for a translation invariant model, in accordance with (2.63),

$$t_{\mathbf{l}_0} H_\Delta = H_{\Delta'},$$

where Δ' is obtained as a translation of Δ on the vector $\mathbf{l}_0$.

Definition 27: A Gibbs measure μ is called translation invariant if, for every $f \in \mathcal{F}$ and $\mathbf{l}_0 \in \mathbb{Z}^d$,

$$\langle f \rangle_\mu = \langle t_{\mathbf{l}_0}(f) \rangle_\mu.$$

One can show that, for any $\mathbf{l_0} \in \mathbb{Z}^d$ and every $\mu \in \mathcal{P}(\Omega)$, there exists $\tilde{\mu} \in \mathcal{P}(\Omega)$ such that, for every $f \in \mathcal{F}$,

$$\langle f \rangle_{\tilde{\mu}} = \langle t_{\mathbf{l}_0}(f) \rangle_\mu.$$

If this μ is translation invariant, then $\tilde{\mu} = \mu$. On the other hand, since $\mathcal{F}$ separates the points of $\mathcal{P}(\Omega)$, there exists exactly one such $\tilde{\mu}$. By Definition 21, for every $\mu \in \mathcal{G}_\beta$ and every $\mathbf{l}_0 \in \mathbb{Z}^d$, its $\tilde{\mu}$ will also belong to $\mathcal{G}_\beta$ provided the model is translation invariant. Therefore if, for a translation invariant model, the set of all Gibbs measures consists of one μ, this element should be a translation invariant measure. R.L. Dobrushin proved in Ref. [55] that, for the three-dimensional Ising model with the zero external field, below its critical point there exist infinitely many non-translation invariant phases. This is a consequence of the fact, also proved by R.L. Dobrushin [56], that in the Ising model on the lattice $\mathbb{Z}^d$ with $d \geq 3$, below T_C different phases may coexist separated by stable surfaces, which is impossible in the case $d = 2$ (see also Refs. [57], [58]).

Now we present some facts about the infinite volume convergence of thermodynamic functions. By means of estimates proven by D. Ruelle [59] , J.L. Lebowitz and E. Presutti [60] proved the existence of the infinite-volume free energy density, which is independent of the boundary conditions. Below we present this result in a simplified version, for more details and generalizations we refer the reader to the original work Ref. [60]. Given $N \in \mathbb{N}$ and a finite $\Delta \subset \mathbb{Z}^d$, we set

$$\Omega(N, \Delta) = \{\sigma \in \Omega \mid \sum_{\mathbf{l} \in \Delta} \sigma_{\mathbf{l}}^2 \leq N^2 |\Delta|\}. \tag{2.64}$$

The set of *tempered configurations* Ω^{t} by definition consist of those $\sigma \in \Omega$ for which there exists $N \in \mathbb{N}$ such that, for any finite Δ, $\sigma \in \Omega(N, \Delta)$. Of course, the zero configuration, for which all $\sigma_{\mathbf{l}} = 0$, is tempered. Let $\mathcal{L}$ be the sequence of cubes (2.1) defined by a sequence $\{L_n\}_{n \in \mathbb{N}}$ such that $L_{n+1} > L_n$, henceforth $L_n \to +\infty$. We say that the external field is bounded if there exists a constant a such that for all $\mathbf{l} \in \mathbb{Z}^d$, $|h_{\mathbf{l}}| \leq a$.

Proposition 28: *For the spin model described by the Hamiltonian (2.2) with a bounded external field and with the single-spin measure ϱ which has the form (2.7), (2.8) with $r \geq 2$ or (2.5), the free energy density*

$$F_{\beta,\Lambda}(\xi) = -\frac{1}{\beta|\Lambda|} \ln Z_{\beta,\Lambda}(\xi), \tag{2.65}$$

with $\xi \in \Omega^{\mathrm{t}}$ converges, as $\Lambda \to \mathbb{Z}^d$ along any sequence $\mathcal{L}$, to a limit, which does not depend on the choice of $\mathcal{L}$ and on ξ, hence it is the same as for the zero boundary condition.

We complete this subsection by describing a special kind of translation invariant Gibbs measures. Of course, they may be constructed for translation invariant models only. We suppose that the interaction potential has the form $J_{\mathbf{l}\mathbf{l}'} = \phi(|\mathbf{l} - \mathbf{l}'|)$. These measures are built by means of cubes (2.1). Given such a cube Λ , we define

$$|\mathbf{l} - \mathbf{l}'|_\Lambda = \sqrt{|\, l_1 - \; l_1'|_\Lambda^2 + \cdots + |\, l_d - \; l_d'|_\Lambda^2},$$

where $|\, l_j - \; l_j'|_\Lambda = \min\{|\, l_j - \; l_j'|; 2L - |\, l_j - \; l_j'|\}$, $j = 1, \ldots, d$. The above introduced function gives a periodic distance between the points $\mathbf{l}, \mathbf{l}' \in \Lambda$, it may be interpreted as a distance on the torus which one obtains by identifying the opposite walls of the cube Λ. Thereby, we set $J_{\mathbf{l}\mathbf{l}'}^\Lambda = \phi(|\mathbf{l} - \mathbf{l}'|_\Lambda)$ with the same ϕ. The Hamiltonian

$$H_\Lambda = -\frac{1}{2} \sum_{\mathbf{l}, \mathbf{l}' \in \Lambda} J_{\mathbf{l}\mathbf{l}'}^\Lambda \sigma_{\mathbf{l}} \sigma_{\mathbf{l}'} - \sum_{\mathbf{l} \in \Lambda} h \sigma_{\mathbf{l}}, \tag{2.66}$$

is invariant with respect to translations on the mentioned torus. Such Hamiltonians are said to satisfy the Born-von Karman boundary condition. By means of this Hamiltonian, one may construct the corresponding

(periodic) local Gibbs state $\nu_\Lambda^{(p)}$ exactly as it was done in (2.3) for the zero boundary condition. The only difference is that such periodic local Gibbs states may be defined only for boxes like (2.1) or their translates. The following statement may be proven on the base of the results of Refs. [54], [61].

Proposition 29: *There exists a tending to $+\infty$ sequence $\{L_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$, which defines by (2.1) the sequence of boxes $\{\Lambda_n\}_{n \in \mathbb{N}}$, such that the sequence of periodic local Gibbs states $\{\nu_{\beta,\Lambda_n}^{(p)}\}_{n \in \mathbb{N}}$ locally weakly converges to an element of $\mathcal{G}_\beta$, which is a translation invariant Gibbs state.*

2.5. Phase transitions and critical points

As it has been pointed out in the preceding subsection, phase transitions correspond to the fact that the set $\mathcal{G}_\beta$ contains more than one element – i.e., more than one phase. There exist several methods to prove that $|\mathcal{G}_\beta| > 1$, depending on the model considered. A profound and extended description to this problem is done in Part IV of the book Ref. [4]. We recommend also Refs. [8], [62]– [66] for further information on this subject.

In the case of the Ising model, the most known result, which inspired many of the approaches developed in the sequel, is due to R. Peierls [67] who proposed his famous contour method. One of its offsprings is now known as the Pirogov-Sinai theory, first publications on which are due to S.A. Pirogov and Ya. G. Sinai [68]. Its further development was done by several mathematicians, a complete description of this theory may be found in the article Ref. [69].

If the model possesses a symmetry, this symmetry should be preserved in the case $|\mathcal{G}_\beta| = 1$, i.e., a phase should possess this symmetry if it is unique. A typical example here is translation invariance, which was discussed in the preceding subsection. If $|\mathcal{G}_\beta| > 1$, the symmetry may be "distributed" among the different phases, whereas each of them does not possess this symmetry. Then the phase transition is connected with the loss of symmetry and is often called "spontaneous symmetry breaking". Another example, appropriate for our model (2.2), is the Z_2-symmetry, i.e., the symmetry with respect to the transformation $\sigma_{\mathbf{l}} \to -\sigma_{\mathbf{l}}$ for all $\mathbf{l} \in \mathbb{Z}^d$. Of course, to have this symmetry one has to take $h_{\mathbf{l}} = 0$ at all $\mathbf{l} \in \mathbb{Z}^d$. In the following section we show that in the case of the Ising model $|\mathcal{G}_\beta| = 1$ at all temperatures for a nonzero external field. Therefore, the only possibility to get a phase transition in this model is connected with the Z_2-symmetry breaking.

Here one has to point out that phase transitions may occur without symmetry breaking (see Chapter 19 in Ref. [4] and the references therein). An example here is the model (2.2) with $d \geq 3$, a nearest neighbor interaction, zero external field and with the polynomial single-spin measure (2.7) for which the polynomial P has two asymmetric wells. If one well is deep and steep and the other one is wide and shallow, and if the barrier which separates the wells is high enough, there exist the phases in which the particles are located mostly near the deep wells (one phase) and near the shallow wells (the other one). This result was proved by R.L. Dobrushin and S.B. Shlosman [70].

Now let us discuss how to detect that $|\mathcal{G}_\beta| > 1$, i.e., how to prove a phase transition. One way may be described as follows. For the model (2.2) with the homogeneous external field which takes two values $h_{\mathbf{l}} = \pm h$ and with a symmetric single-spin measure ϱ, one calculates the mean magnetization $M^{\pm}_{\Delta}(h)$, corresponding to these values (see (2.18)). Then one has to show that there exists β_C such that, for $\beta > \beta_C$

$$\lim_{h\to 0+} \lim_{\Delta\to\mathbb{Z}^d} M^{+}_{\Delta}(h) \neq \lim_{h\to 0+} \lim_{\Delta\to\mathbb{Z}^d} M^{-}_{\Delta}(h),$$

which obviously contradicts uniqueness of Gibbs states. In fact, if the phase is unique, both limits ought to coincide with the mean magnetization calculated for this phase. This way is very natural from the physical point of view but its mathematical realization may be technically impossible. Of course, instead of calculating $M^{\pm}_{\Delta}(h)$ and then the above limits, one may just to show that the difference $M^{+}_{\Delta}(h) - M^{-}_{\Delta}(h)$ is separated from zero for all $h > 0$ and Δ, but this task may be too difficult as well.

Another way to show that $|\mathcal{G}_\beta| > 1$ is based on the following important notion. For a probability measure μ on the set Ω, defined by (2.52), and a subset $A \subset \Omega$, we write

$$\mu(A) = \int_A \mathrm{d}\mu, \tag{2.67}$$

if the above integral exists[5]. Given such a subset A and $\mathbf{l}_0 \in \mathbb{Z}^d$, we set $\vartheta_{\mathbf{l}_0} A = \{\vartheta_{\mathbf{l}_0}\sigma \mid \sigma \in A\}$, i.e., this set is obtained by translating on $\mathbf{l}_0$ every configuration which belongs to A. For $A, B \subset \Omega$, we denote by $A \triangle B = (A \cup B) \setminus (A \cap B)$ their symmetric difference, i.e, the new set consists of those configurations which belong to exactly one of these sets.

[5] Such integrals exist for *measurable* subsets, which form a σ -algebra – a family of subsets of Ω, which contains Ω and is closed with respect to taking complements and countable unions. In our case this is the Borel σ-algebra – the smallest σ-algebra which contains all open subsets of Ω. To define which subsets are open one uses the metric (2.53).

Definition 30: A translation invariant probability measure μ on the set of all configurations Ω is said to be ergodic, if it has the following property. For every subset A such that, for all $\mathbf{l}_0 \in \mathbb{Z}^d$

$$\mu((\vartheta_{\mathbf{l}_0} A) \triangle A) = 0, \tag{2.68}$$

one has $\mu(A) = 0$ or $\mu(A) = 1$.

Below we present a number of statements, proven in the book Ref. [13], which describe the properties of such ergodic measures.

Proposition 31: *A translation invariant Gibbs state $\mu \in \mathcal{G}_\beta$ is ergodic if and only if it is a pure state. If $|\mathcal{G}_\beta| = 1$, then its unique element is ergodic.*

Thus, in order to prove that $|\mathcal{G}_\beta| > 1$ it is enough to show that there exists a nonergodic Gibbs state. This may be done on the base of the following von Neumann ergodic theorem (see Theorem III.1.8 in Ref. [13]).

Proposition 32: *A translation invariant state $\mu \in \mathcal{G}_\beta$ is ergodic if and only if, for every $f, g \in \mathcal{F}$,*

$$\lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|} \left\{ \sum_{\mathbf{l}_0\in\Lambda} \left[\langle (t_{\mathbf{l}_0} f) g\rangle_\mu - \langle f\rangle_\mu \langle g\rangle_\mu \right] \right\} = 0, \tag{2.69}$$

where Λ is defined by (2.1) and the above limit is taken in the sense $L \to +\infty$.

Now let $\mu \in \mathcal{G}_\beta$ be the periodic Gibbs measure to which a sequence of periodic local Gibbs states converges by Proposition 29. Let us take in (2.69) $f(\sigma) = g(\sigma) = \sigma_{\mathbf{0}}$. Set in (2.66) $h = 0$, then the state μ is Z_2-invariant, hence $\langle f\rangle_\mu = \langle g\rangle_\mu = 0$. By Proposition 29, one has

$$\langle \sigma_{\mathbf{l}_0}\sigma_{\mathbf{0}}\rangle_\mu = \lim_{n\to+\infty} \langle \sigma_{\mathbf{l}_0}\sigma_{\mathbf{0}}\rangle_{\nu^{(p)}_{\beta,\Lambda_n}}. \tag{2.70}$$

In view of the von Neumann ergodic theorem this immediately yields that the state μ is nonergodic if

$$\lim_{n\to+\infty} \frac{1}{|\Lambda_n|} \sum_{\mathbf{l}_0\in\Lambda_n} \langle \sigma_{\mathbf{l}_0}\sigma_{\mathbf{0}}\rangle_{\nu^{(p)}_{\beta,\Lambda_n}} > 0, \tag{2.71}$$

for a certain sequence of boxes $\{\Lambda_n\}_{n\in\mathbb{N}}$. Since the local Gibbs state $\nu^{(p)}_{\beta,\Lambda}$ is invariant with respect to the translations on the corresponding torus, the above condition is equivalent to the following

$$P(\beta) \stackrel{\mathrm{def}}{=} \lim_{n\to+\infty} \frac{1}{|\Lambda_n|^2} \sum_{\mathbf{l},\mathbf{l}'\in\Lambda_n} \langle \sigma_{\mathbf{l}}\sigma_{\mathbf{l}'}\rangle_{\nu^{(p)}_{\beta,\Lambda_n}} > 0. \tag{2.72}$$

This $P(\beta)$ is called *an order parameter*. By (2.28), $P(\beta) \geq 0$. If $P(\beta) > 0$, there exists a periodic nonergodic Gibbs state, which contradicts $|\mathcal{G}_\beta| = 1$ since if $\mathcal{G}_\beta$ is a singleton, its unique element has to be ergodic. Therefore, $P(\beta) > 0$ implies nonuniqueness of phases, i.e., a phase transition. On the other hand, by (2.18),

$$\chi = \lim_{\Lambda \to \mathbb{Z}^d} \chi_\Lambda = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{\mathbf{l},\mathbf{l}' \in \Lambda} \langle \sigma_{\mathbf{l}} \sigma_{\mathbf{l}'} \rangle_{\nu^{(p)}_{\beta,\Lambda}}, \tag{2.73}$$

is the infinite-volume susceptibility if the above order parameter $P(\beta)$ is zero, i.e., above the Curie temperature $T_C = \beta_C^{-1}$. The Curie temperature is defined as the supremum of the values of β^{-1} for which $P(\beta) > 0$. Comparing (2.72) and (2.73) one comes to the following conclusion:

- If $\chi < \infty$, then certainly $P(\beta) = 0$ and $T > T_C$.
- If $P(\beta) > 0$, then certainly the right-hand side of (2.73) is equal to $+\infty$, then there exist many phases, i.e., $T < T_C$ and one has to calculate susceptibility taking into account this fact.
- If $\chi = \infty$ but $P(\beta) = 0$, then $T = T_C$?

The answer on the latter question depends on the model. It may be negative if one has a first order phase transition at $T = T_C$. In this case the third possibility does not occur. If it occurs, one has a second order phase transition at $T = T_C$, or T_C is also called *a critical point*. At this point one may find $\lambda \in (0, 1)$ such that

$$0 < \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|^{1+\lambda}} \sum_{\mathbf{l},\mathbf{l}' \in \Lambda} \langle \sigma_{\mathbf{l}} \sigma_{\mathbf{l}'} \rangle_{\nu^{(p)}_{\beta,\Lambda}} < \infty. \tag{2.74}$$

Such λ is known as *a critical exponent*. Calculation of such exponents for a given model is the main goal of many works in this field. Finally, let us remark that for the model described by (2.2) with $h_{\mathbf{l}} = h \neq 0$ and with the single-spin measure possessing the Lee-Yang property, $\chi < \infty$ for all temperatures. This fact was proven in Ref. [71].

2.6. Uniqueness of Gibbs states for the Ising model

In this subsection we prove that the set of Gibbs states $\mathcal{G}_\beta$ of the ferromagnetic Ising model with a homogeneous external field h is unique at all temperatures and all dimensions of the lattice if $h \neq 0$. Our proof will be an extended version of the proof given by J.L Lebowitz and A.

Martin-Löf [72]. On the other hand, it is a simplified version of the proof given in Refs. [73], [74] for quantum Gibbs states.

We consider a ferromagnetic Ising model with the homogeneous external field, i.e., $h_{\mathbf{l}} = h$ for all $\mathbf{l} \in \mathbb{Z}^d$, with the interaction $J_{\mathbf{l}\mathbf{l}'} \geq 0$, which satisfies the condition (2.15).

Theorem 33: For every $\beta > 0$ and $d \in \mathbb{N}$, the set of Gibbs measures $\mathcal{G}_\beta$ of the Ising model with a homogeneous external field h consists of exactly one element if $h \neq 0$.

To prove this statement we will need a number of new notions and auxiliary facts. For the Ising model, the set of local continuous functions $\mathcal{F}$ defined on the space of all configurations Ω is measure defining, i.e., it has the following property. If for given measures $\mu_1, \mu_2 \in \mathcal{G}_\beta$, the expectations $\langle f \rangle_{\mu_1}$, $\langle f \rangle_{\mu_2}$ coincide for all $f \in \mathcal{F}$, then $\mu_1 = \mu_2$. One could use this property in proving our theorem but the set $\mathcal{F}$ is too big and it would be natural to look for a smaller set possessing the same property. Before formulating a statement, which describes the properties of such sets, we remark that $\mathcal{F}$ is closed under pointwise multiplication, i.e., if one defines that for $f, g \in \mathcal{F}$, their product fg has the value $(fg)(\sigma) = f(\sigma)g(\sigma)$ at every $\sigma \in \Omega$, then $fg \in \mathcal{F}$. The following statement is a corollary of the monotone class theorems (see e.g., Ref. [21], p.6).

Proposition 34: *A subset $\Phi \subset \mathcal{F}$ is measure defining if it has the following properties: (a) it is countable; (b) it is closed under pointwise multiplication; (c) for any $\sigma \in \Omega, \sigma' \in \Omega$ one finds $f \in \Phi$ such that $f(\sigma) \neq f(\sigma')$ if $\sigma \neq \sigma'$.*

Given $n \in \mathbb{N}$ and $\mathbf{l}_1, \ldots, \mathbf{l}_n \in \mathbb{Z}^d$, we set

$$f(\sigma) = \sigma_{\mathbf{l}_1} \ldots \sigma_{\mathbf{l}_n}. \tag{2.75}$$

Every such a function is continuous and local, the set Φ of all such functions (all possible choices of $n \in \mathbb{N}$ and $\mathbf{l}_1, \ldots, \mathbf{l}_n \in \mathbb{Z}^d$) is a subset of $\mathcal{F}$. This set is countable since the set of all finite subsets of the countable set $\mathbb{Z}^d$ is countable. For every $f, g \in \Phi$, $fg \in \Phi$. Finally, it separates the points of Ω. In fact, if $\sigma \neq \sigma'$, then one finds $\mathbf{l} \in \mathbb{Z}^d$ such that $\sigma_{\mathbf{l}} \neq \sigma'_{\mathbf{l}}$. The function $f(\sigma) = \sigma_{\mathbf{l}}$ takes different values on such σ, σ'.

Now we introduce two specific configurations of spins, i.e., two specific elements of Ω. Recall that for the Ising model, all spins take values ± 1, thus the set Ω consists of vectors $\sigma = (\sigma_{\mathbf{l}})_{\mathbf{l} \in \mathbb{Z}^d}$, $\sigma_{\mathbf{l}} = \pm 1$. We set $\xi_+ = (\sigma_{\mathbf{l}})_{\mathbf{l} \in \mathbb{Z}^d}$, with all $\sigma_{\mathbf{l}} = 1$ and $\xi_- = (\sigma_{\mathbf{l}})_{\mathbf{l} \in \mathbb{Z}^d}$, with all $\sigma_{\mathbf{l}} = -1$.

Let $\mathcal{L}$ be a sequence of boxes Λ defined by (2.1). The following lemma plays a key role in the proof of our uniqueness theorem.

Lemma 35: *Suppose that for any* $\mathbf{l} \in \mathbb{Z}^d$ *and for a sequence of boxes* $\mathcal{L}$, *each of which contains this* $\mathbf{l}$, *the following convergence*

$$\langle \sigma_{\mathbf{l}} \rangle_{\nu_{\beta,\Delta}(\cdot|\xi_+)} - \langle \sigma_{\mathbf{l}} \rangle_{\nu_{\beta,\Delta}(\cdot|\xi_-)} \to 0, \quad \text{as} \quad \Lambda \to \mathbb{Z}^d, \tag{2.76}$$

holds. Then the set of all Gibbs measures $\mathcal{G}_\beta$ *contains exactly one element.*

Here the expectations are taken with respect to the conditional Gibbs measures (2.55) with the corresponding boundary conditions $\xi_\pm$.

Proof: We prove this lemma by showing that for any $f \in \Phi$ and two arbitrarily chosen Gibbs measures μ_1 and μ_2, one has $\langle f \rangle_{\mu_1} = \langle f \rangle_{\mu_2}$ if the condition (2.76) holds. This should yield $\mu_1 = \mu_2$ since Φ is measure defining. Hence all elements of $\mathcal{G}_\beta$ coincide, which means $|\mathcal{G}_\beta| = 1$.

Take an arbitrary $f \in \Phi$ and write it in the form (2.75) with certain $\mathbf{l}_1, \dots, \mathbf{l}_n$. For this f, one may pick up $\lambda > 0$ such that the function

$$\phi(\sigma) = \lambda \sum_{j=1}^{n} \sigma_{\mathbf{l}_j} + \theta f(\sigma), \tag{2.77}$$

will be monotone for both values $\theta = \pm 1$. Indeed, for any $\sigma, \sigma' \in \Omega$, such that $\sigma' \geq \sigma$, one has $\sigma'_{\mathbf{l}_j} \geq \sigma_{\mathbf{l}_j}$ with $j = 1, 2, \dots n$. Then by means of the following identity

$$a'_1 a'_2 \dots a'_n - a_1 a_2 \dots a_n = \sum_{j=1}^{n} a'_1 a'_2 \dots a'_{j-1} [a'_j - a_j] a_{j+1} \dots a_n,$$

which holds for any n and any sets of numbers $a_1, \dots, a_n$, $a'_1, \dots a'_n$, we get

$$\phi(\sigma') - \phi(\sigma) = \sum_{j=1}^{n} (\sigma'_{\mathbf{l}_j} - \sigma_{\mathbf{l}_j}) \left[\lambda + \theta \sigma'_{\mathbf{l}_1} \dots \sigma'_{\mathbf{l}_{j-1}} \sigma_{\mathbf{l}_{j+1}} \dots \sigma_{\mathbf{l}_n} \right]. \tag{2.78}$$

Obviously, the latter sum is non-negative if $\lambda > 1$.

We recall that $\mathcal{G}_\beta^{\mathrm{ex}}$ denotes the set of extreme elements of $\mathcal{G}_\beta$, i.e., the elements which cannot be written as nontrivial convex combinations (2.61) of other elements of $\mathcal{G}_\beta$. By Proposition 25, $|\mathcal{G}_\beta| = 1$ if $|\mathcal{G}_\beta^{\mathrm{ex}}| = 1$[6]. Thus, the lemma may be proven by showing that for any $f \in \Phi$ and any two $\mu_1, \mu_2 \in \mathcal{G}_\beta^{\mathrm{ex}}$, one has $\langle f \rangle_{\mu_1} = \langle f \rangle_{\mu_2}$ if (2.76) holds. By (2.55), (2.56),

$$\langle f \rangle_{\nu_\Lambda(\cdot|\xi_\pm)} = \langle f \rangle_{\pi_\Lambda(\cdot|\xi_\pm)}, \tag{2.79}$$

[6] Like a triangle with just one vertex.

which holds for any Λ such that $\mathbf{l}_1, \mathbf{l}_2, \dots \mathbf{l}_n \in \Lambda$. Then by Proposition 25,

$$\langle f \rangle_{\mu_1} = \langle f \rangle_{\mu_2} \tag{2.80}$$

provided

$$\langle f \rangle_{\nu_\Delta(\cdot|\xi_+)} - \langle f \rangle_{\nu_\Delta(\cdot|\xi_-)} \to 0, \qquad \text{as} \quad \Lambda \to \mathbb{Z}^d, \tag{2.81}$$

along a sequence $\mathcal{L}$ each element of which contains $\mathbf{l}_1, \mathbf{l}_2, \dots \mathbf{l}_n$. Thus, it remains to show that (2.76) implies (2.81). To this end we employ Corollary 19. Recall that for the case considered, the Hamiltonian which determines $\nu_\Lambda(\cdot|\xi)$ has the form (2.57), see also (2.55), with the external field $h_{\mathbf{l}} = h$ for all $\mathbf{l} \in \mathbb{Z}^d$. Then for $\xi = \xi_\pm$, it can be rewritten

$$\begin{aligned} H_\Lambda &= -\frac{1}{2} \sum_{\mathbf{l},\mathbf{l}' \in \Lambda} J_{\mathbf{l}\mathbf{l}'} \sigma_{\mathbf{l}} \sigma_{\mathbf{l}'} - \sum_{\mathbf{l} \in \Lambda} \left[h \pm \sum_{\mathbf{l}' \in \Lambda^c} J_{\mathbf{l}\mathbf{l}'} \right] \sigma_{\mathbf{l}} = \\ &= -\frac{1}{2} \sum_{\mathbf{l},\mathbf{l}' \in \Lambda} J_{\mathbf{l}\mathbf{l}'} \sigma_{\mathbf{l}} \sigma_{\mathbf{l}'} - \sum_{\mathbf{l} \in \Lambda} h^\pm_{\mathbf{l}}(\Lambda) \sigma_{\mathbf{l}}, \end{aligned} \tag{2.82}$$

where $h^\pm_{\mathbf{l}}(\Lambda) = h \pm \sum_{\mathbf{l}' \in \Lambda^c} J_{\mathbf{l}\mathbf{l}'}$. Our model is ferromagnetic, i.e., all $J_{\mathbf{l}\mathbf{l}'}$ are nonnegative, which yields $h^+_{\mathbf{l}}(\Lambda) \geq h^-_{\mathbf{l}}(\Lambda)$. Then by Corollary 19,

$$\langle \phi \rangle_{\nu_\Lambda(\cdot|\xi_+)} \geq \langle \phi \rangle_{\nu_\Lambda(\cdot|\xi_-)}, \tag{2.83}$$

since the function ϕ defined by (2.77) is monotone. Then by (2.77), one gets

$$\lambda \sum_{j=1}^{n} \left[\langle \sigma_{\mathbf{l}_j} \rangle_{\nu_\Lambda(\cdot|\xi_+)} - \langle \sigma_{\mathbf{l}_j} \rangle_{\nu_\Lambda(\cdot|\xi_-)} \right] \geq \theta \left[\langle f \rangle_{\nu_\Lambda(\cdot|\xi_+)} - \langle f \rangle_{\nu_\Lambda(\cdot|\xi_-)} \right].$$

Since the latter estimate holds for both $\theta = \pm 1$, it may be rewritten

$$\lambda \sum_{j=1}^{n} \left[\langle \sigma_{\mathbf{l}_j} \rangle_{\nu_\Lambda(\cdot|\xi_+)} - \langle \sigma_{\mathbf{l}_j} \rangle_{\nu_\Lambda(\cdot|\xi_-)} \right] \geq \left| \langle f \rangle_{\nu_\Lambda(\cdot|\xi_+)} - \langle f \rangle_{\nu_\Lambda(\cdot|\xi_-)} \right|, \tag{2.84}$$

which yields (2.81) if (2.76) holds.

The next step in proving Theorem 33 is to show that (2.76) always holds if $h \neq 0$. To this end we employ the infinite-volume free energy density and its properties as a function of the external field h. We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if for every $\theta \in [0, 1]$ and any $s, t \in \mathbb{R}$, one has $f(\theta t + (1 - \theta)s) \leq \theta f(t) + (1 - \theta) f(s)$. A wide variety of the properties of convex functions and their applications in the theory of lattice models may be found in Refs. [5], [13]. In particular, they have the following property (see Ref. [13], pp.35 -39).

Proposition 36: *Let a sequence of convex function $\{f_n\}_{n\in\mathbb{N}}$ converge pointwise on $\mathbb{R}$ to a function f. Then this function is also convex, it is differentiable for all but countable values of its argument. If all f_n and f are differentiable at a given $x_0 \in \mathbb{R}$, then $f_n'(x_0) \to f'(x_0)$.*

For the model considered, the partition function in a box Λ with the external field $h = z$ and with the zero boundary condition is given by (2.23) with the single-spin measure (2.5), it also may be written in the form (2.24). We set

$$p_\Lambda(z) = \frac{1}{|\Lambda|}\ln Z_{\beta,\Lambda}(z). \tag{2.85}$$

In the lattice gas terminology [13], such a function is called *pressure*. Its connection with the free energy density $F_{\beta,\Lambda}(0)$, where 0 means the zero boundary conditions, may be established by means of (2.65). Taking into account Proposition 28, we can state the following.

Proposition 37: *For every $z \in \mathbb{R}$ and $\beta > 0$, the sequence $\{p_\Lambda(z)\}_{\Lambda\in\mathcal{L}}$ converges as $\Lambda \to \mathbb{Z}^d$. Its limit $p(z)$ is a convex function of z. Moreover, for any $\xi \in \Omega$,*

$$p(z) = -\beta \lim_{\Lambda\to\mathbb{Z}^d} F_{\beta,\Lambda}(\xi) \tag{2.86}$$

where $F_{\beta,\Lambda}(\xi)$ is the free energy density of the model with the homogeneous external field $h_l = z$.

Let us prove that the only point at which $p(z)$ may fail to be differentiable is $z = 0$. To this end we employ the Lee-Yang theorem in the form of Proposition 3, which yields (2.24), as well as the following well-known theorem of complex analysis (Vitali's theorem, see e.g., Theorem VIII.19 in Ref. [11]).

Proposition 38: *Given a domain $D \subset \mathbb{C}$, let a sequence of functions $f_n : D \to \mathbb{C}$, $n \in \mathbb{N}$ have the following properties: (a) each f_n is holomorphic on D; (b) for every bounded closed subset $K \subset D$, there exists $C_K > 0$ such that $|f_n(z)| \le C_K$ for all $n \in \mathbb{N}$ and $z \in K$; (c) there exists a subset $F \subset D$, which has an accumulation point, such that, for every $z \in F$, the sequence $\{f_n(z)\}_{n\in\mathbb{N}}$ converges to a function $f : D \to \mathbb{C}$. Then this function is also holomorphic on D.*

This theorem has the following interpretation. If a sequence of holomorphic on D functions has the properties: (a) it is uniformly bounded on compact subsets of D; (b) it converges pointwise on D to a function, which is just defined on D. Then this sequence converges to the latter function uniformly on compact subsets of D, hence the limiting function is holomorphic on D.

Now we may prove the following result.

Lemma 39: *For every $\beta > 0$, the limit of the sequence $\{p_\Lambda(z)\}_{\Lambda\in\mathcal{L}}$ is an infinitely differentiable function at any $z \in \mathbb{R}\setminus\{0\}$.*

Proof: For the Ising model, one has in (2.24) $\gamma_0(\beta,\Lambda) = 0$ (see the analysis following after (2.24)), which yields

$$p_\Lambda(z) = \ln Z_{\beta,\Lambda}(0) + \sum_{j=1}^{\infty} \ln\left[1 + \gamma_j(\beta,\Lambda)z^2\right],$$

and hence

$$\frac{p_\Lambda'(z)}{z} = \sum_{j=1}^{\infty} \frac{2\gamma_j(\beta,\Lambda)}{1+\gamma_j(\beta,\Lambda)z^2}. \tag{2.87}$$

This means that all the functions p_Λ are holomorphic in the domain $\mathbb{C}\setminus A$, where $A = A_+ \cup A_-$, $A_\pm = \{z = \pm it \mid t \in [(\gamma_1(\beta,\Lambda))^{-1/2}, +\infty)\}$, which includes the whole real line $\mathbb{R}$. Then, for $z = x + iy \in \mathbb{C}\setminus A$, one has

$$\begin{aligned}\left|\frac{2\gamma_j(\beta,\Lambda)}{1+\gamma_j(\beta,\Lambda)z^2}\right|^2 &= \frac{4\gamma_j^2(\beta,\Lambda)}{[1+\gamma_j(\beta,\Lambda)(x^2-y^2)]^2 + 4\gamma_j^2(\beta,\Lambda)x^2y^2} \le \\ &\le \frac{4\gamma_j^2(\beta,\Lambda)}{[1+\gamma_j(\beta,\Lambda)(x^2-y^2)]^2}.\end{aligned}$$

Given $\theta > 0$, we set

$$B_\theta = \{z = x+iy \in \mathbb{C} \mid x \ge 0, \quad x^2 - y^2 \ge \theta^2\}.$$

Applying the above estimate in (2.87) we get for $z \in B_\theta$

$$\left|\frac{p_\Lambda'(z)}{z}\right| \le \sum_{j=1}^{\infty} \frac{2\gamma_j(\beta\Lambda)}{1+\gamma_j(\beta,\Lambda)\theta^2} = \frac{p_\Lambda'(\theta)}{\theta}. \tag{2.88}$$

By Proposition 37, the limiting function $p(z)$ is convex on $\mathbb{R}$ hence it is not differentiable on a subset $E \subset \mathbb{R}$, which is at most countable. This means that for any $\varepsilon > 0$, the interval $(0,\varepsilon)$ contains points at which $p'(z)$ exists. Moreover, by the same statement, $p_\Lambda'(z) \to p'(z)$, as $\Lambda \to \mathbb{Z}^d$, at each such a point. Thus, we take an arbitrary ε and pick up

$\theta \in (0, \varepsilon)$ such that $p'(\theta)$ exists. Then the sequence $\{p'_\Lambda(\theta)\}$ converges to $p'(\theta)$ hence this sequence is bounded. Now we take $t > \theta$ and set

$$B_{\theta,t} = \{z = x + iy \in \mathbb{C} \mid x^2 - y^2 \geq \theta^2, \quad x \in [0, t]\}. \tag{2.89}$$

This set contains $[\theta, t] \subset \mathbb{R}$. Then, for $z \in B_{\theta,t}$, one has

$$|z| = \sqrt{x^2 + y^2} \leq \sqrt{2x^2 - \theta^2} \leq \sqrt{2t^2 - \theta^2},$$

and, by the estimate (2.88),

$$|p'_\Lambda(z)| \leq \left(\sqrt{2(t/\theta)^2 - 1}\right) p'_\Lambda(\theta).$$

Since the sequence $\{p'_\Lambda(\theta)\}_{\Lambda\in\mathcal{L}}$ is bounded, the sequence of holomorphic in $B_{\theta,t}$ functions $\{p'_\Lambda\}_{\Lambda\in\mathcal{L}}$ is uniformly bounded on $B_{\theta,t}$. Moreover, one has $p'_\Lambda(z) \to p'(z)$ for all $z \in [\theta, t]$ except possibly for a countable subset of this interval. Thus, the subset of $[\theta, t]$ on which $p'_\Lambda(z) \to p'(z)$ has an accumulation point, which yields by Proposition 38 that p' is holomorphic on $B_{\theta,t}$, hence p is infinitely differentiable on (θ, t). Since this is true for any $t > \theta$ and θ may be chosen arbitrarily close to zero (recall that $\theta \in (0, \varepsilon)$ with any $\varepsilon > 0$), this is true for all $z \in (0, +\infty)$. Since all the functions p_Λ and p are even, the same is true also for $z \in (-\infty, 0)$. Thus, the only point where it may fail to hold is $z = 0$.

The next step is based on the following simple result.

Lemma 40: *The Ising model with the homogeneous external field h and the zero boundary condition has the following property. For any Δ and $f \in \mathcal{F}_\Delta$,*

$$\lim_{h\to+\infty} \langle f \rangle_{\nu_\Delta(h)} = f(\sigma^+_\Delta), \tag{2.90}$$

where σ^+_Δ is the configuration $\sigma_\Delta = (\sigma_{\mathbf{l}})_{\mathbf{l}\in\Delta}$ for which $\sigma_{\mathbf{l}} = 1$ for all $\mathbf{l} \in \Delta$.

Proof: Set $S_\Delta = \sum_{\mathbf{l}\in\Delta} \sigma_{\mathbf{l}}$. The values of this total spin variable constitute the set $\mathcal{S}_\Delta = \{-|\Delta|, -(|\Delta| - 2), \ldots, |\Delta| - 2, |\Delta|\}$. Furthermore, let $\mathcal{S}'_\Delta = \mathcal{S}_\Delta \setminus \{|\Delta|\}$ (i.e., the latter set coincides with the former one up to the element $|\Delta|$). Then

$$\langle f \rangle_{\nu_\Delta(h)} = \tag{2.91}$$
$$= \frac{1}{Z_{\beta,\Delta}(h)} \sum_{S\in\mathcal{S}_\Delta} \sum_{\sigma_\Delta:\ S_\Delta = S} f(\sigma_\Delta) \exp\left(\beta h S_\Delta + \frac{\beta}{2} \sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'}\right).$$

When dealing with the Ising model, for which spin variables take discrete values, it is more convenient to use notations in which expectations with respect to the corresponding local Gibbs measures are written as sums over the values of spins instead of integrals. Thus, for a given Δ, the sum $\sum_{\sigma_\Delta}$ is taken over all values of $\sigma_{\mathbf{l}}$ with $\mathbf{l} \in \Delta$. The second sum in the above expression is taken over all such values but with the restriction $\sum_{\mathbf{l}\in\Delta} \sigma_{\mathbf{l}} = S_\Delta = S$. In such notations the partition function is

$$Z_{\beta,\Delta}(h) = \sum_{S\in\mathcal{S}_\Delta} \sum_{\sigma_\Delta:\ S_\Delta = S} \exp\left(\beta h S_\Delta + \frac{\beta}{2} \sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'}\right),$$

which may be written in the form

$$Z_{\beta,\Delta}(h) = \exp\left[\beta h|\Delta| + \frac{\beta}{2} \sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}\right] \times \tag{2.92}$$
$$\times \left\{1 + \sum_{S\in\mathcal{S}'_\Delta} \exp\left(\beta h(S - |\Delta|)\right) \sum_{\sigma_\Delta:\ S_\Delta = S} \exp\left(\frac{\beta}{2} \sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}[\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'} - 1]\right)\right\}.$$

Similarly (2.91) may be written as

$$\langle f \rangle_{\nu_\Delta} = f(\sigma^+_\Delta) + \frac{1}{1 + A_\Delta(h)} \sum_{S\in\mathcal{S}'_\Delta} \exp\left(\beta h(S - |\Delta|)\right) \times \tag{2.93}$$
$$\times \sum_{\sigma_\Delta:\ S_\Delta = S} f(\sigma_\Delta) \exp\left(\frac{\beta}{2} \sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}[\sigma_{\mathbf{l}}\sigma_{\mathbf{l}'} - 1]\right),$$

where $A_\Delta(h)$ stands for the second term in the figure brackets in (2.92). For all $S \in \mathcal{S}'_\Delta$, one has $S < |\Delta|$, which yields $\exp[\beta h(S - |\Delta|)] \to 0$ as $h \to +\infty$. Furthermore, the sums over $\mathcal{S}'_\Delta$ are finite, hence both $A_\Delta(h)$ and the second term in (2.93) tend to zero as $h \to +\infty$, which completes the proof.

Another result which we use to prove our theorem describes the expectations $\langle f \rangle_{\nu_\Delta(\cdot|\xi_+)}$.

Lemma 41: *The ferromagnetic Ising model with an arbitrary external field $h_\Delta = (h_{\mathbf{l}})_{\mathbf{l}\in\Delta}$ has the following property. For every $\beta > 0$, any finite subsets Δ, Δ', such that $\Delta' \subset \Delta$, and for any monotone function $f \in \mathcal{F}_{\Delta'}$,*

$$\langle f \rangle_{\nu_\Delta(\cdot|\xi_+)} \leq \langle f \rangle_{\nu_{\Delta'}(\cdot|\xi_+)}. \tag{2.94}$$

Proof For $\Delta' \subset \Delta$, we write $\Delta'' = \Delta \setminus \Delta'$. By $\xi_{\Delta'} \times \eta_{\Delta''}$ we denote the configuration ζ_Δ such that $\zeta_{\mathbf{l}} = \xi_{\mathbf{l}}$ for $\mathbf{l} \in \Delta'$, and $\zeta_{\mathbf{l}} = \eta_{\mathbf{l}}$ for $\mathbf{l} \in \Delta''$. Then for the mentioned Δ, Δ', we may write $\sigma_\Delta = \sigma_{\Delta'} \times \sigma_{\Delta''}$. Obviously, for appropriate functions,

$$\sum_{\sigma_\Delta} f(\sigma_\Delta) = \sum_{\sigma'_\Delta} \sum_{\sigma''_\Delta} f(\sigma_{\Delta'} \times \sigma_{\Delta''}).$$

Then the Hamiltonian H_Δ (2.2) may be written

$$H_\Delta = H_{\Delta'} + H_{\Delta''} - \sum_{\mathbf{l}_1 \in \Delta', \mathbf{l}_2 \in \Delta''} J_{\mathbf{l}_1 \mathbf{l}_2} \sigma_{\mathbf{l}_1} \sigma_{\mathbf{l}_2}. \tag{2.95}$$

For $t \in [0, +\infty)$, we set

$$\begin{aligned} \phi(t) &= \frac{1}{Z_{\beta,\Delta}(t)} \sum_{\sigma_\Delta} f(\sigma_\Delta) \times \\ &\times \exp\left(-\beta H_\Delta + \beta \sum_{\mathbf{l} \in \Delta} \sigma_{\mathbf{l}} \sum_{\mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'} + t \sum_{\mathbf{l} \in \Delta''} \sigma_{\mathbf{l}}\right), \end{aligned} \tag{2.96}$$

where

$$Z_{\beta,\Delta}(t) = \sum_{\sigma_\Delta} \exp\left(-\beta H_\Delta + \beta \sum_{\mathbf{l} \in \Delta} \sigma_{\mathbf{l}} \sum_{\mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'} + t \sum_{\mathbf{l} \in \Delta''} \sigma_{\mathbf{l}}\right) \tag{2.97}$$

Here the terms $\sum_{\mathbf{l} \in \Delta} \sigma_{\mathbf{l}} \sum_{\mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'}$ describe interaction with the external spins ξ, fixed at $\xi = \xi_+$ (recall that $\Delta^c = \mathbb{Z}^d \setminus \Delta$). Thus, we have

$$\phi(0) = \langle f \rangle_{\nu_\Delta(\cdot|\xi_+)}. \tag{2.98}$$

Taking into account (2.95) and the fact that $f \in \mathcal{F}_{\Delta'}$ (which means f is independent of the components of σ_Δ with $\mathbf{l} \in \Delta''$, i.e., $f(\sigma_\Delta) = f(\sigma_{\Delta'})$), one can rewrite the above expressions as follows.

$$\begin{aligned} \phi(t) &= \frac{1}{Z_{\beta,\Delta}(t)} \sum_{\sigma_{\Delta'}} f(\sigma_{\Delta'}) \exp\left(-\beta H_{\Delta'} + \beta \sum_{\mathbf{l} \in \Delta'} \sigma_{\mathbf{l}} \sum_{\mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'}\right) \times \\ &\times \sum_{\sigma_{\Delta''}} \exp\left(-\beta H_{\Delta''} + \beta \sum_{\mathbf{l} \in \Delta''} \sigma_{\mathbf{l}} \sum_{\mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'} + \right. \\ &+ \left. \beta \sum_{\mathbf{l}_1 \in \Delta', \mathbf{l}_2 \in \Delta''} J_{\mathbf{l}\mathbf{l}'} \sigma_{\mathbf{l}_1} \sigma_{\mathbf{l}_2} + t \sum_{\mathbf{l} \in \Delta''} \sigma_{\mathbf{l}}\right), \end{aligned} \tag{2.99}$$

and

$$\begin{aligned} Z_{\beta,\Delta}(t) &= \sum_{\sigma_{\Delta'}} \exp\left(-\beta H_{\Delta'} + \beta \sum_{\mathbf{l} \in \Delta'} \sigma_{\mathbf{l}} \sum_{\mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'}\right) \times \\ &\times \sum_{\sigma_{\Delta''}} \exp\left(-\beta H_{\Delta''} + \beta \sum_{\mathbf{l} \in \Delta''} \sigma_{\mathbf{l}} \sum_{\mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'} + \right. \\ &+ \left. \beta \sum_{\mathbf{l}_1 \in \Delta', \mathbf{l}_2 \in \Delta''} J_{\mathbf{l}\mathbf{l}'} \sigma_{\mathbf{l}_1} \sigma_{\mathbf{l}_2} + t \sum_{\mathbf{l} \in \Delta''} \sigma_{\mathbf{l}}\right). \end{aligned} \tag{2.100}$$

The external field in $\exp(\cdot)$ in (2.96) is $h'_\Delta = (h_{\mathbf{l}} + t)_{\mathbf{l} \in \Delta}$, where $h_{\mathbf{l}}$, $\mathbf{l} \in \Delta$ is the external field in H_Δ. Since $t \geq 0$, $h'_\Delta \geq h_\Delta$, which by Corollary 19 yields

$$\phi(t) \geq \phi(0) = \langle f \rangle_{\nu_\Delta(\cdot|\xi_+)}. \tag{2.101}$$

We recall that f is monotone. Set

$$\begin{aligned} F(\sigma_{\Delta''}) &= \frac{1}{Z_{\beta,\Delta'}(\sigma_{\Delta''})} \sum_{\sigma_{\Delta'}} f(\sigma_{\Delta'}) \exp\left(\beta \sum_{\mathbf{l}_1 \in \Delta', \mathbf{l}_2 \in \Delta''} J_{\mathbf{l}_1 \mathbf{l}_2} \sigma_{\mathbf{l}_1} \sigma_{\mathbf{l}_2} - \right. \\ &- \left. \beta H_{\Delta'} + \beta \sum_{\mathbf{l} \in \Delta'} \sigma_{\mathbf{l}} \sum_{\mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'}\right), \end{aligned} \tag{2.102}$$

where

$$\begin{aligned} Z_{\beta,\Delta'}(\sigma_{\Delta''}) &= \sum_{\sigma_{\Delta'}} \exp\left(\beta \sum_{\mathbf{l}_1 \in \Delta', \mathbf{l}_2 \in \Delta''} J_{\mathbf{l}_1 \mathbf{l}_2} \sigma_{\mathbf{l}_1} \sigma_{\mathbf{l}_2} - \right. \\ &- \left. \beta H_{\Delta'} + \beta \sum_{\mathbf{l} \in \Delta'} \sigma_{\mathbf{l}} \sum_{\mathbf{l}' \in \Delta^c} J_{\mathbf{l}\mathbf{l}'}\right). \end{aligned} \tag{2.103}$$

Let $\nu_{\Delta''}(t)$ be the local Gibbs measure of the Ising model in the set Δ'' corresponding to the Hamiltonian $H_{\Delta''} - \beta^{-1} t \sum_{\mathbf{l} \in \Delta''} \sigma_{\mathbf{l}}$ (we have fixed β which, by the end of this proof, is just a parameter) and to the zero boundary condition. Its partition function is

$$Z_{\beta,\Delta''}(t) = \sum_{\sigma_{\Delta''}} \exp\left(-\beta H_{\Delta''} + t \sum_{\mathbf{l} \in \Delta''} \sigma_{\mathbf{l}}\right).$$

By means of this measure, the above expectations may be rewritten as follows

$$Z_{\beta,\Delta'}(t) = \sum_{\sigma_{\Delta''}} \left\{ Z_{\beta,\Delta'}(\sigma_{\Delta''}) \exp\left(\beta \sum_{\mathbf{l}\in\Delta''} \sigma_{\mathbf{l}} \sum_{\mathbf{l}'\in\Delta^c} J_{\mathbf{l}\mathbf{l}'}\right)\right\} \times \exp\left(-\beta H_{\Delta''} + t\sum_{\mathbf{l}\in\Delta''}\sigma_{\mathbf{l}}\right) =$$

$$= Z_{\beta,\Delta''}(t) \quad \langle Z_{\beta,\Delta'}(\sigma_{\Delta''}) \exp\left(\beta \sum_{\mathbf{l}\in\Delta''} \sigma_{\mathbf{l}} \sum_{\mathbf{l}'\in\Delta^c} J_{\mathbf{l}\mathbf{l}'}\right)\rangle_{\nu_{\Delta''}(t)} \quad (2.104)$$

And similarly

$$\phi(t) = \left[Z_{\beta,\Delta''}(t)\langle Z_{\beta,\Delta'}(\sigma_{\Delta''}) \exp\left(\beta \sum_{\mathbf{l}\in\Delta''} \sigma_{\mathbf{l}} \sum_{\mathbf{l}'\in\Delta^c} J_{\mathbf{l}\mathbf{l}'}\right)\rangle_{\nu_{\Delta''}(t)}\right]^{-1} \times$$
$$\times \sum_{\sigma_{\Delta''}} \left\{ \sum_{\sigma_{\Delta'}} f(\sigma_{\Delta'}) \exp\left(-\beta H_{\Delta'} + \beta \sum_{\mathbf{l}_1\in\Delta', \mathbf{l}_2\in\Delta''} J_{\mathbf{l}_1\mathbf{l}_2}\sigma_{\mathbf{l}_1}\sigma_{\mathbf{l}_2} + \right.\right.$$
$$\left.\left. + \beta \sum_{\mathbf{l}\in\Delta'} \sigma_{\mathbf{l}} \sum_{\mathbf{l}'\in\Delta^c} J_{\mathbf{l}\mathbf{l}'}\right)\right\} \exp\left(-\beta H_{\Delta''} + t\sum_{\mathbf{l}\in\Delta''}\sigma_{\mathbf{l}}\right) =$$
$$= \left[\langle Z_{\beta,\Delta'}(\sigma_{\Delta''}) \exp\left(\beta \sum_{\mathbf{l}\in\Delta''} \sigma_{\mathbf{l}} \sum_{\mathbf{l}'\in\Delta^c} J_{\mathbf{l}\mathbf{l}'}\right)\rangle_{\nu_{\Delta''}(t)}\right]^{-1} \times$$
$$\times \langle Z_{\beta,\Delta'}(\sigma_{\Delta''}) F(\sigma_{\Delta''}) \exp\left(\beta \sum_{\mathbf{l}\in\Delta''} \sigma_{\mathbf{l}} \sum_{\mathbf{l}'\in\Delta^c} J_{\mathbf{l}\mathbf{l}'}\right)\rangle_{\nu_{\Delta''}(t)}. \quad (2.105)$$

Passing here to the limit $t \to +\infty$ and employing Lemma 40 we arrive at

$$\lim_{t\to+\infty}\phi(t) = \frac{Z_{\beta,\Delta'}(\sigma^+_{\Delta''})F(\sigma^+_{\Delta''})\exp\left(\beta\sum_{\mathbf{l}\in\Delta''}\sum_{\mathbf{l}'\in\Delta^c}J_{\mathbf{l}\mathbf{l}'}\right)}{Z_{\beta,\Delta'}(\sigma^+_{\Delta''})\exp\left(\beta\sum_{\mathbf{l}\in\Delta''}\sum_{\mathbf{l}'\in\Delta^c}J_{\mathbf{l}\mathbf{l}'}\right)} = F(\sigma^+_{\Delta''}).$$

By (2.102), this yields

$$\lim_{t\to+\infty}\phi(t) = \sum_{\sigma_{\Delta'}} f(\sigma_{\Delta'}) \exp\left(-\beta H_{\Delta'} + \beta \sum_{\mathbf{l}_1\in\Delta'} \sigma_{\mathbf{l}_1} \sum_{\mathbf{l}_2\in\Delta''} J_{\mathbf{l}_1\mathbf{l}_2} + \right.$$
$$\left. +\beta \sum_{\mathbf{l}_1\in\Delta'} \sigma_{\mathbf{l}_1} \sum_{\mathbf{l}_2\in\Delta^c} J_{\mathbf{l}_1\mathbf{l}_2}\right) \times$$
$$\times \left[\sum_{\sigma_{\Delta'}} \exp\left(-\beta H_{\Delta'} + \beta \sum_{\mathbf{l}_1\in\Delta'} \sigma_{\mathbf{l}_1} \sum_{\mathbf{l}_2\in\Delta''} J_{\mathbf{l}_1\mathbf{l}_2} + \beta \sum_{\mathbf{l}_1\in\Delta'} \sigma_{\mathbf{l}_1} \sum_{\mathbf{l}_2\in\Delta^c} J_{\mathbf{l}_1\mathbf{l}_2}\right)\right]^{-1} =$$
$$= \sum_{\sigma_{\Delta'}} f(\sigma_{\Delta'}) \exp\left(-\beta H_{\Delta'} + \beta \sum_{\mathbf{l}_1\in\Delta'} \sigma_{\mathbf{l}_1} \sum_{\mathbf{l}_2\in(\Delta')^c} J_{\mathbf{l}_1\mathbf{l}_2}\right) \times$$
$$\times \left[\sum_{\sigma_{\Delta'}} \exp\left(-\beta H_{\Delta'} + \beta \sum_{\mathbf{l}_1\in\Delta'} \sigma_{\mathbf{l}_1} \sum_{\mathbf{l}_2\in(\Delta')^c} J_{\mathbf{l}_1\mathbf{l}_2}\right)\right]^{-1} = \langle f\rangle_{\nu_{\Delta'}(\cdot|\xi_+)}.$$

Taking into account (2.101) we obtain (2.94).

Recall that we are studying the Ising model with a non-zero homogeneous external field h. Since the interaction term in the Hamiltonian (2.2) and the single-spin measure ϱ^{I} are symmetric with respect to the change $\sigma \to -\sigma$, it is enough to consider the case $h > 0$ only.

A function $f : \mathbb{R} \to \mathbb{R}$ is called *right-continuous* at $x_0 \in \mathbb{R}$ if $\lim_{x\to x_0+0} f(x) = f(x_0)$. Such a function is *left-continuous* at x_0 if $g(x) \stackrel{\mathrm{def}}{=} f(-x)$ is right-continuous at $-x_0$. If a sequence of monotone increasing continuous functions $\{f_n\}_{n\in\mathbb{N}}$, $f_n : \mathbb{R} \to \mathbb{R}$ converges pointwise on $\mathbb{R}$ to a function f, then this f is right-continuous.

The next statement follows from the one just proven.

Corollary 42: *For every $\beta > 0$ and $h \in \mathbb{R}$, the infinite-volume limits of the expectations*

$$M_\pm(h) = \lim_{\Lambda\to\mathbb{Z}^d} \langle\sigma_{\mathbf{l}}\rangle_{\nu_\Lambda(\cdot|\xi_\pm)}, \quad (2.106)$$

exist. They do not depend on $\mathbf{l}$. *Moreover, $M_+(h)$ (respectively $M_-(h)$) is right-continuous (respectively left-continuous).*

Proof: Let us prove the above statement for $M_+(h)$. Since the function $f(\sigma) = \sigma_{\mathbf{l}}$ is monotone, the above lemma yields

$$\langle\sigma_{\mathbf{l}}\rangle_{\nu_\Lambda(\cdot|\xi_+)} \le \langle\sigma_{\mathbf{l}}\rangle_{\nu_{\Lambda'}(\cdot|\xi_+)}, \quad (2.107)$$

for any two boxes $\Lambda' \subset \Lambda$. On the other hand, since $\sigma_{\mathbf{l}} = \pm 1$,

$$-1 \leq \langle \sigma_{\mathbf{l}} \rangle_{\nu_\Lambda(\cdot|\xi_\pm)} \leq 1,$$

hence $\{\langle \sigma_{\mathbf{l}} \rangle_{\nu_\Lambda(\cdot|\xi_+)}\}_\Lambda$ converges as a bounded monotone decreasing sequence. By the FKG inequality,

$$\frac{\partial}{\partial h} \langle \sigma_{\mathbf{l}} \rangle_{\nu_\Lambda(\cdot|\xi_+)} = \beta \sum_{\mathbf{l}' \in \Lambda} K^\Lambda_{\mathbf{l}\mathbf{l}'}(h) \geq 0,$$

hence $\langle \sigma_{\mathbf{l}} \rangle_{\nu_\Lambda(\cdot|\xi_+)} \geq 0$ for $h \geq 0$. Since for every Λ, $\langle \sigma_{\mathbf{l}} \rangle_{\nu_\Lambda(\cdot|\xi_+)}$ is a continuous and monotone function of h, the limit of the above sequence is right-continuous. Finally, since the locally weak limits of $\pi_\Lambda(\cdot|\xi_\pm)$ are the extreme translation invariant states $\mu_\pm$, the above $M_+(h)$ is independent of $\mathbf{l}$. We remark here, that all our conclusions regarding $M_+(h)$ are valid for all $h \in \mathbb{R}$ since no assumptions restricting h were made in the above lemma. Now let us prove the statement for $M_-(h)$. By symmetry, $\langle \sigma_{\mathbf{l}} \rangle_{\nu_\Lambda(\cdot|\xi_-)} = -\langle \sigma_{\mathbf{l}} \rangle_{\tilde{\nu}_\Lambda(\cdot|\xi_+)}$, where $\tilde{\nu}_\Lambda(\cdot|\xi_+)$ is the local Gibbs measure with the external field equal to $-h$. Then the convergence of the sequence of $\langle \sigma_{\mathbf{l}} \rangle_{\nu_\Lambda(\cdot|\xi_-)}$ and the translation invariance of its limit follow from the above consideration. Moreover, for these limits one has $M_-(h) = -M_+(-h)$. Finally

$$\lim_{h' \to h-0} M_-(h') = - \lim_{-h' \to -h+0} M_+(-h') = -M_+(-h) = M_-(h).$$

Proof of Theorem 33: By (2.65)

$$-\frac{\partial}{\partial h} F_{\beta,\Lambda}(\xi_\pm) = \frac{1}{|\Lambda|} \sum_{\mathbf{l} \in \Lambda} \langle \sigma_{\mathbf{l}} \rangle_{\nu_\Lambda(\cdot|\xi_\pm)}. \tag{2.108}$$

By Proposition 37, both $-F_{\beta,\Lambda}(\xi_\pm)$ have the same limit, which coincide with $\beta p(h)$ studied in Lemma 39. This limit, as a convex function of $h \in \mathbb{R}$, has one-sided derivatives at any h, which are the limits of the derivatives (2.108). This means

$$\lim_{h' \to h\pm 0} \frac{\beta(p(h') - p(h))}{h' - h} = - \lim_{h' \to h\pm 0} \lim_{\Lambda \to \mathbb{Z}^d} \frac{\beta F_{\beta,\Lambda}(\xi_\pm, h') - F_{\beta,\Lambda}(\xi_\pm, h)}{h' - h} =$$

$$= \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{\mathbf{l} \in \Lambda} \langle \sigma_{\mathbf{l}} \rangle_{\nu_\Lambda(\cdot|\xi_\pm)} = M_\pm(h). \tag{2.109}$$

Here we have used the results of Corollary 42. By Lemma 39, $p(h)$ is differentiable for $h \neq 0$, which means that both its one-sided derivatives coincide at such h. Therefore, $M_+(h) = M_-(h)$ if $h \neq 0$, which means (2.76). Then we may apply Lemma 35 which completes the proof. □.

2.7. Self-similarity, one-dimensional and hierarchical models

As it was mentioned in Introduction, the one-dimensional Ising model with the interaction potential of finite range has no phase transition. Moreover, one may show that the set of its Gibbs states is a singleton for all $\beta > 0$ (see Ref. [4], p.164). Is it true for the case of long-range interactions? The answer was given by F.J. Dyson in his articles Refs. [75], [76]. Namely, the one-dimensional ferromagnetic Ising model with the zero external field and the translation invariant interaction potential $J_{\mathbf{l}\mathbf{l}'} = \phi(|\mathbf{l} - \mathbf{l}'|)$ such, that $\phi(x) \sim \phi_0 x^{-1-\lambda}$ as $x \to +\infty$ with $\lambda \in (0, 1)$, has a phase transition. The same model with $\lambda > 1$ has no phase transitions. The borderline case $\lambda = 1$ was studied in the paper Ref. [77], where it was shown that the model has a phase transition, but in contrast to the case $\lambda < 1$, the order parameter has a jump at $\beta = \beta_c$.

To obtain his results Dyson introduced in Ref. [75] a spin model with a specific hierarchical structure. Later it was understood that this structure has a very deep connection with a certain property of lattice spin models, which appears at their critical points. This property is called self-similarity. The first publication where it was discussed, though yet without this name, is L.P. Kadanoff's paper [78] (see also Refs. [79], [80] and the references therein) in which he presented his known block-spin construction. A similar arguments were developed also in R.B. Griffiths' paper Ref. [44]. Later there was an explosion of activity in this direction, a consequence of which is an approach in the theory of critical phenomena known as the renormalization group method. In fact, it is not a method but a vast variety of methods and tools with different levels of mathematical background. The notion of self-similarity, first appeared in Ya. G. Sinai's paper Ref. [81], was formulated as a property of a random field. Later the conception of self-similar random fields became a part of the mathematical theory of critical phenomena in models of statistical physics, lattice spin models in particular. A full and comprehensive description of this conception is given in Refs. [14], [66]. In this connection we mention also papers of K. Gawędski Refs. [82], [83], [84].

Renormalization group methods developed in theoretical physics have produced and still are producing a very strong impact on the theory of critical phenomena. One may say that they created a new philosophy with its own set of concepts. In this context we mention brilliant works of K.G. Wilson Ref. [85] and I.R. Yuknovskii Refs. [86], [87].

At the same time, the mathematical tools used in modern renormalization theories, especially those applied to more realistic models, are not sufficiently elaborated. Moreover, quite often ill defined mathematical objects are employed here. As examples, one can mention nonexisting integrals, series expansions with zero convergence radii, etc. An attempt to bridge the gap between the treatments of renormalization in physics and the mathematically rigorous approach was made in a recent book Ref. [88].

Let us described, briefly and schematically, the mathematical aspects of the renormalization group approach to the models we consider here. Recall that we study spin models defined on the lattice $\mathbb{Z}^d$ and described by the Hamiltonian (2.2) and single-spin measures ϱ, which belong to one of the types described in the preceding subsections. Recall also that Ω , $\mathcal{P}(\Omega)$ and $\mathcal{G}_\beta$ stand for the set of all spin configurations, the set of all probability measures on this set and the set of all Gibbs states of our model existing at a given β respectively. Suppose there is defined a surjection $\varkappa : \mathbb{Z}^d \to \mathbb{Z}^d$ (a map such that for every $\mathbf{l} \in \mathbb{Z}^d$, there exist $\mathbf{l}' \in \mathbb{Z}^d$, for which $\varkappa(\mathbf{l}') = \mathbf{l}$). Suppose also that, for every $\mathbf{l} \in \mathbb{Z}^d$, the set

$$V_{\mathbf{l}} \stackrel{\rm def}{=} \{\mathbf{l}' \in \mathbb{Z}^d \mid \varkappa(\mathbf{l}') = \mathbf{l}\}, \tag{2.110}$$

contains a fixed number of elements, the same for all such sets, i.e., $|V_{\mathbf{l}}| = v \in \mathbb{N}$. We will call such $V_{\mathbf{l}}$ *blocks*. Given a configuration $\sigma \in \Omega$, we set

$$\omega_{\mathbf{l}} = v^{-\zeta} \sum_{\mathbf{l}' \in V_{\mathbf{l}}} \sigma_{\mathbf{l}'}, \tag{2.111}$$

where $\zeta > 0$ is a parameter of our theory. Clearly, the vector $\omega = (\omega_{\mathbf{l}})_{\mathbf{l} \in \mathbb{Z}^d}$ is again an element of Ω , i.e., it is a configuration. This defines a transformation $O_{\varkappa,\zeta} : \Omega \to \Omega$, which maps configurations into configurations and depends on the initial map $\varkappa$ and on the parameter ζ. As a linear bounded (in the metric $\mathbf{d}$ defined by (2.53)) transformation, $O_{\varkappa,\zeta}$ is continuous, hence for every open $A \subset \Omega$, its preimage $O^{-1}_{\varkappa,\zeta}(A) = \{\sigma \in \Omega \mid O_{\varkappa,\zeta}(\sigma) \in A\}$ is also an open subset. This immediately yields that $O_{\varkappa,\zeta}$ is a measurable map, i.e., for every Borel set $A \subset \Omega$, its preimage $O^{-1}_{\varkappa,\zeta}(A)$ is also a Borel set (see subsection 2.5). For a probability measure $\mu \in \mathcal{P}(\Omega)$, we define a new measure $\tilde{\mu}$ by its values on Borel subsets of Ω as follows:

$$\tilde{\mu}(A) = \mu\left(O^{-1}_{\varkappa,\zeta}(A)\right), \tag{2.112}$$

which is correct in view of the properties of $O_{\varkappa,\zeta}$ discussed above. This new measure is again an element of $\mathcal{P}(\Omega)$, thus we have defined the map

$$R_{\varkappa,\zeta}(\mu) = \tilde{\mu}. \tag{2.113}$$

This map is called *the renormalization transformation*. The above definition of this transformation, although correct mathematically, may seem

to be too formal from the “physical point of view". Let us try to explain it. First one takes an “initial" spin configuration σ and transforms it into the configuration of renormalized block-spins, i.e., into ω given by (2.111). Then one fixes ω and “integrates out" the measure μ under the condition that the sums (2.111) are fixed. The new measure $\tilde{\mu}$ obtained in such a way is called *a renormalized measure* $R_{\varkappa,\zeta}(\mu)$. This program of passing from the distribution of spins to the distribution of renormalized block-spins was a key element of all constructions mentioned above, starting form the pioneering paper of L.P. Kadanoff. A measure $\mu \in \mathcal{P}(\Omega)$ is called *self-similar* if it is a fixed point of $R_{\varkappa,\zeta}$, that is it is a solution of the following equation

$$R_{\varkappa,\zeta}(\mu) = \mu. \tag{2.114}$$

The basic idea of the renormalization group theory of critical phenomena is the so called *universality hypothesis*, which states that at the critical point $\beta = \beta_c$, the set of all Gibbs states of the model $\mathcal{G}_\beta$ consists of one element and this element is self-similar. In other words, at the critical point the individual spins have the same probability distribution as the renormalized sums of such spins, as well as the renormalized sums of these sums, and so on. If this is the case, the critical point properties of the model may be obtained by studying $R_{\varkappa,\zeta}$. Here we are at the point of this theory where it is natural to show why it may fail to give the description of the mentioned properties. Since $R_{\varkappa,\zeta}$ is defined on the whole set $\mathcal{P}(\Omega)$, it is also defined on its subset $\mathcal{G}_\beta$, but it may not be a self-map of the latter set. This means, that for a given Gibbs measure $\mu \in \mathcal{G}_\beta$, the renormalized measure $R_{\varkappa,\zeta}(\mu)$ may not be Gibbsian, i.e., it may be out of $\mathcal{G}_\beta$. Since, except for simple situations, one renormalizes local Gibbs measures, not the elements of $\mathcal{G}_\beta$ directly, it is not clear whether or not the renormalized local Gibbs measures may give the elements of $\mathcal{G}_\beta$. The problem of this kind did appear in practice, for example it appears in the case of the Ising model with the interaction potential of finite range. In order to be able to proceed with the renormalization, physicists apply certain approximate “decouplings" that effectively corresponds to considering not the model itself but its caricature, for which such decouplings may be justified. This gives a hint that there might exist the models for which the mentioned problems do not appear and the renormalization scheme may be realized rigorously. Proceeding along this line of arguments, the authors of the paper Ref. [89] discovered that the approximate scheme developed in Refs. [86], [87] for the three-dimensional Ising model becomes rigorous being applied to Dyson's hierarchical model, which was invented and used by F.J. Dyson as a tool in the study of the one-dimensional translation invariant spin model. First critical point properties of Dyson's hierarchical model were studied in the papers of P.M. Bleher and Ya. G. Sinai Refs. [90], [91] (a complete description of these results is presented in the article Ref. [64]). Among other papers on this subject we mention here Refs. [82], [83], [84], [92].

It is not surprising that the hierarchical model (we describe it below) has that nice property. Instead of translation invariance possessed by the models we have considered so far this model has a symmetry which ideally fits the renormalization scheme – it is self-similar in some sense. The idea to substitute translation invariance by such a symmetry was (more or less consciously) used in the method developed by I.R. Yukhnovskii. In the paper Ref. [89] (slightly different version of the construction described in that paper was used in Ref. [42]) it was explicitly shown how does it lead from translation invariance to self-similarity possessed by hierarchical models. In this subsection we show that the translation invariant one-dimensional spin model with the power-like decaying interaction potential also fits the renormalization scheme. A preliminary version of the theory given below has appeared in Ref. [93].

We consider the model defined on the lattice $\mathbb{Z}$ and described by the Hamiltonian (2.2) with the interaction potential

$$J_{\mathbf{l}\mathbf{l}'} = [|\mathbf{l}-\mathbf{l}'|+1]^{-1-\lambda}. \qquad (2.115)$$

Set

$$V_{\mathbf{l}}^{(n)} = \{\mathbf{l}' \in \mathbb{Z} \mid 2^n\mathbf{l} \le \mathbf{l}' \le 2^n\mathbf{l} + 2^n - 1\}, \quad n \in \mathbb{N}_0, \quad \mathbf{l} \in \mathbb{Z}. \qquad (2.116)$$

The subsets $V_{\mathbf{l}}^{(1)}$ are the blocks mentioned above, for which $v = 2$. Then $\varkappa$ maps the two elements of such a block $V_{\mathbf{l}}^{(1)}$ into $\mathbf{l}$. Considering the blocks $V_{\mathbf{l}}^{(1)}$, $\mathbf{l} \in \mathbb{Z}$ as elements of a new lattice, which however is the same $\mathbb{Z}$, we may apply the map $\varkappa$ to these elements, which will map $V_{\mathbf{l}'}^{(1)}$ into $\mathbb{Z}$ exactly as it was above. This produces a hierarchy of subsets $V_{\mathbf{l}}^{(n)}$ of the lattice $\mathbb{Z}$, defined by (2.116), organized according to the following rule

$$V_{\mathbf{l}}^{(n)} = \bigcup_{\mathbf{l}' \in V_{\mathbf{l}}^{(n-k)}} V_{\mathbf{l}'}^{(k)}; \quad n \in \mathbb{N}, \quad k = 0, 1, \ldots, n-1. \qquad (2.117)$$

Given $n \in \mathbb{N}_0$, we set $\mathcal{V}^{(n)} = \{V_{\mathbf{l}}^{(n)}\}_{\mathbf{l}\in\mathbb{Z}}$.

Definition 43: The family $\{\mathcal{V}^{(n)}\}_{n\in\mathbb{N}_0}$ is called a hierarchical structure on the lattice $\mathbb{Z}$.

For $\alpha, \alpha' = 0, 1$ and $\mathbf{l}, \mathbf{l}' \in \mathbb{Z}$, we set

$$I(2\mathbf{l}+\alpha, 2\mathbf{l}'+\alpha') = \qquad (2.118)$$
$$= \frac{1}{[|(2\mathbf{l}+\alpha)-(2\mathbf{l}'+\alpha')|+1]^{1+\lambda}} - \frac{2^{-(1+\lambda)}}{[|\mathbf{l}-\mathbf{l}'|+1]^{1+\lambda}},$$

which defines $I(\mathbf{l}, \mathbf{l}')$ for all $\mathbf{l}, \mathbf{l}' \in \mathbb{Z}$. In order to introduce a lattice spin model one often uses its formal Hamiltonian written on the whole lattice, which has no rigorous mathematical meaning but shows how one can define local Hamiltonians (2.2). Such a formal Hamiltonian in the case of the model we consider now is

$$H = -\frac{1}{2}\sum_{\mathbf{l},\mathbf{l}'\in\mathbb{Z}} \frac{1}{[|\mathbf{l}-\mathbf{l}'|+1]^{1+\lambda}} \cdot \sigma_{\mathbf{l}}\sigma_{\mathbf{l}'}, \qquad (2.119)$$

which by means of (2.118) may be rewritten (more details on how to pass to the expression below may be found in Ref. [93])

$$H = -\frac{1}{2}\sum_{n=0}^{\infty}\sum_{\mathbf{l},\mathbf{l}'\in\mathbb{Z}} 2^{-n(1+\lambda)} I(\mathbf{l},\mathbf{l}')\sigma(V_{\mathbf{l}}^{(n)})\sigma(V_{\mathbf{l}'}^{(n)}), \qquad (2.120)$$

where for a set $\Delta \subset \mathbb{Z}$,

$$\sigma(\Delta) = \sum_{\mathbf{l}\in\Delta} \sigma_{\mathbf{l}}. \qquad (2.121)$$

The essence of the above representation of H is that it has a block-spin structure, which enables us to apply here the renormalization scheme described above. The local Hamiltonians may be defined on the base of (2.120) by restricting the sums to finite subsets of the lattice $\mathbb{Z}$, which now are to be taken in a special way if one wants to preserve the block-spin structure of these Hamiltonians. In particular, the local Hamiltonian in

$$\Lambda_m \stackrel{\text{def}}{=} \{\mathbf{l} \in \mathbb{Z} \mid -2^m \le \mathbf{l} \le 2^m - 1\}, \quad m \in \mathbb{N}_0,$$

with the zero boundary conditions is

$$H_{\Lambda_m} = \frac{1}{2}\sum_{n=0}^{m}\sum_{\mathbf{l},\mathbf{l}'\in\Lambda_{m-n}} 2^{-n(1+\lambda)} I(\mathbf{l},\mathbf{l}')\sigma(V_{\mathbf{l}}^{(n)})\sigma(V_{\mathbf{l}'}^{(n)}), \quad m \in \mathbb{N}. \qquad (2.122)$$

It should be stressed here that this Hamiltonian does not coincide with the one which can be obtained from the formal Hamiltonian in the form (2.119) by restricting the sums to Λ_m. But, in order to construct Gibbs states according to the scheme based on the DLR equation (see subsection 2.4) we would need such local Hamiltonians defined on different subsets $\Delta \subset \mathbb{Z}$ and with boundary conditions outside such Δ. This means that following this scheme we could not preserve the block-spin structure of local Hamiltonians, which would make the above construction useless. This problem may be overcome as follows. Given $m \in \mathbb{N}$, we set

$$H_m = -\frac{1}{2} \sum_{n=0}^{m} \sum_{\mathbf{l},\mathbf{l}' \in \mathbb{Z}} 2^{-n(1+\lambda)} I(\mathbf{l},\mathbf{l}') \sigma(V_{\mathbf{l}}^{(n)}) \sigma(V_{\mathbf{l}'}^{(n)}). \tag{2.123}$$

In contrast to (2.122) this is still a formal Hamiltonian since the sums over $\mathbf{l}, \mathbf{l}'$ run through the whole lattice. In view of the way how we produced it one can call such an expression *a truncated Hamiltonian*. By means of (2.121) we may rewrite it in the form (2.119) but with a certain interaction potential $J_m(\mathbf{l},\mathbf{l}')$ instead of $[|\mathbf{l}-\mathbf{l}'|+1]^{-1-\lambda}$. The effect of the above truncation is that this potential has the following asymptotics

$$J_m(\mathbf{l},\mathbf{l}') \sim J_m^{(0)} |\mathbf{l}-\mathbf{l}'|^{-2-\lambda}, \quad |\mathbf{l}-\mathbf{l}'| \to +\infty. \tag{2.124}$$

Due to Dyson's results mentioned above the model with such a formal Hamiltonian has no phase transitions. Moreover, for a similar model with the above asymptotics of the interaction potentials, it was proven that their sets $\mathcal{G}_\beta$ are singletons for all β (see the paper Ref. [94] and subsection 8.3 in the book Ref. [4]). This opens a possibility to construct Gibbs states of the model described by (2.120) in the following way. First one constructs the states μ_m for (2.123), which are unique for every $m \in \mathbb{N}$, and then passes to the limit $m \to +\infty$. By construction, these states satisfy the recursion

$$R_{\varkappa,1+\lambda}(\mu_{m+1}) = \mu_m, \tag{2.125}$$

and the limit of the sequence $\{\mu_m\}$ should be a fixed point of $R_{\varkappa,1+\lambda}$. We are going to realize this program in a separate work.

Now let us turn to hierarchical models. The simplest way to get such a model from the translation invariant model we consider in this subsection is to put in (2.120) $I(\mathbf{l},\mathbf{l}') = 0$ for all $\mathbf{l} \neq \mathbf{l}'$. As a consequence, the formal Hamiltonian of this new model is

$$H = -\frac{1}{2} \sum_{n=0}^{\infty} \sum_{\mathbf{l} \in \mathbb{Z}} 2^{-n(1+\lambda)} I(\mathbf{l},\mathbf{l}) \left[\sigma(V_{\mathbf{l}}^{(n)})\right]^2. \tag{2.126}$$

Since $I(\mathbf{l},\mathbf{l}') \geq 0$, such an action decreases the interaction potential of the model (2.120). Thus, by Corollary 20, the order parameter of the

hierarchical model $P^{\mathrm{h}}(\beta)$ (see (2.72) will not exceed the corresponding parameter $P(\beta)$ of the model (2.120). Hence if the former one is positive for big enough β, $P(\beta)$ should be positive as well. This argumentation was used by F.J. Dyson in Ref. [75], although no direct comparison of these models was made in that paper. Comparing the Hamiltonians (2.120) and (2.126) one can conclude that in the hierarchical version the interaction between different blocks of the same hierarchy level is neglected. The effect of this is that the Hamiltonian (2.126) is additive in $\mathbf{l}$, which in turn implies that instead of considering measures on infinite dimensional spaces of configurations, such as the measures μ_m, μ_{m+1} in (2.125), one considers measures just on the space $\mathbb{R}$, which define the probability distributions of block-spins. The corresponding recurrence will have the form of (2.125) but with the renormalization transformation acting on such one-dimensional measures. For further details the reader is referred to the articles describing hierarchical models mentioned above. Finally, we remark that the hierarchical models are studied very well, one can say that almost all critical point properties of such models are known. In particular, a self-similar Gibbs state of such a model was constructed in the paper Ref. [95].

3. Quantum Models

3.1. Local Gibbs states

For quantum lattice systems, Gibbs states are defined by means of their Hamiltonians, which now are operators on certain complex Hilbert spaces. These spaces consist of wave functions, as usual they have countable bases – complete orthonormal systems of functions, such that every wave function may be written as countable linear combination of the elements of such a system. For a given finite subset $\Delta \subset \mathbb{Z}^d$, let $\mathcal{H}_\Delta$ be such a Hilbert space and $\{\psi_n\}_{n\in\mathbb{N}}$ be its base. Let also $\mathbf{L}(\mathcal{H}_\Delta)$ be the set of all linear operators acting from $\mathcal{H}$ to $\mathcal{H}$.

An example of a quantum lattice system is the spin model described by the following Hamiltonian

$$H_\Delta = -\frac{1}{2} \sum_{\mathbf{l},\mathbf{l}' \in \Delta} \sum_{\alpha=x,y,z} J_{\mathbf{l}\mathbf{l}'}^{\alpha} \sigma_{\mathbf{l}}^{\alpha} \sigma_{\mathbf{l}'}^{\alpha} - \sum_{\mathbf{l} \in \Delta} \sum_{\alpha=x,y,z} h_{\mathbf{l}}^{\alpha} \sigma_{\mathbf{l}}^{\alpha}, \tag{3.1}$$

where, as above, $J_{\mathbf{l}\mathbf{l}'}^{\alpha}$, $h_{\mathbf{l}}^{\alpha}$ are real parameters of the model, defined for all $\mathbf{l}, \mathbf{l}' \in \mathbb{Z}^d$ and $\alpha = x, y, z$, and $\sigma_{\mathbf{l}}^{x}$, $\sigma_{\mathbf{l}}^{y}$, $\sigma_{\mathbf{l}}^{z}$ are the Pauli matrices. With every $\mathbf{l}$ we associate the space $\mathcal{H}_{\mathbf{l}}$ of two-dimensional complex vectorcolumns with the scalar product

$$\phi = \begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix}, \qquad <\psi, \phi> = \bar{\psi}^{(1)}\phi^{(1)} + \bar{\psi}^{(2)}\phi^{(2)},$$

where $\bar{\psi}^{(j)}$, $j = 1, 2$ stands for complex conjugate. The action of the Pauli matrices on such vectors is defined as usual multiplication of matrices. The Hilbert space $\mathcal{H}_\Delta$ is defined as a tensor product

$$\mathcal{H}_\Delta = \bigotimes_{\mathbf{l}\in\Delta} \mathcal{H}_\mathbf{l}. \tag{3.2}$$

Its canonical base consists of the following vectors

$$\epsilon(s_\Delta) = \bigotimes_{\mathbf{l}\in\Delta} \epsilon(s_\mathbf{l}), \qquad s_\mathbf{l} = \pm 1, \quad \epsilon(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \epsilon(-1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.3}$$

Here, similarly as it was above, s_Δ is a vector with the components $s_\mathbf{l}$, $\mathbf{l} \in \Delta$ taking values ± 1. Thus, the space $\mathcal{H}_\Delta$ is finite-dimensional and $\dim \mathcal{H}_\Delta = 2^{|\Delta|}$. Every $\sigma^\alpha_\mathbf{l}$ acts on the above vectors as follows

$$\sigma^\alpha_\mathbf{l}\epsilon(s_\Delta) = \left(\sigma^\alpha_\mathbf{l}\epsilon(s_\mathbf{l}) \bigotimes\right) \bigotimes_{\mathbf{l}'\in\Delta\setminus\{\mathbf{l}\}} \epsilon(s_{\mathbf{l}'}).$$

In particular

$$\sigma^z_\mathbf{l}\epsilon(s_\Delta) = s_\mathbf{l}\epsilon(s_\Delta).$$

Each element of the space $\mathcal{H}_\Delta$ may be associated with a $2^{|\Delta|}$-dimensional complex vector-column. Then every element of $\mathbf{L}(\mathcal{H}_\Delta)$ may be represented by a $2^{|\Delta|} \times 2^{|\Delta|}$-complex matrix with the standard definition of its action on the above vectors.

Various versions of the model (3.1) are employed as the so called *quasi-spin models* [96]. If in the Hamiltonian (3.1) all $J^x_{\mathbf{l}\mathbf{l}'} = J^y_{\mathbf{l}\mathbf{l}'} = 0$ and $h^x_\mathbf{l} = h^y_\mathbf{l} = 0$, it turns into the Ising model in the external field $h^z_\mathbf{l}$. The Heisenberg model, the Ising model in a transverse field may be obtained from (3.1) in an evident way.

Another example of quantum lattice models which is widely employed in the theory of structural phase transitions (see e.g., Ref. [32]) is the model of interacting quantum anharmonic oscillators, described by the following Hamiltonian (c.f., (2.10))

$$H_\Delta = \frac{1}{2m}\sum_{\mathbf{l}\in\Delta} p^2_\mathbf{l} + \sum_{\mathbf{l}\in\Delta} P(q_\mathbf{l}) - \sum_{\mathbf{l}\in\Delta} h_\mathbf{l} q_\mathbf{l} - \frac{1}{2}\sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'} q_\mathbf{l} q_{\mathbf{l}'}. \tag{3.4}$$

Here m is the particle mass, P is the same polynomial as in (2.7)-(2.10), the external field $h_\mathbf{l}$ and the interaction potential $J_{\mathbf{l}\mathbf{l}'}$ are also the same as in the classical case. But now $p_\mathbf{l}$ and $q_\mathbf{l}$ are canonical momentum and position operators, defined in the complex Hilbert space $L^2(\mathbb{R})$ of functions, which are square integrable on $\mathbb{R}$ with respect to Lebesgue's measure. These operators obey the canonical commutation relation

$$p_\mathbf{l} q_\mathbf{l} - q_\mathbf{l} p_\mathbf{l} = -i\hbar,$$

and are unbounded (see below), which strongly complicates the theory of this model. The Hamiltonian (3.4) is also unbounded, all these operators are essentially self-adjoint.

For every $\psi \in \mathcal{H}_\Delta$, we define its norm as usual $\|\psi\| = \sqrt{<\psi, \psi>}$. An operator $T \in \mathbf{L}(\mathcal{H}_\Delta)$ is said to be bounded if there exists a constant $C > 0$ such that, for every ψ,

$$\|T\psi\| \leq C\|\psi\|.$$

The least such C will be denoted $\|T\|$. We will also denote by $\mathbf{L}_b(\mathcal{H}_\Delta)$ the set of all bounded linear operators acting from $\mathcal{H}_\Delta$ into $\mathcal{H}_\Delta$. As usual, every such an operator is defined on the whole space $\mathcal{H}_\Delta$. For the quantum spin models described above, $\mathbf{L}_b(\mathcal{H}_\Delta) = \mathbf{L}(\mathcal{H}_\Delta)$ since the corresponding Hilbert space is finite-dimensional. For every $T \in \mathbf{L}_b(\mathcal{H}_\Delta)$, we define its Hermitian conjugate T^* as an operator satisfying the following relation $<T^*\phi, \psi> = <\phi, T\psi>$ for all $\phi, \psi \in \mathcal{H}_\Delta$. An operator $T \in \mathbf{L}_b(\mathcal{H}_\Delta)$ is said to be positive if

$$<\psi, T\psi> \geq 0,$$

for all $\psi \in \mathcal{H}_\Delta$. For such an operator, one may define $\sqrt{T}$. For every $T \in \mathbf{L}_b(\mathcal{H}_\Delta)$, the operator T^*T is positive, hence one may set $|T| = \sqrt{T^*T}$. An operator $T \in \mathbf{L}_b(\mathcal{H}_\Delta)$ is said to be trace-class if, for an orthonormal base $\{\psi_n\}_{n\in\mathbb{N}}$,

$$\sum_{n=1}^{\infty} <\psi_n, |T|\psi_n> < \infty. \tag{3.5}$$

Then one may set

$$\mathrm{trace}(T) = \sum_{n=1}^{\infty} <\psi_n, T\psi_n>. \tag{3.6}$$

The latter series converges absolutely, its sum is independent of the particular choice of the base. The set of all trace-class operators will be denoted by $\mathbf{L}_t(\mathcal{H}_\Delta)$. For every $T \in \mathbf{L}_t(\mathcal{H}_\Delta)$ and $Q \in \mathbf{L}_b(\mathcal{H}_\Delta)$, the products TQ and QT are trace-class. Clearly $\mathbf{L}_t(\mathcal{H}_\Delta) \subset \mathbf{L}_b(\mathcal{H}_\Delta) \subset \mathbf{L}_(\mathcal{H}_\Delta)$. In finite-dimensional spaces all these sets coincide, but in the infinite-dimensional case the inclusions are proper.

For $T \in \mathbf{L}_b(\mathcal{H}_\Delta)$, the above introduced $\|T\|$ is a norm, we shall call it *operator norm*. The set $\mathbf{L}_b(\mathcal{H}_\Delta)$ equipped with the operator norm turns into a complete normed space – a Banach space. In addition to the linear space structure we have also multiplication on the latter space. By definition, for any $T, Q \in \mathbf{L}_b(\mathcal{H}_\Delta)$,

$$\|TQ\| \leq \|T\|\|Q\|, \quad \|T^*\| = \|T\|. \tag{3.7}$$

Banach spaces with multiplication and an operation $T \mapsto T^*$, which possess the above properties, are called C^*-algebras. A detailed presentation of the theory of these algebras and their application in quantum statistical physics may be found in the book Ref. [2].

For an entire function $F : \mathbb{C} \to \mathbb{C}$ and $T \in \mathbf{L}_b(\mathcal{H}_\Delta)$, we set

$$F(T) = \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(0) T^n, \quad F^{(n)} = \frac{d^n F}{dz^n}(0).$$

If this series converges in the operator norm, it defines a bounded operator $F(T)$. In both above examples the Hamiltonians (not necessarily belonging to $\mathbf{L}_b(\mathcal{H}_\Delta)$) are such that $\exp(-\beta H_\Delta) \in \mathbf{L}_t(\mathcal{H}_\Delta)$ for any finite subset Δ and every $\beta > 0$. For such models, one may set

$$Z_{\beta,\Delta} = \mathrm{trace}[\exp(-\beta H_\Delta)]; \quad \rho_{\beta,\Delta} = Z_{\beta,\Delta}^{-1} \exp(-\beta H_\Delta). \tag{3.8}$$

The latter trace-class operator is called a density matrix. Then, for every $T \in \mathbf{L}_b(\mathcal{H}_\Delta)$, one may define

$$T \mapsto \langle T \rangle_{\beta,\Delta} = \mathrm{trace}\,(T \rho_{\beta,\Delta}), \tag{3.9}$$

which is a normalized positive linear functional on the C^*-algebra $\mathbf{L}_b(\mathcal{H}_\Delta)$. It is called a local Gibbs state of the model and the elements of $\mathbf{L}_b(\mathcal{H}_\Delta)$ are called observables. Comparing with the case of classical models one can conclude that here taking trace corresponds to integration, the density matrix corresponds to the local Gibbs measure (2.3), the algebra of observables $\mathbf{L}_b(\mathcal{H}_\Delta)$ corresponds to the set of functions $\mathcal{F}_\Delta$.

To simplify notations we set

$$\mathfrak{A}_\Delta = \mathbf{L}_b(\mathcal{H}_\Delta).$$

As above, for $\Delta \subset \Delta'$, we may include $\mathfrak{A}_\Delta$ into $\mathfrak{A}_{\Delta'}$ and define

$$\mathfrak{A} = \bigcup_{\Delta \in \mathcal{D}} \mathfrak{A}_\Delta, \tag{3.10}$$

where $\mathcal{D}$ is an increasing sequence of subsets, which exhausts the lattice $\mathbb{Z}^d$.

3.2. Green and Matsubara functions

The time evolution of a quantum system is described by the corresponding Schrödinger equation, the solutions of which define the evolution of the states (3.9). In the Heisenberg approach wave functions, and hence the states, do not evolute. Instead the evolution of the system is described by the evolution of observables, which constitute the algebras $\mathfrak{A}_\Delta$. It is described by the following map. Given $t \in \mathbb{R}$, considered as time, we set

$$\mathfrak{a}_\Delta^t(T) = \exp(i(t/\hbar)H_\Delta) T \exp(-i(t/\hbar)H_\Delta), \quad T \in \mathfrak{A}_\Delta. \tag{3.11}$$

In what follows, an observable T at time $t = 0$ evolutes into the observables $\mathfrak{a}_\Delta^t(T)$. The evolution maps $\mathfrak{a}_\Delta^t$ have the following properties. Since $U_t = \exp(i(t/\hbar)H_\Delta)$, $t \in \mathbb{R}$ is a unitary operator, one has

$$\|\mathfrak{a}_\Delta^t(T)\| = \|T\|, \tag{3.12}$$

i.e., $\mathfrak{a}_\Delta^t$ are norm-preserving hence continuous as maps acting between normed spaces. Furthermore, they are linear, that is

$$\mathfrak{a}_\Delta^t(\kappa T + \lambda Q) = \kappa \mathfrak{a}_\Delta^t(T) + \lambda \mathfrak{a}_\Delta^t(Q),$$

for all $\kappa, \lambda \in \mathbb{C}$ and $T, Q \in \mathfrak{A}_\Delta$. For any $t, s \in \mathbb{R}$ and $T \in \mathfrak{A}_\Delta$, one has

$$\mathfrak{a}_\Delta^s\left(\mathfrak{a}_\Delta^t(T)\right) = \mathfrak{a}_\Delta^{t+s}(T), \tag{3.13}$$

which means that they constitute a group of algebraic isomorphisms. Finally, they are called time automorphisms since they map the algebra of observables $\mathfrak{A}_\Delta$ into itself. For $T_1, \ldots, T_n \in \mathfrak{A}_\Delta$ and $t_1, \ldots, t_n \in \mathbb{R}$, we set

$$G_{T_1,\ldots,T_n}^{\beta,\Delta}(t_1, \ldots, t_n) = \mathrm{trace}\left(\mathfrak{a}_\Delta^{t_1}(T_1) \ldots \mathfrak{a}_\Delta^{t_n}(T_n) \rho_{\beta,\Delta}\right), \tag{3.14}$$

where the density matrix $\rho_{\beta,\Delta}$ was defined in (3.8). For fixed $T_1, \ldots, T_n \in \mathfrak{A}_\Delta$, this is a function of $t_1, \ldots, t_n$ defined on the whole $\mathbb{R}^n$. It is called a Green function for those observables. Clearly, the whole information about the evolution is contained in these functions defined for all observables $T \in \mathfrak{A}_\Delta$. Here it would be quite natural to try to find a smaller set of observables such that the Green functions defined for the elements of this set completely describe the evolution of the whole algebra. For the models considered in this section, such a smaller set was found by R. Høegh-Krohn in his paper Ref. [97]. In order to formulate its results we have to introduce new notions. A subset $\mathfrak{M} \subset \mathfrak{A}_\Delta$ is called subalgebra if it is an algebra with respect to the linear operations and multiplication, which means that it is closed with respect to these operations. A subalgebra is called abelian if all its elements commute with each other. For the model of interacting quantum anharmonic oscillators described by the Hamiltonian (3.4), such a subalgebra consists of multiplication operators on bounded continuous functions. An operator $T : L^2(\mathbb{R}^{|\Delta|}) \to L^2(\mathbb{R})$ is called a multiplication operator on the bounded continuous function $F : \mathbb{R}^{|\Delta|} \to \mathbb{C}$ if for every $\psi \in L^2(\mathbb{R}^{|\Delta|})$,

$$(T\psi)(x_\Delta) = F(x_\Delta)\psi(x_\Delta). \tag{3.15}$$

In the sequel, we will denote such operators also by F. Of course, linear combinations and products of multiplication operators are again multiplication operators, they commute with each other. The algebra of such operators will be denoted by $\mathfrak{F}_\Delta$. Now we are at a position to present the result of R. Høegh-Krohn.

Proposition 44: *Let $\mathfrak{A}_\Delta^{(0)} \subset \mathfrak{A}_\Delta$ be the set of all observables which are linear combinations of the operators*

$$\mathfrak{a}_\Delta^{t_1}(F_1) \ldots \mathfrak{a}_\Delta^{t_1}(F_n)$$

for all possible choices $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$ and $F_1, \ldots F_n \in \mathfrak{F}_\Delta$. Then the strong closure of this set coincides with the whole algebra $\mathfrak{A}_\Delta$.

The meaning of this statement is that the Green functions defined on the multiplication operators only fully determine the local Gibbs state $\langle\cdot\rangle_{\beta,\Delta}$ defined in (3.8). A similar statement may be proven also for certain quantum spin models described by the Hamiltonian (3.1). In this case the role of $\mathfrak{F}_\Delta$ will be played by the algebra generated by the Pauli matrices $\sigma_{\mathbf{l}}^z$ with $\mathbf{l} \in \Delta$. The next step in developing the tools for studying local Gibbs states of quantum models is to extend analytically the Green functions to imaginary values of $t_1, \ldots, t_n$ and to obtain Matsubara functions. In a general situation the corresponding theorem was proven in the paper Ref. [98]. The proof is quite complicated. For the

model of interacting quantum anharmonic oscillators, the proof was done in the paper Ref. [33], its extended and simplified version may be found in the review article Ref. [34]. Given $n \in \mathbb{N}$ and a domain $\mathcal{O} \subset \mathbb{C}^n$, let $\mathrm{Hol}(\mathcal{O})$ be the set of all functions holomorphic on $\mathcal{O}$. Given $\beta > 0$ and $n \in \mathbb{N}$, we set

$$\mathcal{D}_n^\beta = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid 0 < \Im(t_1) < \Im(t_2) < \cdots < \Im(t_n) < \beta\}, \tag{3.16}$$

where $\Im(t_j)$ stands for the imaginary part of t_j, $j = 1, \ldots, n$. By $\overline{\mathcal{D}_n^\beta}$ we denote the closure of $\mathcal{D}_n^\beta$. Given $\xi_1, \ldots, \xi_n \in \mathbb{R}$, we also set

$$\mathcal{D}_n^\beta(\xi_1, \ldots, \xi_n) = \{(t_1, \ldots, t_n) \in \mathcal{D}_n^\beta \mid \Re(t_j) = \xi_j, \quad \text{for} \quad j = 1, \ldots, n\}. \tag{3.17}$$

Proposition 45: *For every $T_1, \ldots, T_n \in \mathfrak{A}_\Delta$,*

(a) *the function $G_{T_1,\ldots,T_n}^{\beta,\Delta}$ may be extended to a holomorphic function on $\mathcal{D}_n^\beta$;*

(b) *this extension (which will also be written as $G_{T_1,\ldots,T_n}^{\beta,\Delta}$) is continuous on $\overline{\mathcal{D}_n^\beta}$ and for all $(t_1, \ldots, t_n) \in \overline{\mathcal{D}_n^\beta}$,*

$$|G_{T_1,\ldots,T_n}^{\beta,\Delta}(t_1, \ldots, t_n)| \leq \|T_1\| \ldots \|T_n\|, \tag{3.18}$$

where $\|\cdot\|$ stands for the operator norm;

(c) *for every $\xi_1, \ldots, \xi_n \in \mathbb{R}$, the set $\mathcal{D}_n^\beta(\xi_1, \ldots, \xi_n)$ is such that for any $f, g \in \mathrm{Hol}(\mathcal{D}_n^\beta)$, the equality $f = g$ on $\mathcal{D}_n^\beta(\xi_1, \ldots, \xi_n)$ implies that these functions are equal on the whole $\mathcal{D}_n^\beta$.*

The meaning of this result may be explained as follows. If one has the Green functions for all possible choices of $F_j \in \mathfrak{F}_\Delta$, defined on one of such $\mathcal{D}_n^\beta(\xi_1, \ldots, \xi_n)$ only, then one has the complete information about the state. Indeed, by claims (a) and (c) of the above proposition, the values of the Green functions for real $t_1, \ldots, t_n$ may be uniquely determined by their values on such $\mathcal{D}_n^\beta(\xi_1, \ldots, \xi_n)$. Then, by Proposition 44, the values of the Green functions constructed for $F_j \in \mathfrak{F}_\Delta$ only uniquely determine the values of such functions constructed for all operators, which in turn determines the state $\langle\cdot\rangle_{\beta,\Delta}$. By claim (a) of Proposition 45, the Green functions are differentiable for all real $t_1, \ldots, t_n$, which can be used to study them by means of differential equations.

The restrictions of the Green functions $G_{T_1,\ldots,T_n}^{\beta,\Delta}$ to $\mathcal{D}_n^\beta(0, \ldots, 0)$, i.e.,

$$\Gamma_{T_1,\ldots,T_n}^{\beta,\Delta}(\tau_1, \ldots, \tau_n) = G_{T_1,\ldots,T_n}^{\beta,\Delta}(it_1, \ldots, it_n), \tag{3.19}$$

are called Matsubara functions for the observables $T_1, \ldots, T_n$. In the light of the above discussion, these functions constructed for all possible choices of $F_1, \ldots, F_n \in \mathfrak{F}_\Delta$ completely determine the state $\langle\cdot\rangle_{\beta,\Delta}$.

3.3. Euclidean approach

Integration in spaces of functions is one of the most popular and powerful methods of modern quantum theory. It appeared as a result of mathematical development of R. Feynman's ideas [1] to formulate quantum theory in terms of path integrals. In the course of this development deep connections between quantum theory and stochastic analysis has been revealed. A profound description of these connections, as well as of the method and its various applications, is given in B. Simon's book Ref. [12].

In 1975 in the papers Refs. [33], [97] an approach to the construction of Gibbs states of quantum lattice models of the type of (3.4) based on integration in function spaces has been initiated. In the case of the Ising model with transverse field a similar methods were used in the paper Ref. [65]. The essence of the approach of Ref. [33] may be expressed in the following formula derived in that paper

$$\Gamma^{\beta,\Delta}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) = \int_{\Omega_{\beta,\Delta}} F_1(\sigma_\Delta(\tau_1))\ldots F_n(\sigma_\Delta(\tau_n))\mathrm{d}\nu_{\beta,\Delta}(\sigma_\Delta), \tag{3.20}$$

which strongly reminds expressions from the preceding section like (2.17). The main dissimilarity of (3.20) and (2.17) is that the above integral is taken in an infinite dimensional space. Let us describe all components of the right-hand side of (3.20). First we introduce $\Omega_{\beta,\Delta}$. By $C[0,\beta]$ we denote the real linear space of continuous functions $\phi : [0,\beta] \to \mathbb{R}$. This space endowed with the norm $\|\phi\| = \sup\{|\phi(\tau)| \mid \tau \in [0,\beta]\}$ becomes a Banach space. Set

$$\mathcal{C}_\beta = \{\phi \in C[0,\beta] \mid \phi(0) = \phi(\beta)\}. \tag{3.21}$$

This space consists of continuous periodic functions on the interval $[0,\beta]$. This is a closed subspace of $C[0,\beta]$, which means that it is a Banach space with the same norm. Furthermore, we set

$$\Omega_{\beta,\Delta} = \{\sigma_\Delta = (\sigma_{\mathbf{l}})_{\mathbf{l}\in\Delta} \mid \sigma_{\mathbf{l}} \in \mathcal{C}_\beta\}. \tag{3.22}$$

Each element of $\Omega_{\beta,\Delta}$ is a vector $\sigma_\Delta = (\sigma_{\mathbf{l}})_{\mathbf{l}\in\Delta}$ with the components $\sigma_{\mathbf{l}}$, which may also be called *spins*, but this time the spins are periodic continuous functions defined on $[0,\beta]$, i.e., they are infinite dimensional.

Now we describe the measure $\mathrm{d}\nu_{\beta,\Delta}$, which plays here a similar role as the local Gibbs measure $\mathrm{d}\nu_\Delta$ (2.3) does in the classical case. First we define a reference measure γ_β. It is a symmetric Gaussian measure on the Banach space $\mathcal{C}_\beta$ (the theory of such measures may be found in the books Refs. [12], [19], [20]), which is completely determined by its covariation operator. The latter is an integral operator with the following integral kernel

$$S(\tau,\tau') = \frac{1}{2\sqrt{m}} \cdot \frac{\exp\left(-|\tau-\tau'|/\sqrt{m}\right) + \exp\left(-(\beta-|\tau-\tau'|)/\sqrt{m}\right)}{1-\exp\left(-\beta/\sqrt{m}\right)}, \tag{3.23}$$

where m is the same as in the Hamiltonian (3.4), i.e., it is the particle mass. It appears that this is nothing else but the Matsubara function of the quantum harmonic oscillator of mass m described by the Hamiltonian

$$H^{\mathrm{har}} = \frac{1}{2m}p^2 + \frac{1}{2}q^2.$$

On the other hand, $S(\tau,\tau')$ is the correlation function of the so called *periodic Ornstein-Uhlenbeck process with period* β . First this process has appeared in the pioneering paper Ref. [97], the study of such processes and their applications in quantum statistical physics is given in the papers Refs. [98], [99].

In what follows, the Gaussian measure γ_β describes the states of a single quantum harmonic oscillator. The states of interacting quantum anharmonic oscillators located at sites of the subset Δ are described by the measure which is constructed from γ_β and the energy functions $E_{\beta,\Delta}$ on the base of the famous Feynman-Kac formula (see e.g., Ref. [12])

$$\mathrm{d}\nu_{\beta,\Delta}(\sigma_\Delta) = \frac{1}{Z_{\beta,\Delta}}\exp\{-E_{\beta,\Delta}(\sigma_\Delta)\}\prod_{\mathbf{l}\in\Delta}\mathrm{d}\gamma_\beta(\sigma_{\mathbf{l}}), \tag{3.24}$$

where

$$Z_{\beta,\Delta} = \int_{\Omega_{\beta,\Delta}}\exp\{-E_{\beta,\Delta}(\sigma_\Delta)\}\prod_{\mathbf{l}\in\Delta}\mathrm{d}\gamma_\beta(\sigma_{\mathbf{l}}), \tag{3.25}$$

and

$$\begin{aligned} E_{\beta,\Delta}(\sigma_\Delta) &= -\frac{1}{2}\sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'}\int_0^\beta \sigma_{\mathbf{l}}(\tau)\sigma_{\mathbf{l}'}(\tau)\mathrm{d}\tau - \sum_{\mathbf{l}\in\Delta} h_{\mathbf{l}}\int_0^\beta \sigma_{\mathbf{l}}(\tau)\mathrm{d}\tau + \\ &+ \sum_{\mathbf{l}\in\Delta}\int_0^\beta \tilde{P}(\sigma_{\mathbf{l}}(\tau))\mathrm{d}\tau, \end{aligned} \tag{3.26}$$

where $\tilde{P}(t) = P(t) - (t^2/2)$ – we have extracted $t^2/2$ into the Gaussian measure γ_β. The measure (3.24) is called *the local Euclidean Gibbs measure*. Since this measure completely determines the Matsubara functions (3.19) for all $F_1, \ldots, F_n \in \mathfrak{F}_\Delta$, it determines the local Gibbs state $\langle\cdot\rangle_{\beta,\Delta}$, it is also called *the local Euclidean Gibbs state*. In what follows, the Euclidean approach allows one to study local Gibbs states of the model (3.4) by means of probability measures as if it is a system of classical spins with the only difference that these spins are infinite dimensional. This approach was developed in the papers Refs. [34], [54], [61], [73], [74], [100]–[111]. Its full description and an extended related bibliography is given in the review article Ref. [34]. Here we mention certain results obtained in this approach. First of all it would make sense to study these states in the quasi-classical limit $m \to +\infty$. In Ref. [101] (see also section 3 in Ref. [34]) a statement describing such a limit was proved. Its corollary may be formulated as follows. We recall that in section 2 we have introduced the set $\mathcal{F}_\Delta$ of all continuous polynomially bounded functions $f : \Omega_\Delta = \mathbb{R}^{|\Delta|} \to \mathbb{R}$. Let $\mathcal{F}_\Delta^{(0)} \subset \mathcal{F}_\Delta$ be the set of such functions which are bounded. We shall use the set $\mathbf{F}_{\beta,\Delta}$ consisting of all bounded continuous functions $F : \Omega_{\beta,\Delta} \to \mathbb{R}$, where $\Omega_{\beta,\Delta}$ is defined by (3.22). By $\Omega^{c}_{\beta,\Delta}$ we denote the subset of $\Omega_{\beta,\Delta}$ consisting of all constant vectors, i.e.,

$$\Omega^{c}_{\beta,\Delta} = \{\sigma_\Delta = (\sigma_{\mathbf{l}})_{\mathbf{l}\in\Delta} \in \Omega_{\beta,\Delta} \mid \exists \xi_\Delta = (\xi_{\mathbf{l}})_{\mathbf{l}\in\Delta} \in \Omega_\Delta \quad \forall \tau \in [0,\beta] \ \ \forall \mathbf{l} \in \Delta : \quad \sigma_{\mathbf{l}}(\tau) \equiv \xi_{\mathbf{l}}\}.$$

For the elements of this set we write $\sigma_\Delta(\tau) \equiv \xi_\Delta$ meaning that all the components of σ_Δ, which are constant functions of τ, coincide with the corresponding components of the vector $\xi_\Delta \in \Omega_\Delta$. Given a function $f \in \mathcal{F}_\Delta^{(0)}$, we set

$$\Psi_f = \{F \in \mathbf{F}_{\beta,\Lambda} \mid \forall \sigma_\Delta \in \Omega^{c}_{\beta,\Delta} : \quad F(\sigma_\Delta) = f(\xi_\Delta)\}. \tag{3.27}$$

In other words, the above set consists of the functions which have on constant σ_Δ values coinciding with the corresponding values of this chosen function f.

Proposition 46: *For any finite Δ, for every $\beta > 0$, for any $f \in \mathcal{F}_\Delta^{(0)}$ and all $F \in \Psi_f$,*

$$\lim_{m\to+\infty} \int_{\Omega_{\beta,\Delta}} F(\sigma_\Delta) \mathrm{d}\nu_{\beta,\Delta}(\sigma_\Delta) = \int_{\Omega_\Delta} f(\xi_\Delta) \mathrm{d}\nu_\Delta(\xi_\Delta), \tag{3.28}$$

where the measure ν_Δ is defined by (2.11) with the same P, $h_{\mathbf{l}}$ and $J_{\mathbf{l}\mathbf{l}'}$ as in (3.24) - (3.26).

We recall that the Gibbs states of classical systems were introduced as solutions of the DLR equation (see Definition 21). In the quantum case the equilibrium states are defined by means of the *Kubo-Martin-Schwinger (KMS) conditions* (see the second volume of Ref. [2]). We are not going to pay here more attention to this condition and just remark that for the models with unbounded operators, like the one described by (3.4), this construction is impossible (see the discussion in Ref. [34]). The only possibility for such models, at least so far, is to construct Euclidean Gibbs states following the scheme:

$$\textsf{local Gibbs measures} \ \Rightarrow \ \textsf{DLR equation} \ \Rightarrow \ \textsf{Gibbs measure, its solution}$$

described in the preceding section. We refer to the article Ref. [34] where this scheme has been realized.

3.4. Phase transitions and critical points

Since the main place in our consideration of quantum models belongs to the model (3.4), we restrict ourselves to presenting here results on phase transitions and critical phenomena on the base of this model only. The corresponding results for a number of quantum spin models described by the Hamiltonian (3.1) may be found in the papers Refs. [65] and [112].

Thus, we consider the model described by the Hamiltonian (3.4) with the zero external field and an even polynomial P. All the methods employed to prove the existence of the long range order for the model (3.4) are based on the so called infrared bounds [113] (see also Ref. [112] for more details and appropriate modifications to the quantum case). As in the classical case the order parameter is defined by the following expression

$$P(\beta) = \lim_{n\to+\infty} \frac{1}{|\Lambda_n|^2} \sum_{\mathbf{l},\mathbf{l}'\in\Lambda_n} \langle q_{\mathbf{l}} q_{\mathbf{l}'} \rangle^{(p)}_{\beta,\Lambda}, \tag{3.29}$$

where, as in (2.71), $\{\Lambda_n\}_{n\in\mathbb{N}}$ is a sequence of boxes and the state $\langle\cdot\rangle^{(p)}_{\beta,\Lambda}$ is defined by (3.8), (3.9) with the Hamiltonian (3.4) in which the interaction potential has been modified to take into account the periodic boundary conditions, exactly as it was done in the classical case. Here one has to mention that the states $\langle\cdot\rangle_{\beta,\Lambda}$, and hence the periodic state $\langle\cdot\rangle^{(p)}_{\beta,\Lambda}$, were defined for bounded operators only, whereas the displacement operators $q_{\mathbf{l}}$ are unbounded. In general, this is a problem, which takes some efforts to be overcome, see Refs. [114], [115]. But in the case considered we may use the representation (3.20) (fortunately $q_{\mathbf{l}}$ is a multiplication operator), which yields

$$\langle q_{\mathbf{l}} q_{\mathbf{l}'} \rangle^{(p)}_{\beta,\Lambda} = \int_{\Omega_{\beta,\Lambda}} \sigma_{\mathbf{l}}(0)\sigma_{\mathbf{l}'}(0) \mathrm{d}\nu^{(p)}_{\beta,\Lambda}(\sigma_\Lambda), \tag{3.30}$$

where the Euclidean Gibbs measure $\nu^{(p)}_{\beta,\Lambda}$ is defined by (3.24) - (3.26) with $h_{\mathbf{l}} = 0$ and $J^{\Lambda}_{\mathbf{l}\mathbf{l}'}$ instead of $J_{\mathbf{l}\mathbf{l}'}$ (see (2.66)). We also suppose that $J^{\Lambda}_{\mathbf{l}\mathbf{l}'} \geq 0$ and the condition (2.15) is satisfied. The following statement was proven in Ref. [105], see also Refs. [61] and [107].

Proposition 47: *Let the polynomial P (2.8) in (3.26) be even, with $r \geq 2$ and possess two nondegenerate minima at some points $\pm t_0$ with $t_0 > 0$. Then for $d \geq 3$, there exists $m_* > 0$ such that for the particle mass $m > m_*$, there exists $\beta_C > 0$ such that: (a) for $\beta < \beta_C$, the order parameter (3.29) is zero; (b) for $\beta > \beta_C$, $P(\beta) > 0$.*

A particular case of this statement, where the polynomial P was as above but with $r = 2$ was proven in Refs. [65] and [116].

The only theorem describing a critical point of a model of this type was proven in Ref. [103], where a hierarchical version of the model (3.4) was considered. Its formal Hamiltonian may be written in the following form

$$\begin{aligned} H &= \frac{1}{2m}\sum_{\mathbf{l}\in\mathbb{Z}} p^2_{\mathbf{l}} + \sum_{\mathbf{l}\in\mathbb{Z}} \left[a q^2_{\mathbf{l}} + b q^4_{\mathbf{l}}\right] - \\ &- \frac{1}{2}\sum_{n=0}^{\infty}\sum_{\mathbf{l}\in\mathbb{Z}} 2^{-n(1+\lambda)} I(\mathbf{l},\mathbf{l}) \left[q(V^{(n)}_{\mathbf{l}})\right]^2, \end{aligned} \tag{3.31}$$

where $I(\mathbf{l},\mathbf{l})$ and $V^{(n)}_{\mathbf{l}}$ are the same as in (2.126), $a \in \mathbb{R}$, $b > 0$ and

$$q(V^{(n)}_{\mathbf{l}}) = \sum_{\mathbf{l}\in V^{(n)}_{\mathbf{l}}} q_{\mathbf{l}}.$$

The statement below is a corollary of the main theorem of Ref. [103].

Proposition 48: *For the model described by (3.31) with $\lambda \in (0, 1/2)$, there exist such values of the parameters m, a and b that the following holds. There exists $\beta_* > 0$ such that: (a) for $\beta = \beta_*$ (c.f., (2.74)),*

$$0 < \lim_{n\to+\infty} 2^{-n(1+\lambda)} \sum_{\mathbf{l},\mathbf{l}'\in V^{(n)}_{\mathbf{0}}} \langle q_{\mathbf{l}} q_{\mathbf{l}'} \rangle_{\beta, V^{(n)}_{\mathbf{0}}} < \infty; \tag{3.32}$$

(b) for $\beta < \beta_*$

$$2^{-n} \sum_{\mathbf{l},\mathbf{l}'\in V^{(n)}_{\mathbf{0}}} \langle q_{\mathbf{l}} q_{\mathbf{l}'} \rangle_{\beta, V^{(n)}_{\mathbf{0}}} \leq C < \infty; \tag{3.33}$$

for all $n \in \mathbb{N}$.

We remark here that (3.33) means that the static susceptibility $\chi_{V^{(n)}_{\mathbf{0}}}$ (c.f., (2.73) and the final part of subsection 2.5) remains bounded as $n \to +\infty$. Here for a finite subset Δ, we set

$$\chi_\Delta = \frac{1}{|\Delta|}\sum_{\mathbf{l},\mathbf{l}'\in\Delta} \langle q_{\mathbf{l}} q_{\mathbf{l}'} \rangle_{\beta,\Delta}. \tag{3.34}$$

As follows from Proposition 47, the long-range order appears when the particle mass is big enough, which corresponds to the quasi-classical limit (see Proposition 46). What can be said about the opposite limit $m \to 0$? In other words, which quantum effects one may expect in such models. This question was first studied in the paper Ref. [117], where it was shown that the long-range order does not appear in the small mass limit. A mathematically rigorous proof of this effect was done in Ref. [118]. Here we present a result, proven in Ref. [100], which shows that not only the long-range order, but any critical point anomaly, are suppressed if a certain condition involving the particle mass is satisfied.

The Hamiltonian (3.4) may be written in the form

$$H_\Delta = \sum_{\mathbf{l}\in\Delta} H_{\mathbf{l}} - \frac{1}{2}\sum_{\mathbf{l},\mathbf{l}'\in\Delta} J_{\mathbf{l}\mathbf{l}'} q_{\mathbf{l}} q_{\mathbf{l}'}, \tag{3.35}$$

where the one-particle Hamiltonian is

$$H_{\mathbf{l}} = \frac{1}{2m} p^2_{\mathbf{l}} + P(q_{\mathbf{l}}). \tag{3.36}$$

It is well-known that its spectrum consists of non-degenerate eigenvalues λ_n, $n \in \mathbb{N}_0$. Set

$$\delta = \min_{n\in\mathbb{N}} (\lambda_n - \lambda_{n-1}), \tag{3.37}$$

which is the minimal distance between the one-particle energy levels. For the quantum harmonic oscillator described by (3.36) with $P(q_{\mathbf{l}}) = (b/2)q^2_{\mathbf{l}}$, where $b > 0$ is its *rigidity*, one has

$$\delta_{\mathrm{h}} = \hbar\sqrt{b/m}. \tag{3.38}$$

The following statement is a corollary of the main theorem proven in Ref. [100].

Proposition 49: *For the model described by the Hamiltonian (3.4), (3.35) let the following condition be satisfied*

$$(m\delta^2/\hbar^2) > \|J\| \stackrel{\mathrm{def}}{=} \sup_{\mathbf{l}\in\mathbb{Z}^d} \sum_{\mathbf{l}'\in\mathbb{Z}^d} J_{\mathbf{l}\mathbf{l}'}. \tag{3.39}$$

Then for every $\beta > 0$ and for any increasing sequence $\mathcal{D}$ of subsets which exhausts the lattice $\mathbb{Z}^d$, the sequence of static susceptibilities $\{\chi_\Delta\}_{\Delta\in\mathcal{D}}$ defined by (3.34) remains bounded, i.e., no critical point anomalies are possible at all temperatures.

As it was shown in Ref. [100], if in (3.35) P is an even polynomial of degree $2r \geq 4$, then $m\delta^2$ is a continuous function of m such that $m\delta^2 \sim Cm^{-(r-1)/(r+1)}$ as $m \to 0$, which means that there should exist a constant m_*, depending on $\|J\|$ and on the coefficients of the polynomial P, such that the condition (3.39) is satisfied for $m < m_*$. This yields the following corollary of the above statement.

Corollary 50: *For the model described by the Hamiltonian (3.4), (3.35), there exists a constant $m_* > 0$, which depends solely on the coefficients of the polynomial P and on the interaction parameter $\|J\|$ and is independent of β, such that for $m < m_*$, no critical point anomalies, and all the more no long-range order, are possible at all temperatures.*

The extension of the above results to the case of vector quantum oscillators was given in the papers Refs. [108], [109].

Let us analyze these statements. By (3.38), for the harmonic oscillators, one has $m\delta_{\mathrm{h}}^2 = \hbar^2 b$. Then the condition (3.39) gets the form

$$b > \|J\|,$$

which is nothing else but the stability condition (2.16). Then for the anharmonic oscillators, the parameter $m\delta^2$ may be considered as a measure of its quantum rigidity and the effect described by the above statements may be called *a quantum stabilization* of the system of quantum anharmonic oscillators described by (3.4), (3.35). Stronger statements of this kind establishing uniqueness of Euclidean Gibbs states for this system, were proven in Refs. [73], [74], [102].

Acknowledgments

The author is grateful to his teachers Igor Stasyuk and Igor Yukhnovskii who introduced him into the subject of this article. He thanks Sergio Albeverio, Yuri Kondratiev, Agnieszka Kozak, Mykhailo Kozlovskii, Taras Krokhmalskii, Mykola Melnyk, Michael Röckner and Lech Wołowski for collaboration – certain parts of the results described in this article were obtained and/or discussed together. He also thanks Yuri Holovatch and Igor Stasyuk for organizing in March 2002 in ICMP, Lviv the Ising Lectures on which the main part of the above material was presented. Helpful comments and suggestions of Yuri Holovatch substantially improved the presentation of the article. The author is also grateful for the hospitality of the Research Centre BiBoS, Bielefeld University where this article was brought into its final form. His research was supported in part by the Deutscheforschungsgemeinschaft through the German-Polish project 436 POL 113/98/0-1 "Probability Measures" that is gratefully acknowledged.

References

1. S. Albeverio and R. Høegh-Krohn, *Mathematical Theory of Feynman Path Integrals.* (Lect. Notes in Math. 523, Springer-Verlag, Berlin, 1976).
2. O. Bratteli and D.W. Robinson, *Operators Algebras and Quantum Statistical Mechanics, I.* (Springer-Verlag, New York Heidelberg, 1979); *Operators Algebras and Quantum Statistical Mechanics, II.* (Springer-Verlag, New York Berlin, 1981).
3. R. Fernández, J. Fröhlich and A.D. Sokal, *Random Walks, Critical Phenomena and Triviality in Quantum Field Theory.* (Springer-Verlag, Berlin, 1992).
4. H.-O. Georgii, *Gibbs Measures and Phase Transitions.* (Walter de Gruyter, Berlin New York, 1988).
5. R.B. Israel, *Convexity in the Theory of Lattice Gases.* (Princeton Series in Physics, Princeton University Press, Princeton, N.J., 1979).
6. V.A. Malyshev and R.A. Minlos *Gibbsovskie sluchaĭnye polya. Metod klasternykh razlozheniĭ.* ("Nauka", Moscow, 1985); translation: *Gibbs random fields. Cluster expansions.* (Mathematics and its Applications (Soviet Series), 44. Kluwer Academic Publishers Group, Dordrecht, 1991).
7. V.A. Malyshev and R.A. Minlos *Linear infinite-particle operators.* (Translations of Mathematical Monographs, 143. American Mathematical Society, Providence, RI, 1995).
8. R. A. Minlos, *Introduction to Mathematical Statistical Physics.* (University Lecture Series, 19, Amer. Math. Soc. Providence, RI, 2000).

9. G. Roepstorff, *Path Integral Approach to Quantum Physics. An Introduction.* (Springer-Verlag, Berlin Heidelberg, 1994).
10. D. Ruelle, *Statistical Mechanics: Rigorous Results.* (W. A. Benjamin, Inc., New York-Amsterdam, 1969).
11. B. Simon, *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory.* (Princeton University Press, Princeton, N.J., 1974).
12. B. Simon, *Functional Integration and Quantum Physics.* (Academic Press, New York London, 1979).
13. B. Simon, *The Statistical Mechanics of Lattice Gases. I.* (Princeton University Press, Princeton, N.J., 1993).
14. Ya. G. Sinai, *Theory of Phase Transitions: Rigorous Results.* (Pergamon Press, Oxford-Elmsford, N.Y., 1982).
15. M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis.* (Academic Press, New York London, 1972).
16. L. Iliev, *Laguerre Entire Functions.*(Second edition, Publishing House of the Bulgarian Academy of Sciences, Sofia, 1987).
17. B. Ya. Levin, *Lectures on Entire Function.* (American Mathematical Society, Providence, RI, 1996).
18. P. Billingsley, *Probability and Measure.*(Third edition. Wiley Series in Probability and Mathematical Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995).
19. K.R. Parthasarathy, *Probability Measures on Metric Spaces.* (Academic Press, New York, 1967).
20. A.V. Skorohod, *Intergration in Hilbert Space.* (Springer-Verlag, Berlin, 1974).
21. Vivek S. Borkar, *Probability Theory. An Advanced Course.* (Springer-Verlag, New York, 1995).
22. E. Ising, *Z. Phys.* **31**, 253 (1925).
23. S.G. Brush, *Rev. Mod. Phys.* **39**, 883 (1967).
24. S. Kobe, *J. Statist. Phys.* **88**, 991 (1997).
25. L. Onsager, *Phys. Rev.* **65**, 117 (1944).
26. D. Gandolfo, Private communication, 2002.
27. A.K. Hartmann and H. Rieger, *Optimization Algorythms in Physics.* (Wiley-VCH Verlag Berlin GmbH, Berlin, 2002).
28. http://www.physik.uni-bielefeld.de/complexity/
29. B. Simon and R. Griffiths, *Comm. Math. Phys.* **33** , 145 (1973).
30. P.M. Bleher and N.N. Ganikhodzhaev, *Teor. Veroyatnost. i Primenen.* **35**, 220 (1990); translation in *Theory Probab. Appl.* **35**, 216 (1990).
31. N.N. Ganikhodzhaev, *Teoret. Mat. Fiz.* **130**, 493 (2002); translation in *Theoret. and Math. Phys.* **130**, 419 (2002).

32. S. Stamenković, *Condens. Matter Phys.* **1(14)** 257 (1998).
33. S. Albeverio and R. Høegh-Krohn, *J. Func. Anal.* **19**, 242 (1975).
34. S. Albeverio, Yu. Kondratiev, Yu. Kozitsky, and M. Röckner, *Rev. Math. Phys.* **14**, 1335 (2002).
35. J. Abdullaev and R.A. Minlos, in *Probability Contributions to Statistical Mechanics.*(Adv. Soviet Math., 20, Amer. Math. Soc., Providence, RI, 1994), p 1.
36. T.D. Lee and C.N. Yang, *Phys. Rev.* **87**, 410 (1952).
37. E.H. Lieb and A.D. Sokal, *Comm. Math. Phys.* **80** , 153 (1981).
38. C.M. Newman, *Comm. Pure Appl. Math.* **27**, 143 (1974).
39. C.M. Newman, *J. Statist. Phys.* **15**, 399 (1976).
40. Yu. Kozitsky and N.O. Melnik, *Teoret. Mat. Fiz.* **78**, 177 (1989); translation in *Theoret. and Math. Phys.* **78**, 127 (1989).
41. Yu. Kozitsky, *Teoret. Mat. Fiz.* **58**, 96 (1984).
42. Yu. Kozitsky, *Rep. Math. Phys.* **26**, 429 (1988).
43. C. Fortuin, P. Kastelyn and J. Ginibre, *Comm. Math. Phys.* **22**, 89 (1971).
44. R.B. Griffiths, *J. Math. Phys.* **10**, 1559 (1969).
45. G. Sylvester, *J. Statist. Phys.* **15**, 327 (1976).
46. R.S. Ellis and J.S Rosen, *Ann. Probab.* **10** , 47 (1982).
47. M. Aizenman, *Comm. Math. Phys.* **86**, 1 (1982).
48. S.B. Shlosman, *Comm. Math. Phys.* **102**, 679 (1986).
49. C.M. Newman, *Comm. Math. Phys.* **74**, 119 (1980).
50. N.N. Bogolyubov, in *Problems in the Theory and History of Differential Equations,* (Naukova Dumka, Kiev, 1968), p 13.
51. R.L. Dobrushin, *Teor. Veroyatnost. i Primenen.* **13**, 201 (1968); translation in *Theor. Probab. Appl.* **13**, 179 (1968).
52. R.L. Dobrushin, *Funcional. Anal. i Priložen.* **3**, 27 (1969).
53. O.E. Lanford and D. Ruelle, *Comm. Math. Phys.* **13**, 194 (1969).
54. S. Albeverio, Yu. Kondratiev, T. Pasurek and M. Röckner, *A Priori Estimates and Existence for Euclidean Gibbs Measures.* (Preprint BiBoS Nr 02-06-089, Bielefeld, 2002).
55. R.L. Dobrushin, *Teor. Veroyatnost. i Primenen.* **18**, 261 (1973); translation in *Theor. Probab. Appl.* **18**, 253 (1973).
56. R.L. Dobrushin, *Teor. Veroyatnost. i Primenen.* **17**, 619 (1972); translation in *Theor. Probab. Appl.* **17**, 619 (1972).
57. M. Aizenman, *Comm. Math. Phys.* **73**, 83 (1980).
58. M. Aizenman, D.J. Barsky and R. Fernández, *J. Statist. Phys.* **47**, 343 (1987).
59. D. Ruelle, *Comm. Math. Phys.* **50**, 189 (1976).
60. J.L. Lebowitz and E. Presutti, *Comm. Math. Phys.* **50**, 195 (1976).

Erratum: *Comm. Math. Phys.* **78**, 151 (1980).

61. V.S. Barbulyak and Yu. G. Kondratiev, *Dokl. Akad. Nauk Ukrain. SSR* **8**, 31 (1991).
62. M. Aizenman, in *Statistical Physics and Dynamical Systems (Kőszeg, 1984)* (Progr. Phys., 10, Birkhäuser Boston, Boston, MA, 1985), p 453.
63. M. Aizenman and R. Fernández, *J. Statist. Phys.* **44**, 393 (1986).
64. P.M. Bleher and P. Major, *Ann. Probab.* **15**, 153 (1987).
65. W. Driessler, L. Landau and J.F. Perez, J. Statist. Phys. **20**, 123 (1979).
66. Ya. G. Sinai, in *Mathematical problems in theoretical physics (Proc. Internat. Conf., Univ. Rome. Rome, 1977)* (Lecture Notes in Phys., 80, Springer, Berlin New York, 1978), p 303.
67. R. Peierls, *Proc. Cambridge Phil. Soc.* **32** , 477 (1936).
68. S.A. Pirogov and Ja. G. Sinai, *Teoret. Mat. Fiz.* **25**, 358 (1975); *ibid.* **26**, 61 (1976).
69. M. Zahradník, *Rend. Math. Appl. (7)* **18**, 411 (1998).
70. R.L. Dobrushin and S.B. Shlosman, *Selecta Math. Sov.* **1**, 317 (1981).
71. D. Iagolnitzer and B. Souillard, *Phys. Rev.,* **B19**, 1515 (1979).
72. J.L. Lebowitz and A. Martin-Löf, *Comm. Math. Phys.* **25**, 276 (1972).
73. S. Albeverio, Yu. Kondratiev, Yu. Kozitsky and M. Röckner, *C.R. Acad. Sci. Paris, Ser I* **335**, 693 (2002).
74. S. Albeverio, Yu. Kondratiev, Yu. Kozitsky and M. Röckner, *Small Mass Implies Uniqueness of Gibbs States of a Quantum Crystal.* (Preprint BiBoS Nr 02-12-104, Bielefeld, 2002, subm. in *Comm. Math. Phys.*).
75. F.J. Dyson, *Comm. Math. Phys.* **12**, 91 (1969).
76. F.J. Dyson, in *Mathematical aspects of statistical mechanics (Proc. Sympos. Appl. Math., New York, 1971)* (SIAM-AMS Proceedings, Vol. V, Amer. Math. Soc., Providence, R. I., 1972), p 1.
77. M. Aizenman, J.T. Chayes, L. Chayes and C.M. Newman, *J. Statist. Phys.* **50**, 1 (1988).
78. L.P. Kadanoff, *Physics* **2**, 263 (1966).
79. L.P. Kadanoff, in *Phase transitions and critical phenomena, Vol. 5a* (Academic Press, London, 1976), p 1.
80. L.P. Kadanoff, *Statistical Physics. Statics, Dynamics and Renormalization.* (World Scientific Publishing Co., Inc., River Edge, NJ, 2000).
81. Ya. G. Sinai, *Teor. Verojatnost. i Primenen.* **21**, 63 (1976).

82. K. Gawędzki, in *Mathematics + physics. Vol. 1,* (World Sci. Publishing, Singapore, 1985), p 99.
83. K. Gawędzki and A. Kupiainen, *Comm. Math. Phys.* **89**, 191 (1983).
84. K. Gawędzki and A. Kupiainen, *Comm. Math. Phys.* **99**, 197 (1985).
85. K.G. Wilson, *Rev. Modern Phys.* **55**, 583 (1983).
86. I.R. Yukhnovskii, *Phase Transitions of the Second Order.* (World Scientific Publishing Co., Singapure, 1987).
87. I.R. Yukhnovskii, *Riv. Nuovo Cimento (3)* **12** , 1 (1989).
88. M. Salmhofer, *Renormalization. An Introduction.* (Springer-Verlag, Berlin Heidelberg 1999).
89. Yu. Kozitsky and I.R. Yukhnovskii, *Teoret. Mat. Fiz.* **51**, 268 (1982).
90. P.M. Bleher and Ya. G. Sinai, *Comm. Math. Phys.* **33**, 23 (1973).
91. P.M. Bleher and Ya. G. Sinai, *Comm. Math. Phys.* **45**, 247 (1975).
92. Yu. Kozitsky, M. Kozlovskii and T. Krokhmalskii, *Condens. Matter Phys.* **2**, 15 (1999).
93. Yu. Kozitsky, *Condens. Matter Phys.* **5**, 74 (1995).
94. E. Olivieri, P. Picco and Yu. M. Suhov, *J. Statist. Phys.* **70**, 985 (1993).
95. P.M. Bleher, *Comm. Math. Phys.* **84**, 557 (1982).
96. I.V. Stasyuk, *Condens. Matter Phys.* **2(19)** 435 (1999).
97. R. Høegh-Krohn, *Comm. Math. Phys.* **38**, 195 (1974).
98. A. Klein and L.D. Landau, *J. Func. Anal.* **42**, 368 (1981).
99. A. Klein and L.D. Landau, *Pacif. Journ. Math.* **94**, 341 (1981).
100. S. Albeverio, Yu. Kondratiev and Yu. Kozitsky, *Comm. Math. Phys.* **194**, 493 (1998).
101. S. Albeverio, Yu. Kondratiev and Yu. Kozitsky, *Lett. Math. Phys.* **48**, 221 (1999).
102. S. Albeverio, Yu. Kondratiev, Yu. Kozitsky and M. Röckner, *Ann. Inst. H. Poincaré, Probab. Statist.* **37**, 43 (2001).
103. S. Albeverio, Yu. Kondratiev, A. Kozak and Yu. Kozitsky, *A System of Quantum Anharmonic Oscillators with Hierarchical Structure: Critical Point Convergence.* (Preprint BiBoS Universitaet Bielefeld, Nr 02-07-092, 2002, submitted in *Comm. Math. Phys.*)
104. V.S. Barbulyak and Yu. G. Kondratiev, *Dokl. Akad. Nauk Ukrain. SSR* **9**, 38 (1991).
105. V.S. Barbulyak and Yu. G. Kondratiev, *Funktsional. Anal. i Prilozhen.* **26**, 61 (1992); translation in *Funct. Anal. Appl.* **26**, 124 (1992).

106. S.A. Globa and Yu. G. Kondratiev, *Selecta Math. Sov.* **9**, 297 (1990).
107. Ju. G. Kondratiev, in *Stochastic Processes, Physics and Geometry II.* (World Scientific, Singapure, 1994), p 465.
108. Yu. Kozitsky, *Lett. Math. Phys.* **51**, 71 (2000).
109. Yu. Kozitsky, *Lett. Math. Phys.* **53**, 289 (2000).
110. Yu. Kozitsky, in *Stochastic Processes, Physics and Geometry: New Interplays, II (Leipzig, 1999),* (Canad. Math. Soc. Conf. Proc., 29, Amer. Math. Soc., Providence, RI, 2000), p 403.
111. Yu. Kozitsky, in *Noncommutative Structures in Mathematics and Physics (Kiev, 2000, S. Duplij and J. Wess eds.)* (NATO Sci Ser. II Math. Phys. Chem. 22. Kluwer Acad. Publ. Dordrecht, 2001), p 415.
112. F.J. Dyson, E.H. Lieb and B. Simon, *J. Statist. Phys.* **18**, 335 (1978).
113. J. Fröhlich, B. Simon and T. Spencer, *Comm. Math. Phys.* **50**, 79 (1976).
114. G.L. Sewell, *J. Math. Phys.* **11**, 1868 (1970).
115. G.L. Sewell, *Quantum Theory of Collective Phenomena.* (Clarendon Press, Oxford, 1986).
116. L.A. Pastur, B. A. Khoruzhenko, *Teoret. Mat. Fiz.* **73**, 111 (1987).
117. T. Schneider, H. Beck and E. Stoll, *Phys. Rev.* **B13**, 1123 (1976).
118. A. Verbeure and V.A. Zagrebnov, *J. Phys.A.: Math. Gen.* **28**, 5415 (1995).

Препринти Інституту фізики конденсованих систем НАН України розповсюджуються серед наукових та інформаційних установ. Вони також доступні по електронній комп'ютерній мережі на WWW-сервері інституту за адресою http://www.icmp.lviv.ua/

The preprints of the Institute for Condensed Matter Physics of the National Academy of Sciences of Ukraine are distributed to scientific and informational institutions. They also are available by computer network from Institute's WWW server (http://www.icmp.lviv.ua/)

Юрій Козицький

Математична теорія моделі Ізінга та її узагальнення: вступ

Роботу отримано 23 квітня 2003 р.

Затверджено до друку Вченою радою ІФКС НАН України

Рекомендовано до друку Вченою радою ІФКС НАН України

Виготовлено при ІФКС НАН України