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Ланцюг кінетичних рівнянь для функцій розподілу
частиночок в простій рідині з врахуванням
нелінійних гідродинамічних флуктуацій

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1. Introduction

The study of nonlinear kinetic and hydrodynamic fluctuations in dense gases, liquids and plasma, in turbulence phenomena and dynamics of phase transitions, in chemical reactions and self-organizing processes are relevant both on kinetic and hydrodynamic levels of description in statistical theory of non-equilibrium processes [1–18]. The non-equilibrium states of such systems are far from equilibrium. Therefore the study of both the processes establishing the stationary states with characteristic times of life and the relaxation processes to the known equilibrium states, that are described by means of molecular hydrodynamics [19–23], is of great importance. An important feature of theoretical modeling of non-equilibrium phenomena in dense gases, liquids, dense plasmas (dusty plasmas) is a consistent description of kinetic and hydrodynamic processes [23–27] and taking into account the characteristic short and long-range interactions between the particles of the systems. In particular, the non-equilibrium gas-liquid phase transition is characterized by nonlinear hydrodynamic fluctuations of mass, momentum and particle energy, which describe a collective nature of the process and define the spatial and temporal behavior of the transport coefficients (viscosity, thermal conductivity), time correlation functions and dynamic structure factor. At the same time, due to heterogeneity in collective dynamics of these fluctuations, liquid drops emerge in the gas phase (in case of transition from the gas phase to the liquid phase), or the gas bubbles emerge in the liquid phase (in case of transition from the liquid phase to the gas phase), formation of which has a kinetic nature described by a redistribution of momentum and energy, i.e. when a certain group of particles in the system receives a significant decrease (in the case of drops), or increase (in the case of bubbles) of kinetic energy. The particles, that form bubbles or droplets, diffuse out of their phases in the liquid or the gas and vice versa. They have different values of momentum, energy and pressure in different phases. All these features are related to the non-equilibrium unary, binary and \( s \)-particle distribution functions (which depend on coordinate, momentum and time) that satisfy the BBGKY chain of equations. Therefore, the construction of kinetic equations that take into account nonlinear hydrodynamic fluctuations [25–32] is an important problem in the theory of transport processes in dense gases and liquids. In particular, this problem arises in the description of low-frequency anomalies in the kinetic equations and related "long tail" correlation functions [33–35].

The main difficulty of the problem is that the kinetics and hydrody-
For a consistent description of kinetic and hydrodynamic fluctuations in a classical one-component fluid it is necessary to select the description parameters for one-particle and collective processes. As these parameters we choose the non-equilibrium one-particle distribution function \( f(x,t) = \langle n_1(x)^t \rangle \) and distribution function of hydrodynamic variables \( f(a,t) = \langle \delta(\hat{a} - a)^t \rangle \). Here the phase function
\[
\hat{a}_1(x) = \sum_{j=1}^{N} \delta(x - x_j) = \sum_{j=1}^{N} \delta(r - r_j)\delta(p - p_j)
\]
is the microscopic particle number density. \( x_j = (r_j, p_j) \) is the set of phase variables (coordinates and momentums), \( N \) is the total number of particles in a volume \( V \). A microscopic phase distribution of hydrodynamic variables is given by
\[
\hat{f}(a) = \delta(\hat{a} - a) = \prod_{m=1}^{3} \prod_{k} \delta(\hat{a}_{mk} - a_{mk}),
\]
where \( \hat{a}_{1k} = \hat{n}_k, \hat{a}_{2k} = \hat{J}_k, \hat{a}_{3k} = \hat{\varepsilon}_k \) are the Fourier components of the densities of particle number, momentum and energy:
\[
\hat{n}_k = \sum_{j=1}^{N} e^{-ikr_j}, \quad \hat{J}_k = \sum_{j=1}^{N} p_j e^{-ikr_j}, \\
\hat{\varepsilon}_k = \sum_{j=1}^{N} \left[ \frac{p_j^2}{2m} + \frac{1}{2} \sum_{l \neq j=1}^{N} \Phi(|r_{lj}|) \right] e^{-ikr_j},
\]
and \( a_{mk} = (n_k, J_k, \varepsilon_k) \) are the corresponding collective variables, \( \Phi(|r_{lj}|) = \Phi(|r_l - r_j|) \) is the pair interaction potential between particles. The average values \( \langle n_1(x)^t \rangle \) and \( \langle \delta(\hat{a} - a)^t \rangle \) are calculated by means of the non-equilibrium \( N \)-particle distribution function \( \varrho(x^N; t) \), that satisfies the Liouville equation. In line with the idea of reduced description of non-equilibrium states this function is the functional
\[
\varrho(x^N; t) = \varrho(\ldots, f_1(x; t), f(a; t), \ldots).
\]
In order to find a non-equilibrium distribution function \( \varrho(x^N; t) \) we use Zubarev’s method [39, 41], in which a general solution of Liouville
equation taking into account a projection procedure can be presented in the form:

\[ g(t^N; t) = g_0(x^N; t) - \int_{-\infty}^t dt' e^{(t-t')} T_q(t, t')(1 - P_q(t'))iL_N g_0(x^N; t'), \]

(2.5)

where \( \epsilon \rightarrow +0 \) after thermodynamic limiting transition. The source selects the retarded solutions of Liouville equation with operator \( iL_N \). \( T_q(t, t') = \exp(-\int_{t'}^t dt'(1 - P_q(t'))iL_N) \) is the generalized time evolution operator taking into account Kawasaki-Gunton projection \( P_q(t') \).

The structure of \( P_q(t') \) depends on the quasi-equilibrium distribution function \( g_q(x^N; t) \), which in method by Zubarev is determined from extremum of the information entropy at simultaneous conservation of normalization condition

\[ \int d\Gamma_N g_q(x^N; t) = 1, \quad d\Gamma_N = \frac{(dx)^N}{N!} = \frac{(dx_1, \ldots, dx_N)}{N!}, \quad dx = d\mathbf{r}d\mathbf{p}, \]

(2.6)

and the fact that the parameters of the reduced description, \( f_1(x; t) \) and \( f(a; t) \) are fixed. Then quasi-equilibrium distribution function can be written as

\[ g_q(x^N; t) = \exp \{ -\Phi(t) - \int dx_1 \gamma(x; t)n_1(x) - \int daF(a; t)f(a) \}, \]

(2.7)

where \( da = \{dn_k, dJ_k, d\varepsilon_k\} \). The Massieu-Planck functional \( \Phi(t) \) is determined from the normalization condition for the quasi-equilibrium distribution function

\[ \Phi(t) = \ln \int d\Gamma_N \exp \{ -\int dx_1 \gamma(x; t)n_1(x) - \int daF(a; t)f(a) \}. \]

The functions \( \gamma(x; t) \) and \( F(a; t) \) are the Lagrange multipliers and are determined from the consistency conditions:

\[ f_1(x; t) = \langle n_1(x) \rangle^t, \]

\[ f(a; t) = \langle \delta(\bar{a} - a) \rangle^t, \]

\[ \gamma(x; t) = \langle \delta(\bar{a} - a) \rangle^t f(a; t) = \langle \delta(\bar{a} - a) \rangle^t, \]

(2.8)

where \( \langle \ldots \rangle^t = \int d\Gamma_N \ldots g(x^N; t) \) and \( \langle \ldots \rangle_q = \int d\Gamma_N \ldots g_q(x^N; t) \). To find the explicit form of non-equilibrium distribution function \( g(x^N; t) \) we exclude the factor \( F(a; t) \) in quasi-equilibrium distribution function and thereafter, by means of consistency conditions (2.8), we have

\[ g_q(x^N; t) = g_q^{kin}(x^N; t) \frac{f(a; t)}{W(a; t)} |_{a = \bar{a}}. \]

(2.9)

Here

\[ W(a; t) = \int d\Gamma_N e^{-\Phi^{kin}(t) - \int dx_1 \gamma(x; t)n_1(x)} f(a) \]

(2.10)

\[ = \int d\Gamma_N g_q^{kin}(x^N; t)f(a) \]

is the structure distribution function of hydrodynamic variables, which could be also considered as a Jacobian for transition from \( f(a) \) into space of collective variables \( n_k, J_k, \varepsilon_k \) averaged with the "kinetic" quasi-equilibrium distribution function

\[ g_q^{kin}(x^N; t) = \exp \{ -\Phi^{kin}(t) - \int dx_1 \gamma(x; t)n_1(x) \}, \]

(2.11)

\[ \Phi^{kin}(t) = \ln \int d\Gamma_N \exp \{ -\int dx_1 \gamma(x; t)n_1(x) \}. \]

Here the entropy

\[ S(t) = -(\ln g_q(x^N; t))^q \]

(2.12)

corresponds to the quasi-equilibrium distribution (2.9). In combination with the self-consistency conditions (2.8), it can be considered as entropy of non-equilibrium state. In accordance with (2.5), in order to obtain the explicit form of non-equilibrium distribution function, it is necessary to disclose the action of Liouville operator on \( g_q(x^N; t) \) and action of the Kawasaki-Gunton projection operator, which in our case has the following structure according to (2.9):

\[ P_q(t)g' = g_q(x^N; t) \int d\Gamma_N g' + \int dx \frac{\partial g_q(x^N; t)}{\partial n_1(x)} e^c \times \left( \int d\Gamma_N n_1(x)g' - \langle n_1(x) \rangle \int d\Gamma_N g' \right) \]

(2.13)

\[ + \int da \frac{\partial g_q(x^N; t)}{\partial F(a; t)} \frac{1}{W(a; t)} \left( \int d\Gamma_N f(a)g' - f(a; t) \int d\Gamma_N g' \right) \]

\[ + \int dx \int da \frac{\partial g_q(x^N; t)}{\partial F(a; t)} \frac{1}{W(a; t)} \left( \int d\Gamma_N f(a)g' - f(a; t) \int d\Gamma_N g' \right) \times \left( \int d\Gamma_N n_1(x)g' - \langle n_1(x) \rangle \int d\Gamma_N g' \right). \]
Next, we consider the action of Liouville operator on quasi-equilibrium distribution function (2.14):

\[ iL_N \varrho_q(x^N; t) = -\int dx \gamma(x; t) \dot{\hat{n}}_1(x) \varrho_q(x^N; t) \] 

(2.14)

where \( \dot{\hat{n}}_1(x) = iL_N \hat{n}_1(x) \). Having used thereafter the relation

\[ iL_N \hat{f}(a) = iL_N \hat{f}(n_k, J_k, \xi_k) = \sum_k \left[ \frac{\partial}{\partial n_{il}} \hat{f}(a) n_{il} + \frac{\partial}{\partial J_k} \hat{f}(a) J_k + \frac{\partial}{\partial \xi_k} \hat{f}(a) \xi_k \right] \]

where \( \hat{n}_k = iL_N \hat{n}_k, \quad \hat{J}_k = iL_N \hat{J}_k, \quad \hat{\xi}_k = iL_N \hat{\xi}_k \), the last expression in (2.14) can be rewritten in following form:

\[
\left[ iL_N \frac{f(a; t)}{W(a; t)} \right]_{a = \hat{a}} \varrho_{q}^{\text{kin}}(x^N; t) = \int da \sum_k W(a; t) \left[ \frac{\partial f(a; t)}{\partial n_{il}} \right]_{a = \hat{a}} \varrho_{q}^{\text{kin}}(x^N; t)
+
\hat{J}_k \frac{\partial f(a; t)}{\partial J_k} W(a; t) + \hat{\xi}_k \frac{\partial f(a; t)}{\partial \xi_k} W(a; t) \] 

(2.15)

Here we introduced new quasi-equilibrium distribution function \( \varrho_L(x^N, a; t) \) with the microscopic distribution of large-scale collective variables

\[ \varrho_L(x^N, a; t) = \varrho_{q}^{\text{kin}}(x^N; t) \frac{\hat{f}(a; t)}{W(a; t)}. \] 

(2.16)

This quasi-equilibrium distribution function is connected with \( \varrho_q(x^N; t) \) by the relation

\[ \varrho_q(x^N; t) = \int df(a; t) \varrho_L(x^N, a; t) \] 

(2.17)

and is obviously normalized to unity

\[ \int d\Gamma_N \varrho_L(x^N, a; t) = 1. \] 

(2.18)

Using then the relation (2.16), the average values with quasi-equilibrium distribution is convenient to represent in following form:

\[ \langle \ldots \rangle_q = \int da \langle \ldots \rangle_L f(a; t), \quad \langle \ldots \rangle_L = \int d\Gamma_N \ldots \varrho_L(x^N, a; t). \] 

(2.19)

Now in accordance with (2.15) and (2.14) we can rewrite the action of the Liouville operator on \( \varrho_q(x^N; t) \) as follows

\[
iL_N \varrho_q(x^N; t) = -\int da \int dx \gamma(x; t) \dot{\hat{n}}_1(x) \varrho_q(x^N, a; t) f(a; t) \] 

(2.20)

\[
+ \int da \sum_k W(a; t) \left[ \frac{\partial f(a; t)}{\partial n_{il}} \varrho_L(x^N, a; t) + \hat{J}_k \frac{\partial f(a; t)}{\partial J_k} \right] + \hat{\xi}_k \frac{\partial f(a; t)}{\partial \xi_k} \varrho_L(x^N, a; t). \]

Substituting this expression into (2.15), one obtains for non-equilibrium distribution function the following result:

\[ \varrho(x^N; t) = \int df(a; t) \varrho_L(x^N, a; t) \] 

(2.21)

\[
+ \int da \int dx \int_{-\infty}^{t} dt' e^{t'(t-t')} T_q(t', t) \left( 1 - P_q(t') \right) \] 

\[
\times \dot{\hat{n}}_1(x) \varrho_L(x^N, a; t') f(a; t') \gamma(x; t') \] 

\[
- \int da \sum_k \int_{-\infty}^{t} dt' e^{t'(t-t')} T_q(t', t) \left( 1 - P_q(t') \right) W(a; t') \left[ \frac{\partial f(a; t')}{\partial n_{il}} \right]_{a = \hat{a}} \varrho_{q}^{\text{kin}}(x^N; t)
+
\hat{J}_k \frac{\partial f(a; t')}{\partial J_k} W(a; t') + \hat{\xi}_k \frac{\partial f(a; t')}{\partial \xi_k} W(a; t') \] 

\[ \varrho_L(x^N, a; t'). \]

and the corresponding generalized transport equations:

\[
\left[ \frac{\partial}{\partial t} + \frac{\mathbf{P}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} \right] f_1(x; t) - \int dx' \frac{\partial}{\partial \mathbf{r}} \Phi(\mathbf{r} - \mathbf{r}') \left[ \frac{\partial}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}'} \right] g_2(x, x'; t) \] 

\[
= \int dx' \int da \int_{-\infty}^{t} dt' e^{t'(t-t')} \phi_m(x, x', a; t, t') f(a; t') \gamma(x'; t') \] 

\[
- \sum_k \int da \int_{-\infty}^{t} dt' e^{t'(t-t')} \left\{ \phi_m(x, k, a; t, t') \cdot \frac{\partial}{\partial J_k} \right\} \] 

(2.22)

\[
+ \phi_{ac}(x, k, a; t, t') \frac{\partial f(a; t')}{\partial J_k} W(a; t'). \]
\[
\dot{I}_a(t) = (1 - P(t)) \hat{J}_k, \tag{2.27}
\]
\[
\dot{L}_a(t) = (1 - P(t)) \hat{J}_k. \tag{2.28}
\]

Here \(P(t)\) is the generalized Mori operator related to Kawasaki-Gunton projection operator \(P_q(t)\) by following relation
\[
P_q(t)a(x)\delta_q(x^N,t) = \delta_q(x^N,t)P(t)a(x).
\]

It should be emphasized that in equations (2.22), the averages are calculated with distribution function \(g_L(x^N,a;t)\), so that the transport kernels are some functions of collective variables \(a_k\). In equation (2.23), the functions (called hydrodynamic velocities) \(v_{n,k}(a;t), v_{j,k}(a;t), v_{c,k}(a;t)\) represent the fluxes in the space of collective variables and are defined as:

\[
v_{n,k}(a;t) = \int d\Gamma_N \dot{n}_k g_L(x^N,a;t) = \langle \dot{n}_k \rangle_L,
\]
\[
v_{j,k}(a;t) = \int d\Gamma_N \dot{J}_k g_L(x^N,a;t) = \langle \dot{J}_k \rangle_L,
\]
\[
v_{c,k}(a;t) = \int d\Gamma_N \dot{\epsilon}_k g_L(x^N,a;t) = \langle \dot{\epsilon}_k \rangle_L.
\]

The presented system of transport equations gives consistent description of kinetic and hydrodynamic processes of classical fluids which take into account long-living fluctuations.

The system of transport equations (2.22), (2.23) is not closed due to the presence of the averages. To Lagrange parameter \(\gamma(x;t)\), which is determined from the corresponding self-consistent conditions. From the kinetic processes standpoint, we must supplement this system of transport equations with the kinetic equation \(f_2(x,x';t)\), and hence for \(f_s(x_1 \ldots x_s;t)\), \(s < N\):

\[
\frac{\partial}{\partial t} f_2(x,x';t) + \int L_2 f_2(x,x';t) \tag{2.30}
\]
\[
- \int d\Gamma NN \{iL(x,x'') + iL(x',x'')\} f_3(x,x',x'';t)
\]
\[
iL_2 \Delta f_2(x,x';t) - \int d\Gamma NN \{iL(x,x'') + iL(x',x'')\} \Delta f_3(x,x',x'';t)
\]
\[
+ \int d\Gamma NN \int da \int dt' e^{(t-t')\phi_Ga(x,x',a';t,t') f(a;t') \gamma(x'';t')}
\]
\[
- \sum_k \int dt' e^{(t-t')\phi_{Gk}(x,x',k,a;\gamma(t')) \frac{\partial}{\partial \Gamma_k} f(a;t')} + \phi_{Gc}(x,x',k,a;\gamma(t')) \frac{\partial}{\partial \Gamma_k} f(a;t') \tag{2.29}
\]

The generalized transport equations (2.22), (2.23) include the quasi-equilibrium binary distribution function of particles \(g_2(x,x';t)\):

\[
g_2(x,x';t) = \langle G_2(x,x') \rangle_q = \langle n_1(x) n_1(x') \rangle_q \tag{2.24}
\]
\[
= \int d\Gamma_N \n_2 g_2(x^N,a;t) = \int d\Gamma_N \phi_{\alpha\beta}(x^N,a;t) f(a;t),
\]

where

\[
g_2(x,x';a;t) = \int d\Gamma_N \phi_{\alpha\beta}(x^N,a;t)
\]

is the binary quasi-equilibrium distribution function of large-scale collective variables. The generalized transport kernels \(\phi_{\alpha\beta} (\alpha, \beta = (n,j,\varepsilon))\), that describe non-Markovian kinetic and hydrodynamic processes, are non-equilibrium correlation functions of generalized fluxes \(I_\alpha, I_\beta:\n\]

\[
\phi_{\alpha\beta}(t,t') = \langle I_\alpha(t) T_q(t,t') I_\beta(t') \rangle_L, \tag{2.25}
\]
\[
\dot{I}_a(x) = \langle (1 - P(t)) \dot{n}_1(x) \rangle, \tag{2.26}
\]
\[
\frac{\partial}{\partial t} f_s(x_1 \ldots x_s; t) + iL_s f_s(x_1 \ldots x_s; t) = \sum_j \int dx_{s+1} iL(x_j, x_{s+1}) f_{s+1}(x_1 \ldots x_s, x_{s+1}; t)
\]
(2.31)

\[
iL_s \Delta f_s(x_1 \ldots x_s; t) = \sum_j \int dx_{s+1} iL(x_j, x_{s+1}) \Delta f_{s+1}(x_1 \ldots x_s, x_{s+1}; t)
\]

\[
+ \int dx'' \int da \int_{-\infty}^t dt' e^{(t'-t)} \phi_{G,n}(x, x'', a; t, t') f(a; t') \gamma(x''; t')
\]

\[
- \sum_k \int da \int_{-\infty}^t dt' e^{(t'-t)} \left\{ \phi_{G,j}(x, x', k; a; t, t') \frac{\partial}{\partial J_k} + \phi_{G,x}(x, x', k; a; t, t') \frac{\partial}{\partial \epsilon_j} \right\} f(a; t') \int dx \int dx \int dx \int dx \int dx \int dx
\]

where \[\Delta f_s(x_1 \ldots x_s; t) = f_s(x_1 \ldots x_s; t) - g_s(x_1 \ldots x_s; t).\]

In Eq. (2.30) the two-particle Liouville operator \[iL_2 = iL_0(x) + iL_0(x^*) + iL(x, x')\]

was introduced. It contains one-particle operator

\[iL_0(x) = \frac{p}{m} \frac{\partial}{\partial r}, \quad x = \{r, p\},\]

and also a potential part \[iL(x, x') = \frac{\partial}{\partial r} \Phi(|r - r'|), \quad x = \{r, p\} \]

Accordingly, in Eq. (2.31), \[iL_s\] is the s-particle Liouville operator, and

\[g_s(x_1 \ldots x_s; t) = (\hat{G}_s(x_1 \ldots x_s))^t = \int da \int dx f_s(x_1 \ldots x_s; a; t) f(a; t),\]

where

\[g_s^t(x_1 \ldots x_s; a; t) = \int dN \hat{G}_s(x_1 \ldots x_s) g_L(x^N; a; t)\]

is the s-particle quasi-equilibrium distribution function of large-scale variables and \[\hat{G}_s(x^s) = \hat{n}_1(x_1) \ldots \hat{n}_1(x_s).\]

Thus we obtained a system of equations for non-equilibrium single, double, s-particle distribution functions which take into account nonlinear hydrodynamic fluctuations.

We now discuss the equation (2.30) that is of Fokker-Planck type one for non-equilibrium distribution function of collective variables which take into account the kinetic processes. The transport kernel in this equation \[\phi_{nn}(x, x'; t, t')\] describes a dissipation of kinetic processes, while the kernels \[\phi_{nj}(x, k; a; t, t'), \phi_{nx}(x, k; a; t, t'), \phi_{jn}(x, k; a; t, t'), \phi_{jn}(x, x', a; t, t')\] describe a dissipation of correlations between kinetic and hydrodynamic processes. To uncover more detailed a structyre of transport kernels \[\phi_{nn}(x, x'; a; t, t'), \phi_{nx}(x, x'; a; t, t'), \phi_{jn}(x, x'; a; t, t')\] we consider action of Liouville operator on \[\hat{n}_1(x)\] and \[\hat{G}(x, x'):\]

\[
iL_N \hat{n}_1(x) = -\frac{\partial}{\partial r} \frac{1}{m} \hat{j}(r, p) + \frac{\partial}{\partial p} \hat{F}(r, p),
\]

(2.32)

\[
iL_N \hat{G}(x, x') = -\frac{\partial}{\partial r} \frac{1}{m} \hat{j}(r, p) \hat{n}_1(x') - \hat{n}_1(x) \frac{\partial}{\partial r} \frac{1}{m} \hat{j}(r', p')
\]

\[
+ \frac{\partial}{\partial p} \hat{F}(r, p) \hat{n}_1(x') + \hat{n}_1(x) \frac{\partial}{\partial p} \hat{F}(r', p'),
\]

(2.33)

where

\[
\hat{j}(r, p) = \sum_{j=1}^N p_j \delta(|r - r_j|) \delta(p - p_j)
\]

(2.34)

is the microscopic density of momentum vector in coordinate-momentum space,

\[
\hat{F}(r, p) = \sum_{i \neq j} \frac{\partial}{\partial r_j} \Phi(|r_j - r_i|) \delta(|r - r_j|) \delta(p - p_j)
\]

(2.35)

is the microscopic density of force vector in coordinate-momentum space.

Taking into account equations (2.32)-(2.35), for the kinetic transport kernels, we obtain:

\[
\phi_{nn}(x, x'; a; t, t') = -\left[ \frac{\partial}{\partial r} \cdot D_{jj}(x, x', a; t, t') \cdot \frac{\partial}{\partial r'} \right.
\]

(2.36)

\[
- \frac{\partial}{\partial p} \cdot D_{jj}(x, x', a; t, t') \cdot \frac{\partial}{\partial r'} + \frac{\partial}{\partial p} \cdot D_{jj}(x, x', a; t, t') \cdot \frac{\partial}{\partial r'}
\]

\[
- \frac{\partial}{\partial r} \cdot D_{jj}(x, x', a; t, t') \cdot \frac{\partial}{\partial p} + \frac{\partial}{\partial p} \cdot D_{jj}(x, x', a; t, t') \cdot \frac{\partial}{\partial p'}
\]
\[ D_{jj}(x, x', a; t, t') = \int d\Gamma_N \tilde{j}(x) T_q(t, t')(1 - P(t')) \tilde{j}(x') p_L(x^N; t'), \]
\[ D_{FF}(x, x', a; t, t') = \int d\Gamma_N \tilde{F}(x) T_q(t, t')(1 - P(t')) \tilde{F}(x') p_L(x^N; t') \]

are the generalized diffusion and the particle friction coefficients in the coordinate-momentum space. Moreover,
\[ \int dp \int dp' D_{jj}(x, x', a; t, t') = D_{jj}(r, r'; t, t'), \]
\[ \int dp \int dp' D_{FF}(x, x', a; t, t') = D_{FF}(r, r'; t, t') \]
are the generalized coefficients of diffusion and friction. Similarly, we obtain the expression for the transport kernel \( \phi_{Gn}(x, x', x''; t') \):
\[ \phi_{Gn}(x, x', x'', a; t, t') = -\left[ \frac{\partial}{\partial r} D_{jj}(x, x', x'', a; t, t') \cdot \frac{\partial}{\partial r'} \right] \]
\[ + \left[ \frac{\partial}{\partial p} \cdot D_{Fj}(x, x', x'', a; t, t') \cdot \frac{\partial}{\partial r'} - \frac{\partial}{\partial p} \cdot D_{Fj}(x, x', x'', a; t, t') \cdot \frac{\partial}{\partial r''} \right] \]
\[ - \frac{\partial}{\partial p} \cdot D_{Fj}(x, x', x'', a; t, t') \cdot \frac{\partial}{\partial r'} + \left[ \frac{\partial}{\partial p} \cdot D_{Fj}(x, x', x'', a; t, t') \cdot \frac{\partial}{\partial r''} - \frac{\partial}{\partial p} \cdot D_{Fj}(x, x', x'', a; t, t') \cdot \frac{\partial}{\partial r''} \right] \]
\[ + \left[ \frac{\partial}{\partial p} \cdot D_{FF}(x, x', x'', a; t, t') \cdot \frac{\partial}{\partial r'} + \frac{\partial}{\partial p} \cdot D_{FF}(x, x', x'', a; t, t') \cdot \frac{\partial}{\partial r'} + \frac{\partial}{\partial p} \cdot D_{FF}(x, x', x'', a; t, t') \cdot \frac{\partial}{\partial r''} \right]. \]

It is remarkable that expression
\[ \int dx' \int da \int_0^t dt' e^{(t'-t)} \phi_{nn}(x, x', a; t', t') f(a; t') \gamma(x'; t') \]
in equation (2.23) with (2.26) is the generalized collision integral of Fokker-Planck type in the coordinate-momentum space. That is, taking into account (2.23) and (2.26), the kinetic equation (2.22) can be written as:
\[
\left[ \frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial r} \right] f_1(x; t) - \int dx' \int da \frac{\partial}{\partial r} \phi(x', a; t') f(a; t') \cdot \frac{\partial}{\partial r} \gamma(x'; t') \]
\[ \cdot dF \cdot e^{(t'-t)} g_2(x, x'; t) = \int dx' \int da \int_0^t dt' e^{(t'-t)} \phi_{nn}(x, x', a; t, t') y(x'; t') \]
\[ + \phi_{nc}(x, k, a; t, t') \frac{\partial}{\partial J_k} \{ f(a; t') \} \cdot \frac{\partial}{\partial J_k} \{ W(a; t') \}. \]

In the equation (2.23) the quantities \( \phi_{jj}(x, y, a; t', t') \), \( \phi_{jj}(k, q, a, a'; t', t') \), \( \phi_{jj}(k, q, a, a'; t', t') \), \( \phi_{tt}(k, q, a, a'; t', t') \) correspond to the dissipative processes connected with the correlations between viscous and heat hydrodynamic processes. The set of equations (2.22), (2.23), (2.30), (2.31) allows two limiting cases. First, if the description of non-equilibrium processes does not take into account nonlinear hydrodynamic fluctuations, we will obtain generalized kinetic equation for the non-equilibrium distribution function of the particles [15]:
\[
\left[ \frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial r} \right] f_1(x; t) - \int dx' \int da \frac{\partial}{\partial r} \phi(x', a; t') f(a; t') \cdot \frac{\partial}{\partial r} \gamma(x'; t') \]
\[ \cdot dF \cdot e^{(t'-t)} g_2(x, x'; t) \]
\[ = \int dx' \int da \int_0^t dt' e^{(t'-t)} \phi_{nn}(x, x', a; t, t') y(x'; t'). \]
Zubarev non-equilibrium statistical operator [43]:
\[
\frac{\partial}{\partial t} f(a; t) = \sum_k \left\{ \frac{\partial}{\partial j_k} v_{n,k}(a; t) + \frac{\partial}{\partial \xi_k} v_{\varepsilon,k}(a; t) \right\} f(a; t)
\]
(2.40)
\[
= \sum_{k,q} \int da' \int dt' e^{(t'-t)} \frac{\partial}{\partial j_k} \phi_{jj}(k, q, a', t', t') \frac{\partial}{\partial j_k} W(a; t') f(a; t')
\]
\[
+ \sum_{k,q} \int da' \int dt' e^{(t'-t)} \frac{\partial}{\partial \xi_k} \phi_{\varepsilon\varepsilon}(k, q, a', t', t') \frac{\partial}{\partial \xi_k} W(a; t') f(a; t')
\]
\[
+ \sum_{k,q} \int da' \int dt' e^{(t'-t)} \left\{ \frac{\partial}{\partial j_k} \phi_{jj}(k, q, a', t', t') \frac{\partial}{\partial j_k} W(a; t') f(a; t')
\]
\[
+ \frac{\partial}{\partial \xi_k} \phi_{\varepsilon\varepsilon}(k, q, a', t', t') \frac{\partial}{\partial \xi_k} W(a; t') f(a; t') \right\}
\]

First, we calculate the structure function \(W(a; t)\) for collective variables. To do this, we use the integral representation for δ-functions:
\[
\hat{f}(a) = \int d\omega \exp \left\{ -i\pi \sum_{l,k} \omega_{l,k} (\hat{a}_{l,k} - a_{l,k}) \right\}, \quad l = n, j, \varepsilon.
\]
(3.1)

Next, using a cumulant expansion [47] for \(W(a; t)\) one obtains:
\[
W(a; t) = \int d\Gamma_N \hat{g}_0^{kin}(x^N; t) \hat{f}(a)
\]
(3.2)
\[
= \int d\omega \exp \left\{ -i\pi \sum_{l,k} \omega_{l,k} \hat{a}_{l,k} \right\}
\]
\[
- \frac{\pi^2}{2} \sum_{i,j,k} \frac{M_{i,j,k}^{l_{i,j},l_{k}}}{\omega_{i,j,k}^2} \omega_{i,j,k} \omega_{i,j,k} \left\{ \sum_{n \geq 3} D_n (\omega; t) \right\}
\]
where
\[
\hat{a}_{l,k} = a_{l,k} - \langle \hat{a}_{l,k} \rangle_{kin}, \quad d\omega = \prod_{l,k} d\omega_{l,k} d\omega_{l,k}^{*},
\]
\[
\omega_{l,k} = \omega_{l,k}^{\dagger}, \quad \omega_{l,k} = \omega_{l,k}^{\dagger},
\]
\[
D_n (\omega; t) = \left( -i\pi \right)^n \sum_{l_1,\ldots,l_n} \sum_{k_1,\ldots,k_n} \frac{M_{l_1,\ldots,l_n}^{k_1,\ldots,k_n}}{\omega_{l_1,k_1} \ldots \omega_{l_n,k_n}}
\]
(3.3)
\[
= \langle \hat{a}_{l_1,k_1} \ldots \hat{a}_{l_n,k_n} \rangle_{kin}^{t,ce}
\]
(3.4)
are the non-equilibrium cumulant averages in approximations of the \(n\)-order, which are calculated using distribution \(\hat{g}_0^{kin}(x^N; t)\) (2.11). We present the structure function \(W(a; t)\) for further calculations in following form:
\[
\hat{W}(a; t) = \int d\omega \exp \left\{ -i\pi \sum_{l,k} \omega_{l,k} \hat{a}_{l,k} \right\}
\]
(3.5)
\[
- \frac{\pi^2}{2} \sum_{i,j,k} \frac{M_{i,j,k}^{l_{i,j},l_{k}}}{\omega_{i,j,k}^2} \omega_{i,j,k} \omega_{i,j,k} \left\{ \sum_{n \geq 3} D_n (\omega; t) \right\}
\]
\[
\times \left( 1 + B + \frac{1}{24} B^2 + \frac{1}{315} B^3 + \ldots + \frac{1}{n!} B^n + \ldots \right),
\]
where \( B = \sum_{n \geq 3} D_n(\omega; t) \). If in series of exponent (3.3), namely, \( \sum_{n \geq 3} D_n(\omega; t) \), one retains only the first term equal to unity, one will obtain the Gaussian approximation for \( W(a; t) \):

\[
W^G(a; t) = \int d\omega \exp\{ i \sum_{l,k} \omega_{l,k} a_{l,k} \} \quad (3.6)
\]

\[
- \frac{\pi^2}{2} \sum_{l_1,l_2,k_1,k_2} \mathfrak{M}_{l_1,l_2}^{(l_2)}(k_1, k_2; t) \omega_{l_1,k_1} \omega_{l_2,k_2},
\]

where \( \mathfrak{M}_{l_1,l_2}^{(l_2)}(k_1, k_2; t) \) are the matrix elements of non-equilibrium correlation functions:

\[
\mathfrak{M}_{l_1,l_2}^{(l_2)}(k_1, k_2; t) = \begin{vmatrix} \langle \hat{n}_l \rangle_{k_1} & \langle \hat{n}_l \hat{n}_m \rangle_{k_1} & \langle \hat{n}_l \hat{n}_m \hat{n}_n \rangle_{k_1} \\ \langle \hat{n}_l \hat{n}_m \rangle_{k_1} & \langle \hat{n}_l \hat{n}_m \hat{n}_n \rangle_{k_1} \\ \langle \hat{n}_l \hat{n}_m \hat{n}_n \rangle_{k_1} & \langle \hat{n}_l \hat{n}_m \hat{n}_n \rangle_{k_1} \end{vmatrix}_{(k_1, k_2)}
\]

and the non-equilibrium cumulant average

\[
\langle \hat{n}_l \hat{n}_m \hat{n}_n \rangle_{k_1} = \langle \hat{n}_l \hat{n}_m \hat{n}_n \rangle_{k_1} - \langle \hat{n}_l \rangle_{k_1} \langle \hat{n}_m \rangle_{k_1} \langle \hat{n}_n \rangle_{k_1}.
\]

For integrating over \( d\omega \) in (3.6) we should transform the quadratic form in exponential expression into a diagonal form with respect to \( \omega_{l,k} \). To this end it is necessary to find the eigenvalues of the matrix (3.7) by solving the equation

\[
\det[\mathfrak{M}_{2}(k_1, k_2; t) - \hat{E}(k_1, k_2; t)] = 0,
\]

\( \hat{E}(k_1, k_2; t) \) is the diagonal matrix. Further, obtained eigenvalues \( E_l(k; t) \), \( l = 1, \ldots, 5 \) of the expression (3.6) are as follows:

\[
W^G(a; t) = \int d\hat{\omega} \det[\hat{\omega}] \exp\{ - i \sum_{l,k} \hat{a}_{l,k} \hat{\omega}_{l,k} \} \quad (3.9)
\]

\[
- \frac{\pi^2}{2} \sum_{l} \sum_{k} E_l(k; t) \hat{a}_l(k) \hat{\omega}_{l,-k},
\]

where new variables \( \hat{a}_l(k), \hat{\omega}_{l,k} \) are connected with the old variables by ratio:

\[
\hat{a}_{l,k} = \sum_{l} a_{l,k} \omega_{l,n}, \quad \hat{\omega}_{l,k} = \sum_{m} \omega_{l,m} \omega_{m,k}.
\]

Integrand in (3.9) is a quadratic function \( \hat{\omega}_{l,k} \hat{\omega}_{l,-k} \) and after integrating over \( d\hat{\omega}_{l,k} \) we will obtain following structural function in Gaussian approximation \( W^G(a; t) \):

\[
W^G(a; t) = \exp\left\{ - \frac{1}{2} \sum_{l,k} E_l^{-1}(k; t) \hat{a}_l(k) \hat{a}_{l,-k} \right\} \quad (3.10)
\]

× \exp\{ - \frac{1}{2} \sum_{k} \ln \pi \det \hat{E}(k; t) \} \exp\{ \sum_{k} \ln \det \hat{W}(k; t) \},

or through variables \( \hat{a}_l(k) \):

\[
W^G(a; t) = Z(t) \exp\{ - \frac{1}{2} \sum_{l,k} E_l(k; t) \hat{a}_l(k) \hat{a}_{l,-k} \},
\]

\( Z(t) = \exp\left\{ - \frac{1}{2} \sum_{k} \ln \pi \det \hat{E}(k; t) \right\} \exp\{ \sum_{k} \ln \det \hat{W}(k; t) \}. \]

The structure function \( W^G(a; t) \) gives a possibility to calculate (3.3) in higher approximations over Gaussian moments (3.7):

\[
W(a; t) = W^G(a; t) \exp\{ \sum_{n \geq 3} \langle \hat{D}_n(a; t) \rangle_G \},
\]

where one presents \( \langle \hat{D}_n(a; t) \rangle_G \) approximately as:

\[
\langle \hat{D}_3(a; t) \rangle_G = \langle \hat{D}_3(a; t) \rangle_G,
\]

\[
\langle \hat{D}_4(a; t) \rangle_G = \langle \hat{D}_4(a; t) \rangle_G,
\]

\[
\langle \hat{D}_6(a; t) \rangle_G = \langle \hat{D}_6(a; t) \rangle_G - \frac{1}{2} \langle \hat{D}_3(a; t) \rangle_G^2,
\]

\[
\langle \hat{D}_8(a; t) \rangle_G = \langle \hat{D}_8(a; t) \rangle_G - \langle \hat{D}_3(a; t) \rangle_G \langle \hat{D}_5(a; t) \rangle_G - \frac{1}{2} \langle \hat{D}_4(a; t) \rangle_G^2,
\]

\[
\langle \hat{D}_n(a; t) \rangle_G = \frac{1}{W^G(a; t)} \sum_{l_1, \ldots, l_n} \mathfrak{M}_{l_1, \ldots, l_n}^{(l_1)}(k_1, \ldots, k_n; t)
\]

\[
\langle \hat{\omega}_{l,k} \hat{\omega}_{l,-k} \rangle_G = \sum_{m} \omega_{l,m} \omega_{m,k}.
\]
\[ W_{G}(a; t) = \frac{1}{(\pi i)^{n}} \delta_{n} \delta(a_{t_{1}, k_{1}} \cdots \delta a_{t_{n}, k_{n}}) W^{G}(a; t). \]

\[ \mathbb{M}_{n}^{l_{1}, \ldots, l_{n}}(k_{1}, \ldots, k_{n}; t) \] are the renormalized non-equilibrium cumulant averages of order \( n \) for the variables \( a_{t_{k}} \). In expression (3.11) the summands are with only even degrees over \( a \) since all odd Gaussian moments vanish.

The method of calculation of the structure function \( W(a; t) \) can be applied for approximate calculations of hydrodynamic velocities \( v_{L,k}(a; t) \). We present general formula of velocities consistent with (2.20) in following form:

\[ v_{L,k}(a; t) = \int d\Gamma_{N} \hat{a}_{t_{L}, k} \hat{q}_{G}^{kin}(x^{N}; t) \hat{f}(a) \]

and introduce function \( W(a, \lambda; t) \):

\[ W(a, \lambda; t) = \int d\Gamma_{N} e^{-i \sum_{L,k} \lambda_{L,k} \hat{a}_{t_{L}, k}} \hat{q}_{G}^{kin}(x^{N}; t) \hat{f}(a), \]

so that

\[ v_{L,k}(a; t) = \frac{\partial}{\partial (-i \pi \lambda_{L,k})} \ln W(a, \lambda; t) \bigg|_{\lambda_{L,k} = 0} \quad (3.13) \]

We calculate the function \( W(a, \lambda; t) \) using the preliminary results of the calculation of the structural function \( W(a; t) \), and rewrite \( W(a, \lambda; t) \) as:

\[ W(a, \lambda; t) = \int d\Gamma_{N} \int d\omega \exp \left\{ -i \pi \sum_{l,k} \lambda_{L,k} \hat{a}_{t_{L}, k} \right\} \times \exp \left\{ -i \pi \sum_{l,k} \omega_{l,k} (\hat{a}_{t_{L}, k} - a_{t_{L}, k}) \right\} \hat{q}_{G}^{kin}(x^{N}; t). \]

Now we carry out an averaging in (3.14) with \( \hat{q}_{G}^{kin}(x^{N}; t) \) using following cumulant expansion:

\[ W(a, \lambda; t) = \int d\omega \exp \left\{ -i \pi \sum_{l,k} \omega_{l,k} \hat{a}_{t_{L}, k} \right\} \times \sum_{n \geq 1} \left[ D_{n}(\omega; t) + D_{n}(\lambda; t) + D_{n}(\omega, \lambda; t) \right], \]

where

\[ D_{n}(\omega; t) = \frac{(-i\pi)^{n}}{n!} \sum_{l_{1}, l_{2}, \ldots, l_{n}, k_{1}, \ldots, k_{n}} \mathbb{M}_{n}^{l_{1}, \ldots, l_{n}}(k_{1}, \ldots, k_{n}; t) \omega_{l_{1}, k_{1}} \cdots \omega_{l_{n}, k_{n}}, \]

\[ D_{n}(\lambda; t) = \frac{(-i\pi)^{n}}{n!} \sum_{l_{1}, l_{2}, \ldots, l_{n}, k_{1}, \ldots, k_{n}} \mathbb{M}_{n}^{(1)} l_{1}, \ldots, l_{n} (k_{1}, \ldots, k_{n}; t) \lambda_{l_{1}, l_{1}} \cdots \lambda_{l_{n}, k_{n}}, \]

\[ D_{n}(\omega, \lambda; t) = \frac{(-i\pi)^{n}}{n!} \sum_{l_{1}, l_{2}, \ldots, l_{n}, k_{1}, \ldots, k_{n}} \mathbb{M}_{n}^{(2)} l_{1}, \ldots, l_{n} (k_{1}, k_{2}, \ldots, k_{n}; t) \omega_{l_{1}, k_{1}} \omega_{l_{2}, k_{2}} \cdots \omega_{l_{n}, k_{n}}. \]

First, we consider a Gaussian approximation for \( W(a, \lambda; t) \), namely we leave in the exponent of an integrand only the summands with \( n = 2 \) and linear over \( \lambda_{L,k} \):

\[ W_{G}(a, \lambda; t) = \int d\omega \exp \left\{ i \pi \sum_{l,k} \omega_{l,k} \hat{a}_{t_{L}, k} - i \pi \sum_{l,k} (\hat{a}_{t_{l}, k})^{t_{l}, e} \lambda_{L,k} \right\} \]

\[ - \frac{\pi^{2}}{2} \sum_{l_{1}, l_{2}, k_{1}, k_{2}} \mathbb{M}_{2}^{l_{1}, l_{2}}(k_{1}, k_{2}; t) \omega_{l_{1}, k_{1}} \omega_{l_{2}, k_{2}} \]

\[ - \frac{\pi^{2}}{2} \sum_{l_{1}, l_{2}, k_{1}, k_{2}} \mathbb{M}_{2}^{(2)} l_{1}, l_{2} (k_{1}, k_{2}; t) \omega_{l_{1}, k_{1}} \lambda_{l_{2}, k_{2}}. \]

Then, transforming this expression in the exponent to diagonal quadratic form over variables \( \omega_{l,k} \), similarly as for \( W(a; t) \), after integrating with respect to the new variables \( \hat{\omega}_{l,k} \), one obtains:

\[ W_{G}(a, \lambda; t) = \int d\omega \exp \left\{ -i \pi \sum_{l,k} (\hat{a}_{t_{l}, k})^{t_{l}, e} \lambda_{L,k} \right\} \]

\[ - \frac{\pi^{2}}{2} \sum_{l,k} E_{l}^{(-1)}(k; t) b_{l,k} b_{l,-k} - \frac{1}{2} \sum_{k} \ln \pi \text{det} \hat{E}(k; t) \]

\[ + \sum_{k} \ln \text{det} \hat{W}(k; t) \].
where
\[ b_{l,k} = \sum_j \omega_j \left[ \hat{a}_{l,j,k} + \frac{i\pi}{2} \sum_{j'} \mathcal{M}_{2}^{(2),j'}(k; t) \lambda_{j',k} \right], \]
and \( \omega_j, \mathcal{M}_{2}^{(2),j'}(k; t) \) and \( E_l(k; t) \) do not depend on \( \lambda_{l,k} \). Here the cumulant \( \mathcal{M}_{2}^{(2),j'}(k; t) \) has the following structure:
\[ \mathcal{M}_{2}^{(2),j'}(k; t) = \langle \hat{a}_{l,j,k} \hat{a}_{l',j',-k} \rangle_{\text{kin}} - \langle \hat{a}_{l,j,k} \rangle_{\text{kin}} \langle \hat{a}_{l',j',-k} \rangle_{\text{kin}}. \]  
(3.18)

Now we calculate the hydrodynamic velocities \( v_{t,k}(a; t) \) in Gaussian approximation according to the formula (3.13):
\[ v_{t,k}(a; t) = \frac{\partial}{\partial(-i\pi \lambda_{l,k})} \ln W^G(a, \lambda; t) \Big|_{\lambda_{l,k}=0} \]
\[ = \langle \hat{a}_{l,j,k} \rangle_{\text{kin}} - \frac{1}{2} \sum_{j,j'} E_l^{-1}(k; t) \omega_j \omega_{j'} \mathcal{M}_{2}^{(2),j',j}(k; t) \hat{a}_{l,j,k}. \]

Specifically, we consider the particular case when one can divide the longitudinal and transverse fluctuations for collective variables. That is, we choose the direction of the wave vector \( k \) along the axis of oz. Thus, one obtains:
\[ W^G(a; t) = \int d\omega \exp \{ i\pi \sum_{l,k} \omega_j \hat{a}_{l,j,k} \} \]
\[ - \frac{\pi^2}{2} \sum_{l,l'=1}^4 \sum_{k_1,k_2} \mathcal{M}_{2}^{(3),l,l'}(k_1, k_2; t) \omega_{l,j,k_1} \omega_{l',j,k_2}, \]
\[ - \frac{\pi^2}{2} \sum_{l,l'=1}^4 \sum_{k_1,k_2} \mathcal{M}_{2}^{(4),l,l'}(k_1, k_2; t) \omega_{l,j,k_1} \omega_{l',j,k_2} \}, \]
where \( \mathcal{M}_{2}^{(3),l,l'}(k_1, k_2; t) \) are the matrix elements of the non-equilibrium correlation functions of longitudinal fluctuations
\[ \mathcal{M}_{2}^{(3),l,l'}(k_1, k_2; t) = \begin{vmatrix} \langle \hat{n}_{l} \rangle_{\text{kin}} & \langle \hat{n}_{l'} \rangle_{\text{kin}} & \langle \hat{n}_{l} \rangle_{\text{kin}} \\ \langle \hat{J}_{l} \rangle_{\text{kin}} & \langle \hat{J}_{l'} \rangle_{\text{kin}} & \langle \hat{J}_{l} \rangle_{\text{kin}} \\ \langle \hat{e}_{l} \rangle_{\text{kin}} & \langle \hat{e}_{l'} \rangle_{\text{kin}} & \langle \hat{e}_{l} \rangle_{\text{kin}} \end{vmatrix}. \]  
(3.21)

and \( \mathcal{M}_{2}^{(4),l,l'}(k_1, k_2; t) \) are the matrix elements of the non-equilibrium correlation functions of transverse and transverse-longitudinal fluctuations.

In this case, the hydrodynamic velocities in the Gaussian approximation are as follows:
\[ v_{n, G}^a(t; a) = \langle \hat{n}_{l} \rangle_{\text{kin}} + E_l^{-1}(k; t) \omega_{l,j,k_1} \omega_{l',j,k_2} \mathcal{M}_{2}^{(3),l,l'}(k_1, k_2; t) \hat{a}_{l,j,k}, \]
\[ v_{n, G}^c(t; a) = \langle \hat{J}_{l} \rangle_{\text{kin}} + E_l^{-1}(k; t) \omega_{l,j,k_1} \omega_{l',j,k_2} \mathcal{M}_{2}^{(4),l,l'}(k_1, k_2; t) \hat{a}_{l,j,k}, \]
\[ v_{n, G}^a(t; a) = \langle \hat{e}_{l} \rangle_{\text{kin}} + E_l^{-1}(k; t) \omega_{l,j,k_1} \omega_{l',j,k_2} \mathcal{M}_{2}^{(3),l,l'}(k_1, k_2; t) \hat{a}_{l,j,k}, \]

where
\[ \Omega_{\alpha}(t; k) = \omega_{l,j,k_1} \omega_{l',j,k_2} \mathcal{M}_{2}^{(3),l,l'}(k_1, k_2; t) \hat{a}_{l,j,k}, \]
\[ \Omega_{\alpha}(t; k) = \omega_{l,j,k_1} \omega_{l',j,k_2} \mathcal{M}_{2}^{(4),l,l'}(k_1, k_2; t) \hat{a}_{l,j,k}, \]
and \( \omega_j \) are the elements of matrix \( W(k; t) \). As one can see, the hydrodynamic velocities (3.23) in the Gaussian approximation for \( W^G(a, \lambda; t) \) are the linear functions of collective variables \( n_k, J_k, \) and \( \varepsilon_k \). It is remarkable that if the kinetic processes are not taken into account, then \( \varepsilon_q(x; t) = 1 \cdot \langle ... \rangle_{\text{kin}} \rightarrow \langle ... \rangle_0 \) is an averaging over a microscopic ensemble \( W(a) \); in this case the expressions (3.23) for hydrodynamic velocities transform into the results of previous work [17], in which the nonlinear hydrodynamic fluctuations in simple fluids were investigated. The collective variable method [15] [47] gives a possibility to calculate the hydrodynamic velocities in approximations higher than the Gaussian one. In particular, the approximation for the Gaussian, based on (3.15) and hydrodynamic velocities (3.23) will be proportional to \( \hat{a}_{l,k} \hat{a}_{l',k} \), and transport kernels in the Fokker-Planck equation will be the fourth-order correlation functions over the variables \( \hat{a}_{l,k} \).

It is important that in Gaussian approximation for \( W^G(k; t) \) and \( v_{n, G}^a(t; a) \), the Fokker-Planck equation leads to the transport equations for \( \langle \hat{a}_{l,k} \rangle \), which are similar in structure to the case of the molecular hydrodynamics, averaged only over \( g_{L}(x^N; a; t) = g_{kin}^{L}(x^N; t) \frac{f(a)}{W^G(a; t)} \). The proposed approach makes possible to go beyond the Gaussian approximation for \( W(k; t) \) and \( v_{t,k}(a; t) \), and hence to do the same in the
transport kernels in Fokker-Planck equation. This allows us to obtain a nonlinear equation system for $\langle \hat{a}_l, k \rangle^t$.

It is noteworthy that kinetic equation (2.23) contains a generalized integral of Fokker-Planck type with generalized coefficients of diffusion and particle friction in the phase space $(r, p, t)$. This region of changes $|r|$ is limited by values $|k|_{hydr}^{-1}$, that correspond to collective nonlinear hydrodynamic processes. This means that in regions of limited $|k|_{hydr}^{-1}$ the processes are described by the generalized coefficients of diffusion and friction, and at small $|k|_{hydr}^{-1}$ they are described by generalized viscosity, thermal conductivity and by cross coefficients $\phi_{j} (k, q, a, a'; t, t')$, $\phi_{\epsilon j} (k, q, a, a'; t, t')$, $\phi_{j} (k, q, a, a'; t, t')$, $\phi_{\epsilon j} (k, q, a, a'; t, t')$. Correlations between these regions are described by cross kernels $\phi_{n j} (x, q, a, a'; t, t')$, $\phi_{\epsilon n j} (x, q, a, a'; t, t')$, $\phi_{\epsilon n j} (x, q, a, a'; t, t')$, $\phi_{\epsilon j} (k, x', a, a'; t, t')$, $\phi_{j} (k, x', a, a'; t, t')$, $\phi_{\epsilon j} (k, x', a, a'; t, t')$, $\phi_{\epsilon j} (k, x', a, a'; t, t')$, that are present both in the kinetic equation and in the Fokker-Planck equation. The calculations of these kernels is very important because they describe the cross-correlations between kinetic and hydrodynamic processes.

4. Conclusions

Using the method of Zubarev non-equilibrium statistical operator, we have developed an approach [12, 13] for consistent description of kinetic and hydrodynamic processes, that are characterized by non-linear fluctuations. We have obtained the non-equilibrium statistical operator of non-equilibrium state of the system when the parameters of the reduced description are a non-equilibrium one-particle distribution function and the non-equilibrium distribution function of the non-linear hydrodynamic variables (densities of mass, momentum and energy). By using this operator we constructed a chain of kinetic equations (of BBGKY type) for non-equilibrium single, double, s-particle distribution functions of particles that take into account the nonlinear hydrodynamic fluctuations. At the same time the non-equilibrium distribution function of hydrodynamic fluctuations satisfy a generalized Fokker-Planck equation.

We proposed a method to calculate the structural distribution function of hydrodynamic collective variables and their hydrodynamic velocities (above Gaussian approximation) contained in a generalized Fokker-Planck equation for the non-equilibrium distribution function of hydrodynamic collective variables. In the future studies, we will go beyond the Gaussian approximation and carry out approximate calculations of kinetic transport coefficients for a specific system of interacting particles.

References

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