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PHASE TRANSITIONS IN
A CONTINUUM CURIE-WEISS SYSTEM:
A QUANTITATIVE ANALYSIS

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Фазові переходи в неперервній системі Кюрі-Вейса: кількісний аналіз

Ю.В.Козицький, М.П.Козловський, О.А.Добуш

Анотація. Чисельно досліджено фазовий перехід у неперервній системі частинок із взаємодією Кюрі-Вейса. Взаємодія визначена поділом простору \mathbb{R}^d на однакові кубічні комірки. Для фіксованих значень температури сили J_1, J_2 та хімічного потенціалу, термодинамічна фаза визначається як ймовірнісна міра на просторі чисел заповнення комірок. Показано, що півплощина $J_1 \times$ хімічний потенціал містить точки фазового співіснування, а тому можуть існувати множинні термодинамічні фази системи при тих самих значеннях температури та хімічного потенціалу. Здійснено числовий розрахунок, що описує таке явище.

Phase transitions in a continuum Curie-Weiss system: a quantitative analysis

Yu.V. Kozitsky, M.P. Kozlovskii, O.A. Dobush

Abstract. Phase transitions in a continuum Curie-Weiss system of interacting particles are studied numerically. The interaction is determined by a division of the underlying space \mathbb{R}^d into congruent cubic cells. For a region $V \subset \mathbb{R}^d$ consisting of $N \in \mathbb{N}$ cells, each two particles contained in V attract each other with intensity J_1/N . The particles contained in the same cell are subject to binary repulsion with intensity $J_2 > J_1$. For fixed values of the temperature, the intensities J_1, J_2 and the chemical potential, the thermodynamic phase is defined as a probability measure on the space of occupation numbers of cells. There is shown that the half-plane $J_1 \times$ chemical potential contains phase coexistence points, and thus multiple thermodynamic phases of the system may exist at the same values of the temperature and chemical potential. The numerical calculations describing such phenomena are presented.

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ФАЗОВІ ПЕРЕХОДИ В НЕПЕРЕРВНІЙ СИСТЕМІ КЮРІ-ВЕЙСА:
 КІЛЬКІСНИЙ АНАЛІЗ

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1. Introduction

The rigorous theory of phase transitions in continuum particle systems has much more modest account of the results than its counterpart dealing with lattices, graphs, etc. It is then quite natural to employ here mean field models. In [1], the mean field approach was mathematically realized by using a Kac-like infinite range attraction combined with a two-body repulsion. By means of rigorous upper and lower bounds obtained in that paper for the canonical partition function, the authors derived the equation of state indicating the possibility of a first-order phase transition. Later on, this result was employed in [2], see also [3], to go beyond the mean field frames. Another mean-field approach is based on the use of Curie-Weiss interactions and appropriate methods of calculating asymptotics of integrals. Recently it was turned into a mathematical theory in the framework of which the thermodynamic phases are constructed as probability measures on an appropriate phase space, see [4, Section 2]. In this context, in [4] there was introduced a simple Curie-Weiss type model of a continuum particle system, for which it was proved in [5] that multiple thermodynamic phases may exist at the same values of the temperature and chemical potential. In the present work, we numerically investigate this model in more detail.

In the model which we study, the interaction is determined by a division of the underlying space \mathbb{R}^d into congruent cubic cells. For a region $V \subset \mathbb{R}^d$ consisting of N such cells, the attraction between each two particles in V is set to be J_1/N , regardless their positions. If such two particles lie in the same cell, they repel each other with intensity $J_2 > J_1$. Unlike to [1], we work in the grand canonical ensemble, and thus the initial thermodynamic variables are the inverse temperature $\beta = 1/k_B T$ and the physical chemical potential. However, for the sake of convenience we employ the variables $p = \beta J_1$ and $\mu = \beta \times \text{physical chemical potential}$ and define single-phase domains of the half-plane $\{(p, \mu) : p > 0, \mu \in \mathbb{R}\}$, see Definition 2.1 below, by a condition that determines a unique probability measure $\mathbf{Q}_{p, \mu}$, given in (2.16) and (2.15). In the grand canonical formalism and the approach of [4], this measure is set to be the thermodynamic phase of the system. The points (p, μ) where the mentioned single-phase condition fails to hold due to the existence of multiple \bar{y} correspond to the coexistence of multiple thermodynamic phases.

Section 3 is dedicated to numerical results related to description of the phase transition in the system with Curie-Weiss interaction. An analysis of the chemical potential behavior $\bar{\mu}(y)$ which meet the condition of maximum of $E(y, p, \mu)$ is provided. To make quantitative analysis we

considered fixed values of the parameters $v = 12$, $p = 6$. The former parameter determine the volume of a cell, the latter is connected to attractive part of the interaction potential. We found the critical value of $p = p_c(a)$ dividing the monotonic and non-monotonic dependence of $\bar{\mu}(y)$. The explicit form of the state equation of the model is obtained in the region of $p < p_c(a)$ (matching to $T > T_c$, T_c is defined in (3.2)) describing a single-phase domain. The pressure as a function of density and temperature is represented. In the region of $p > p_c(a)$ we found values of the chemical potential μ_c at which the first order phase transition occurs for different temperatures. In Section 3.3 the equation of state of the model in the region of low densities and temperatures below the critical one is represented. The second and third cascades of phase transitions are considered in Section 3.4. In this case the pressure as a function of density at $T < T_c$ is obtained. Section 3.5 represents numerical results stated above for the case of another ratio of repulsive and attractive parameters.

2. The Model: a Theoretical Study

By \mathbb{N} , \mathbb{R} we denote the sets of natural and real numbers, respectively. We also put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}$, by \mathbb{R}^d we denote the Euclidean space of vectors $x = (x^1, \dots, x^d)$, $x^i \in \mathbb{R}$. In the sequel, its dimension d will be fixed. By dx we mean the Lebesgue measure on \mathbb{R}^d .

2.1. The grand canonical partition function

For some $c > 0$, we let $\Delta = (-c/2, c/2]^d \subset \mathbb{R}^d$ be a cubic cell of volume $v = c^d$ centered at the origin. Let also $V \subset \mathbb{R}^d$ be the union of $N \in \mathbb{N}$ disjoint translates Δ_ℓ of Δ , i.e.,

$$V = \bigcup_{\ell=1}^N \Delta_\ell.$$

As is usual for Curie-Weiss theories, cf. [4], the form of the interaction energy of the system of particles placed in V depends on V . In our model, the energy of a configuration $\gamma = \{x_1, \dots, x_n\} \subset V$, $n \in \mathbb{N}$, is

$$W_N(\gamma) = \frac{1}{2} \sum_{x, y \in \gamma} \Phi_N(x, y),$$

where

$$\Phi_N(x, y) = -J_1/N + J_2 \sum_{\ell=1}^N \mathbb{I}_{\Delta_\ell}(x) \mathbb{I}_{\Delta_\ell}(y). \quad (2.1)$$

Here \mathbb{I}_{Δ_ℓ} is the indicator of Δ_ℓ , that is, $\mathbb{I}_{\Delta_\ell}(x) = 1$ if $x \in \Delta_\ell$ and $\mathbb{I}_{\Delta_\ell}(x) = 0$ otherwise. For convenience, in W_N above we have included the self-interaction term $\Phi_N(x, x)$, which does not affect the physics of the model. We also write W_N and Φ_N instead of writing W_V and Φ_V since these quantities depend only on the number of cells in V but not on its particular location. The first term in Φ_N with $J_1 > 0$ describes attraction. By virtue of the Curie-Weiss approach, it is taken equal for all particles. The second term with $J_2 > 0$ describes repulsion between two particles contained in one and the same cell. That is, in our model every two particles in V attract each other independently of their location, and repel if they are in the same cell. The intensities J_1 and J_2 in (2.1) are assumed to satisfy the following condition

$$J_2 > J_1. \quad (2.2)$$

The latter is to secure the stability of the interaction, see [6], that is to satisfy

$$\int_V \Phi_N(x, y) dy > 0, \quad \text{for all } x \in V.$$

Let $\beta = 1/k_B T$ be the inverse temperature. To optimize the thermodynamic variables we introduce the following

$$p = \beta J_1, \quad a = J_2/J_1, \quad (2.3)$$

and the dimensionless chemical potential $\mu = \beta \times$ (physical chemical potential). Then $(p, \mu) \in \mathbb{R}_+ \times \mathbb{R}$ is considered as the basic set of thermodynamic variables, whereas a and v are model parameters.

The grand canonical partition function in region V is

$$\Xi_N(p, \mu) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{V^n} \exp\left(\mu n - \frac{\beta}{2} \sum_{i, n=1}^n \Phi_N(x_i, x_j)\right) dx_1 \cdots dx_n. \quad (2.4)$$

In [5], the representation in (2.4) was transformed into the following one

$$\Xi_N(p, \mu) = \sum_{\varrho \in \mathbb{N}_0^N} \exp\left(\frac{p}{2N} \left(\sum_{\ell=1}^N \varrho_\ell\right)^2\right) \prod_{\ell=1}^N \pi(\varrho_\ell, \mu), \quad (2.5)$$

where p is as in (2.3) and

$$\pi(n, \mu) = \frac{v^n}{n!} \exp\left(\mu n - \frac{1}{2} a p n^2\right), \quad n \in \mathbb{N}_0. \quad (2.6)$$

Note that, for $p = 0$, π turns into the (non-normalized) Poisson distribution with parameter ve^μ . Hence, alternating the cell size amounts to shifting μ .

We write Ξ_N instead of Ξ_V for the reasons mentioned above. Such type of distribution was used in [7].

2.2. Single-phase domains and phase transitions

For the reader's convenience, we repeat here some definitions and facts from [5]. By a standard identity involving Gaussian integrals one gets

$$\exp\left(\frac{p}{2N}\left(\sum_{\ell=1}^N \varrho_\ell\right)^2\right) = \sqrt{\frac{N}{2\pi p}} \int_{\mathbb{R}} \exp\left(-N\frac{y^2}{2p} + y\sum_{\ell=1}^N \varrho_\ell\right) dy.$$

Applying this in (2.5) one arrives at

$$\Xi_N(p, \mu) = c_N \int_{\mathbb{R}} \exp\left(NE(y, p, \mu)\right) dy, \quad c_N = \sqrt{\frac{N}{2\pi p}}, \quad (2.7)$$

where

$$E(y, p, \mu) = -\frac{y^2}{2p} + \ln K(y, p, \mu), \quad (2.8)$$

and, cf. (2.3) and (2.6),

$$K(y, p, \mu) = \sum_{n=0}^{\infty} \frac{v^n}{n!} \exp\left((y + \mu)n - \frac{ap}{2}n^2\right). \quad (2.9)$$

Note that E is an infinitely differentiable function of all its arguments. Set

$$P_N(p, \mu) = \frac{1}{vN} \ln \Xi_N(p, \mu). \quad (2.10)$$

By the following evident inequality

$$(y + \mu)n - \frac{ap}{2}n^2 \leq \frac{(y + \mu)^2}{2ap}, \quad n \in \mathbb{N}_0,$$

we obtain from (2.9) and (2.8) that

$$E(y, p, \mu) \leq -\frac{a-1}{2ap}y^2 + \frac{\mu}{2ap}(2y + \mu) + v. \quad (2.11)$$

As is usual for Laplace's method, see [8], the calculation of the large N limit in (2.10) is based on finding global maxima of $E(y, p, \mu)$ as a

function of $y \in \mathbb{R}$ (i.e., for fixed values of $p > 0$ and $\mu \in \mathbb{R}$). By (2.11) we have that $\lim_{|y| \rightarrow +\infty} E(y, p, \mu) = -\infty$; hence, each point \bar{y} of global maximum belongs to a certain interval $(\bar{y} - \varepsilon, \bar{y} + \varepsilon)$, where it is also a maximum point. Since E is everywhere differentiable in y , then \bar{y} is the point of global maximum only if it solves the following equation

$$E_1(y, p, \mu) := \frac{\partial}{\partial y} E(y, p, \mu) = 0. \quad (2.12)$$

By (2.8) and (2.9) this equation can be rewritten in the form

$$-\frac{y}{p} + \frac{K_1(y, p, \mu)}{K(y, p, \mu)} = 0, \quad (2.13)$$

$$[2cm]K_1(y, p, \mu) := \sum_{n=1}^{\infty} \frac{nv^n}{n!} \exp\left((y + \mu)n - \frac{ap}{2}n^2\right).$$

The equation in (2.13) has at least one solution for all $p > 0$ and $\mu \in \mathbb{R}$. Since both K_1 and K take only strictly positive values, these solutions are also strictly positive.

Definition 2.1 *We say that (p, μ) belongs to a single-phase domain if $E(y, p, \mu)$ has a unique global maximum $\bar{y} \in \mathbb{R}$ such that*

$$E_2(\bar{y}, p, \mu) := \frac{\partial^2}{\partial y^2} E(y, p, \mu)|_{y=\bar{y}} < 0. \quad (2.14)$$

Note that \bar{y} can be a point of maximum if $E_1(\bar{y}, p, \mu) = E_2(\bar{y}, p, \mu) = 0$. That is, not every point of global maximum corresponds to a point in a single-phase domain.

The condition in (2.13) determines the unique probability measure $Q_{p, \mu}$ on \mathbb{N}_0 such that

$$Q_{p, \mu}(n) = \frac{1}{K(\bar{y}, p, \mu)n!} v^n \exp\left((\bar{y} + \mu)n - \frac{ap}{2}n^2\right), \quad n \in \mathbb{N}_0, \quad (2.15)$$

which yields the probability law of the occupation number of a single cell. Then the unique thermodynamic phase of the model corresponding to $(p, \mu) \in \mathcal{R}$ is the product

$$\mathbf{Q}_{p, \mu} = \bigotimes_{\ell=1}^{\infty} Q_{p, \mu}^{(\ell)} \quad (2.16)$$

of the copies of the measure defined in (2.15). It is a probability measure on the space of all vectors $\mathbf{n} = (n_\ell)_{\ell=1}^{\infty}$, in which $n_\ell \in \mathbb{N}_0$ is the occupation number of ℓ -th cell.

The role of the condition in (2.14) is to yield the possibility to apply Laplace's method for asymptotic calculating the integral in (2.7). By direct calculations it follows that

$$E_2(y, p, \mu) = -\frac{1}{p} + \frac{1}{2[K(y, p, \mu)]^2} \quad (2.17)$$

$$\times \sum_{n_1, n_2=0}^{\infty} \frac{v^{n_1+n_2}}{n_1!n_2!} (n_1 - n_2)^2 \exp\left((y + \mu)(n_1 + n_2) - \frac{ap}{2}(n_1^2 + n_2^2)\right).$$

In dealing with the equation in (2.13) we will fix $p > 0$ and consider E_1 as a function of $y \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Then, for a given μ , we solve (2.12) to find \bar{y} and then check whether it is the unique point of global maximum and (2.14) is satisfied, i.e., whether (p, μ) belongs to a single-phase domain. As it was shown in [5] each single-phase domain, \mathcal{R} , has the following properties: (a) it is an open subset of $\mathbb{R}_+ \times \mathbb{R}$; (b) for each $(p_0, \mu_0) \in \mathcal{R}$, the function $\mathcal{I}_{p_0} := \{\mu \in \mathbb{R} : (p_0, \mu) \in \mathcal{R}\} \ni \mu \mapsto \bar{y}(\mu)$ is continuously differentiable on \mathcal{I}_{p_0} . Moreover,

$$\frac{d\bar{y}(\mu)}{d\mu} > 0, \quad \text{for all } \mu \in \mathcal{I}_{p_0}. \quad (2.18)$$

By (2.15) and (2.13) we get the $Q_{p,\mu}$ -mean value $\bar{n} = \bar{n}(p, \mu)$ of the occupation number of a given cell in the form

$$\bar{n}(p, \mu) = \sum_{n=0}^{\infty} n Q_{p,\mu}(n) = \frac{K_1(\bar{y}(p, \mu), p, \mu)}{K(\bar{y}(p, \mu), p, \mu)} = \frac{\bar{y}(p, \mu)}{p}. \quad (2.19)$$

Note that, up to the factor v^{-1} , $\bar{n}(p, \mu)$ is the particle density in phase $\mathbf{Q}_{p,\mu}$. For a fixed p , $\bar{n}(p, \cdot)$ is an increasing function on \mathcal{I}_p , which thus can be inverted to give $\bar{\mu}(p, \bar{n})$. By Laplace's method we get that for each $(p, \mu) \in \mathcal{R}$, the limiting pressure $P(p, \mu) = \lim_{N \rightarrow +\infty} P_N(p, \mu)$, see (2.10), exists and is continuously differentiable on \mathcal{R} . Moreover, it is given by the following formula

$$P(p, \mu) = v^{-1} E(\bar{y}(p, \mu), p, \mu). \quad (2.20)$$

Let \mathcal{N}_p be the image of \mathcal{I}_p under the map $\mu \mapsto \bar{n}(p, \mu)$. Then the inverse map $\bar{n} \mapsto \bar{\mu}(p, \bar{n})$ is continuously differential and increasing on \mathcal{N}_p . By means of this map, for a fixed p , the pressure given in (2.20) can be written as a function of \bar{n}

$$P = \bar{P}(\bar{n}) = v^{-1} E(p\bar{n}, p, \bar{\mu}(p, \bar{n})), \quad \bar{n} \in \mathcal{N}_p, \quad (2.21)$$

which is the equation of state.

By virtue of Definition 2.1 each single-phase domain is an open subset of the open right half-plane $\{(p, \mu) : p > 0, \mu \in \mathbb{R}\}$, see (2.18). In the context of this work, a phase transition is understood as the possibility to have different phases at the same value of the pair (p, μ) . If this is the case, (p, μ) is called a *phase coexistence point*. Clearly, such a point should belong to the common topological boundary of at least two distinct single-phase domains. In [5], we demonstrated the existence of phase transitions in this sense. Namely, we proved the following statements.

Theorem 2.2 *There exists $p_0 = p_0(a) > 0$ such that the set $\mathcal{R}(p_0) := \{(p, \mu) : p \in (0, p_0]\}$ is a single-phase domain.*

Theorem 2.3 *For each $a > 1$, there exists $p_1 = p_1(a) > 0$ such that, for each $p \geq p_1$, the line $l_p = \{(p, \mu) : \mu \in \mathbb{R}\}$ contains at least one phase-coexistence point.*

3. The Quantitative Analysis

In the remaining part of the work, we provide our numerical results related to the facts just stated. Let us fix the following values of the parameters

$$a = J_2/J_1 = 1.2, \quad v = 12. \quad (3.1)$$

Below we return to the question how the results depend on the values of a and v .

As follows from Theorems 2.2 and 2.3, for small $p_0 > 0$ the stripe $\{(p, \mu) : \mu \in \mathbb{R}, p \in (0, p_0]\}$ is a single-phase domain, whereas the lines $l_p = \{(p, \mu) : \mu \in \mathbb{R}\}$ contain phase coexistence points if $p \geq p_1$ for sufficiently big $p_1 > p_0$. Thus, there should be some critical $p_c \in (p_0, p_1)$ which separates these two regimes. Our aim now is to find its numerical value for the parameters fixed in (3.1).

Figure 1 below shows the plot of $\bar{\mu}(y)$ which is the function inverse to that in (2.18). That is, given $y > 0$, $\bar{\mu}(y)$ is the value of μ for which y solves (2.13), see [5] for more detail. In view of (2.18), a monotone dependence of $\bar{\mu}(y)$ on y corresponds to a single-phase domain. Case a) of figure 1 corresponds to $p = 3.5$, case b) to $p = p_c = 3.928236$, and case c) to $p = 4.5$. Easily seen that $p_c = 3.928236$ is indeed the critical value. Recall that it corresponds to $a = 1.2$. By means of $p_c(a)$ one may define the corresponding value of the critical temperature, cf. (2.3),

$$T_c(a) = J_1/p_c(a). \quad (3.2)$$

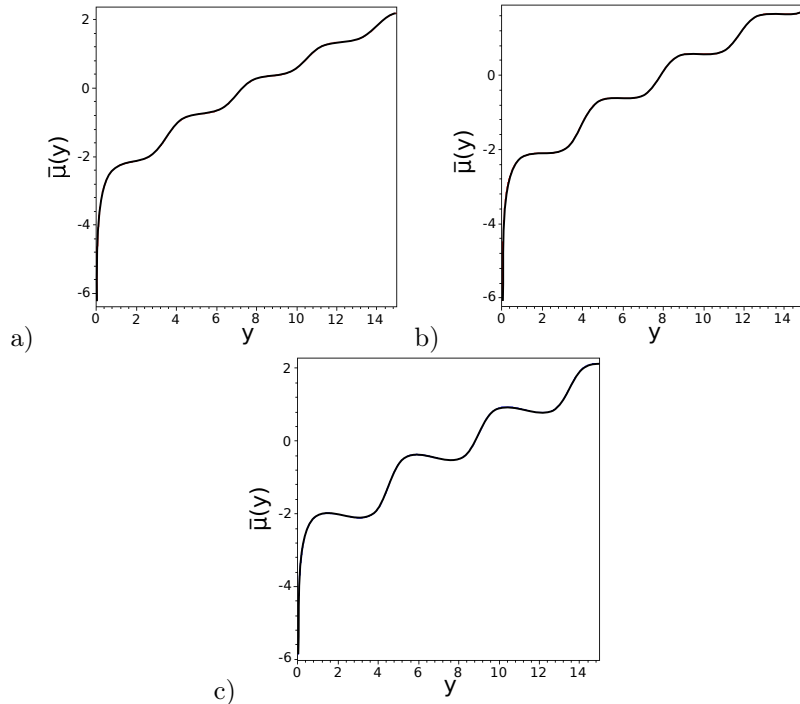


Figure 1: Plot of the chemical potential $\bar{\mu}(y)$ as a function of y for values of the attraction parameter $p = 3.5$ (curve a), $p = p_c(1.2)$ (curve b), $p = 4.5$ (curve c).

At $p < p_c(a)$ the equation (2.13) gives the line of extremes of the function $E(\bar{y}, p, \mu)$. Using this equation the chemical potential $\bar{\mu}(y)$ is a monotonic increasing function of y (see Figure 1a).

According to Theorem 2.3 for all $p > p_c(a)$ multiple solutions \bar{y} correspond to the same value of $\bar{\mu}(y)$, as it is shown on Figure 1c. In this case some part of the curve represented on Figure 1c expresses the $\max E(\bar{y}, p, \mu)$, and the other part of it meets the condition of $\min E(\bar{y}, p, \mu)$ at equal values of μ . As a consequence of inequality (2.14) the parts of the curve for (2.13) meets the condition of $\max E_0(\bar{y}, \mu)$ if $\bar{\mu}(y)$ behaves as an increasing function of y . The condition $\min E(\bar{y}, p, \mu)$ is implemented on the areas where $\bar{\mu}(y)$ is a decreasing function of y , which means that this condition is not applicable since we use the Laplace method for calculating (2.7).

It is easy to make sure about the latter fact directly from calculations.

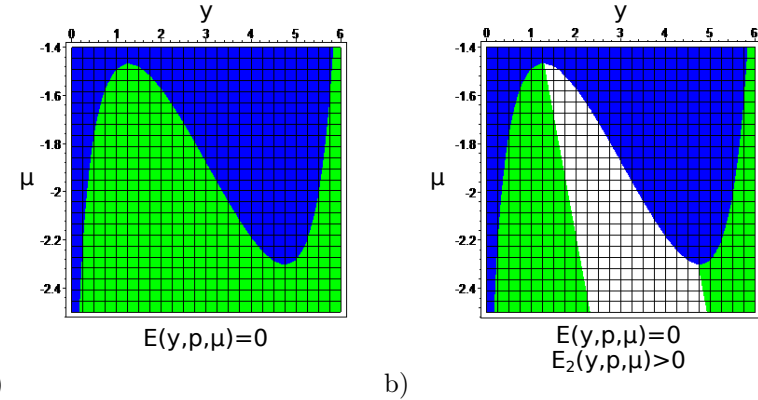


Figure 2: a – division boundary of two surfaces which give the line of extremes (2.12) of the function $E(\bar{y}, p, \mu)$. Figure 2b – the imposed (white) part meet the condition of minimum of $E(\bar{y}, p, \mu)$ ($p = 6$, $a = 1.2$, $v = 12$).

Figure 2a shows the intersection of the surface $E_1(y, p, \mu)$ (2.12) with the plane of zeroth level corresponding to the line of extremes $\bar{\mu}(y)$. On Figure 2b the image of the surface $E_2(y, p, \mu) > 0$ projection (white part) is imposed on the projection from Figure 2a. Positive part of the second derivative of the function $E_0(\bar{y}, y)$ coincide with the area where $\bar{\mu}(y)$ as a function of \bar{y} decrease. The condition $E_2(y, p, \mu) < 0$ is held only on the areas where $\bar{\mu}(y)$, is an increasing function of its argument.

3.1. The equation of state in a single-phase domain

In a single-phase domain, the function in (2.8) has a unique global maximum $\bar{y}(p, \mu)$, and hence the Laplace method yields in (2.7) that

$$\Xi_N \simeq c_N \exp [N E(\bar{y}(p, \mu), p, \mu)],$$

where the asymptotic equality is understood in the large N limit.

Let us write the equation of state of the model in case of $p < p_c(a)$ in the region of temperature $T > T_c$. At this range of parameter p the chemical potential $\bar{\mu}(y)$ behaves as a monotonic increasing function of \bar{y} (Figure 1a). The explicit form of the pressure is given in (2.20) This equation expresses the pressure P as a function of temperature and chemical potential since \bar{y} is a monotonic increasing function of μ as it is seen from (2.13). This equation can be rewritten in terms of the mean density \bar{n} as the mean value of the occupation number n over probability

measure $Q_{p,\mu}(n)$ from (2.15). Taking into account (2.19) we have

$$\bar{n} = \frac{\sum_{m=0}^{\infty} \frac{v^m}{m!} m \exp\left(-\frac{ap}{2}m^2\right) \exp(p\bar{n}m + \beta\mu(p, \bar{n})m)}{\sum_{m=0}^{\infty} \frac{v^m}{m!} \exp\left(-\frac{ap}{2}m^2\right) \exp(p\bar{n}m + \beta\mu(p, \bar{n})m)}. \quad (3.3)$$

The equation (3.3) holds a central place in the grand canonical ensemble formalism. It gives a possibility to find the chemical potential as a function of density and express the pressure in terms of density and temperature. In range of values of the parameter $p < p_c(a)$ for (2.21) we obtain the explicit form of the equation of state

$$Pv = -\frac{1}{2}p\bar{n}^2 + \ln \sum_{m=0}^{\infty} \frac{v^m}{m!} \exp\left(-\frac{ap}{2}m^2\right) \exp(p\bar{n}m + \beta\mu(p, \bar{n})m), \quad (3.4)$$

here $\mu(p, \bar{n})$ is a function of temperature and average density as it can be seen in (3.3).

Figure 3 shows pressure as a function of average density \bar{n} and the parameter p , which is inversely proportional to temperature. Easy to see that pressure is a monotonic increasing function of temperature and density.

3.2. The case of $T < T_c$

Consider calculation of the grand partition function in the region of $p > p_c(a)$, that correspond to temperatures $T < T_c$. Here the chemical potential $\bar{\mu}(y)$ is a monotonic function of \bar{y} . Let us explore the behavior of $\bar{\mu}(y)$ in this range of values of the parameter p . It is represented on Figure 4 at $p = 6$. For example, here we have chosen a large (in comparison to the critical one) value of this parameter so the maximum coordinates $\bar{\mu}(y)$ are not too close to each other.

The points of extremes of $\bar{\mu}(y)$ can be found from the equation (2.13). This equation allows us to determine the local maximums y_1, y_{11}, y_{12} and local minimums y_2, y_{21}, y_{22} of the function $\bar{\mu}(y)(\bar{y})$. Taking into account the equality (2.17) we obtain the following equation

$$K_2(\bar{y}, p, \mu)/K_0(\bar{y}, p, \mu) - (K_1(\bar{y}, p, \mu)/K_0(\bar{y}, p, \mu)) = 1/p \quad (3.5)$$

for the extremum points of the function $\bar{\mu}(y)$. Comparing the latter expression to the condition (2.14) easy to see that the solutions \bar{y} of this

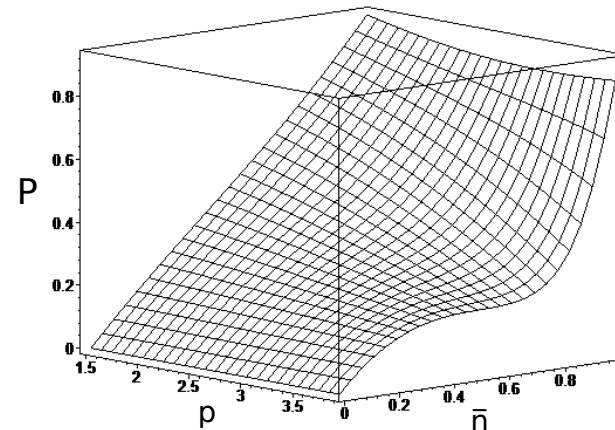


Figure 3: Plot of pressure as a function of density and parameter of attraction p (see equation (3.4)) at temperature higher then critical one ($a = 1.2, v = 12$).

equation do not meet the condition $\max E(\bar{y}, p, \mu)$. However this solutions determine specific points, which divide the region of values \bar{y} into intervals. In particular, in the interval (Figure 4)

$$\bar{y} \in (0, y_1) \quad (3.6)$$

where $y_1 = 1.267510$, the function $E_0(\bar{y}, \mu)$ has its maximum ($E_2(\bar{y}, p, \mu) < 0$), while at

$$\bar{y} \in (y_1, y_2) \quad (3.7)$$

where $y_2 = 4.755127$, this function meet the condition of minimum ($E_2(\bar{y}, p, \mu) > 0$). The following range of values (see also Figure 7)

$$\bar{y} \in (y_2, y_{11}); \quad \bar{y} \in (y_{21}, y_{12}) \quad (3.8)$$

correspond to the maximum of $E(\bar{y}, p, \mu)$ etc. This function has its minimum at

$$\bar{y} \in (y_1, y_2); \quad \bar{y} \in (y_{11}, y_{21})$$

therefore there is no need to consider them anymore. Here $y_{11} = 7.244287$, $y_{21} = 10.763368$. For the purpose of calculating the integral (2.7) we are interested in $\max E(\bar{y}, p, \mu)$ and intervals of values of \bar{y} in (3.6), (3.8).

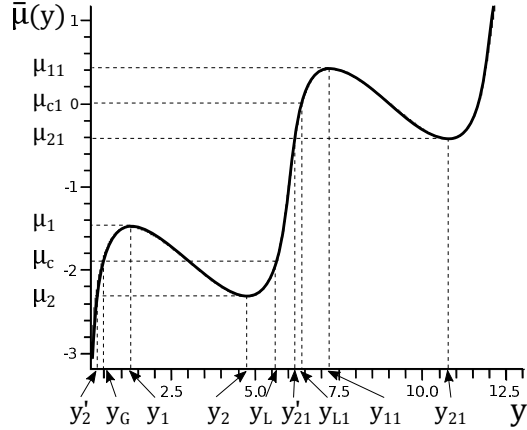


Figure 4: Plot of the chemical potential as a function of \bar{y} at $p = 6$ in the region of monotonic dependence.

In the region of values $\bar{y} \in (0, y_2')$ the function $\mu(\bar{y})$ is monotonic increasing. Here each value of \bar{y} has specific corresponding value of $\bar{\mu}(y) < \mu_2$. So the form of the equation of state for \bar{y} in this interval coincides with (2.20).

Consider in more details the region of the first maximum of the function $\bar{\mu}(y)$ (Figure 4). There are multiple solutions \bar{y} in the interval

$$\bar{\mu}(y) \in (\mu_2, \mu_1) \quad (3.9)$$

where $\mu_2 = -2.308041$, $\mu_1 = -1.470040$. It is necessary to find out which one leads to $\max E(\bar{y}, p, \mu)$.

Let us introduce the function

$$E_G(\bar{y}, p, \bar{\mu}(y)) = E(\bar{y}, p, \bar{\mu}(y))\Theta(y_1 - \bar{y}), \quad (3.10)$$

corresponding to the values $\bar{y} \in (0, y_1)$, and the function

$$E_L(\bar{y}_1, p, \bar{\mu}(y)) = E(\bar{y}, p, \bar{\mu}(y))\Theta(\bar{y} - y_2)\Theta(y_{11} - \bar{y}), \quad (3.11)$$

defined in the interval $\bar{y} \in (y_2, y_{11})$, y_{11} is the coordinate of the second local maximum of $\bar{\mu}(y)$ (Figure 4). Each of these functions is a function of chemical potential. The interval of values (3.7) is not taken into account since $E(\bar{y}, p, \mu)$ fails to meet the condition of maximum there.

The function $E(\bar{y}, p, \mu)$ at $p = 6$ is monotonic for all

$$\bar{\mu}(y) \in (-\infty, \mu_2), \quad (3.12)$$

and also for

$$\bar{\mu}(y) \in (\mu_1, \mu_{21}). \quad (3.13)$$

where $\mu_{21} = -0.4173780$. The value μ_{21} of the chemical potential correspond to the solution of the equation (3.8) at $\bar{y} = y_{21}$. Appliance of the Laplace method for calculation of the integral (2.7) in regions of the chemical potential (3.12) and (3.13) is obvious, since for each \bar{y} one has a single extremum (maximum) value of $E(\bar{y}, p, \mu)$. One should pay additional attention to values of the chemical potential (3.9). It is necessary to find which function either (3.10) or (3.11) has larger value. Easy to make sure that at $\bar{\mu}(y) = \mu_2$ we have

$$E_G(\bar{y}, p, \mu_2) > E_{0L}(\bar{y}, \mu_2),$$

and at $\bar{\mu}(y) = \mu_1$

$$E_G(\bar{y}, p, \mu_1) < E_L(\bar{y}, p, \mu_1).$$

Therefore in range of values (3.9) for all $p > p_c(a)$ there exist such μ_c that

$$E_G(y_G, p, \mu_c) = E_L(y_L, p, \mu_c), \quad (3.14)$$

moreover $y_G \leq y_L$. Here the sign of equality refer to case of $T = T_c$. Easy to see that for all $\mu < \mu_c$ we have $E_G > E_L$, and for $\mu > \mu_c$ the maximal one will be $E_L > E_G$.

Figure 5 represent the plot of $E(\bar{y}, p, \mu)$ at two fixed values of $\bar{\mu}(y)$ (at $p = 6, v = 12, a = 1.2$). The critical value of the chemical potential in case of mentioned above values of the parameters is $\mu_c = -1.890291$. Case a) correspond to $\mu_a < \mu_c$, case b) to $\mu_b > \mu_c$. Note that μ_a is close to μ_2 ($\mu_a \geq \mu_2$), and μ_b is close to μ_1 ($\mu_b \leq \mu_1$). At smaller value of the chemical potential μ_a the maximum on the left-hand side is larger then the one on the right-hand side. At larger one μ_b we have the opposite situation. In other words, at small values of the chemical potential $\mu \in (-\infty, \mu_c)$ the function $E_G > E_L$, and at larger ones $\mu \in (\mu_c, \mu_{21})$ we have $E_L > E_G$. Same situation occurs in the region of values $\mu \in (\mu_{21}, \mu_{11})$. In the next section we consider this in more details.

Quantitative calculation of μ_c is the following. Firstly we consider the coordinates of extremes point of the curve $\mu(\bar{y})$ in the interval

$$\bar{y} \in (0, y_{11}), \quad (3.15)$$

where y_{11} is the coordinate of the second larger maximum of this function (Figure 4). Applying the equation (3.5) for this purpose allows us to find the extremum points $y_1, y_2; y_{11}, y_{21}$ and so on. Using this values in (2.13)

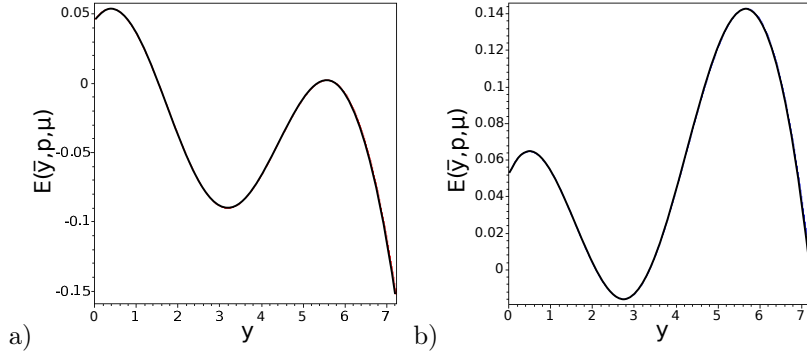


Figure 5: Plot of the function $E(\bar{y}, p, \mu)$ at two fixed values of the chemical potential a) $\mu_a = -1.950$, b) $\mu_b = -1.800$ ($\mu_c = -1.890291$, $p = 6$, $a = 1.2$, $v = 12$).

we obtain corresponding values of the chemical potential $\mu_1, \mu_2, \mu_{11}, \mu_{21}$ (Figure 4).

The following scheme serves to calculate the chemical potential μ_c , see (3.14). The value of $\mu^{(1)}$ matching the coordinate $y_s = (y_1 + y_2)/2$ is obtained from the equation (2.13)

$$\mu^{(1)} = \bar{\mu}(y_s).$$

In the interval (3.15) there are two more values of \bar{y} , namely $y_G^{(1)} < y_s$ and $y_L^{(1)} > y_s$, corresponding to $\mu^{(1)}$

$$\begin{aligned} y_G^{(1)} &= \bar{y}(\mu^{(1)}), & y_G^{(1)} &\in (0, y_1), \\ y_L^{(1)} &= \bar{y}(\mu^{(1)}), & y_L^{(1)} &\in (y_2, y_{11}) \end{aligned}$$

Then $E_G(y_G^{(1)}, p, \mu^{(1)})$ should be compared with $E_L(y_L^{(1)}, p, \mu^{(1)})$. In case they are not equal one has to find $\mu^{(2)}$ from the condition

$$E_G(y_G^{(2)}, p, \mu^{(2)}) = E_L(y_L^{(2)}, p, \mu^{(2)}).$$

and to repeat the applied for $\mu^{(1)}$ procedure of calculation for the value $\mu^{(2)}$. This is the way to find μ_c which meet the condition (3.14) with pre-set accuracy. The value $\mu_c = \lim_{n \rightarrow \infty} \mu^{(n)} = -1.890291$ allows to set the mechanism of the first order phase transition. The chemical potential is the control parameter since we consider the grand canonical partition.

In the limit $\mu \rightarrow -\infty$ we have $\bar{y} = 0$, as follows from (2.13). The increase of μ cause the increase of \bar{y} . In the range of values

$$\mu \in (-\infty, \mu_c)$$

the value \bar{y} increase starting from $\bar{y} = 0$ to $\bar{y} = y_G$ ($y_G = \lim_{n \rightarrow \infty} y_G^{(n)} = 0.420875$). The further increase of μ cause changing the function $E(\bar{y}, p, \mu)$ from $E_G(\bar{y}, p, \mu)$ at $\mu = \mu_c - 0$, to the function $E_G(\bar{y}, p, \mu)$ at $\mu = \mu_c + 0$. This situation is accompanied by jump of \bar{y} starting from y_G to y_L ($y_L = \lim_{n \rightarrow \infty} y_L^{(n)} = 5.621854$). Moreover at $T = T_c$ we have $y_L(T_c) = y_G(T_c)$. With decrease of temperature ($T < T_c$) the value

$$\Delta y(T) = y_L(T) - y_G(T)$$

increases.

3.3. Equation of state in the region of low densities and temperatures $T < T_c$

Taking into account previous results we can write the equation of state in the region of low densities \bar{n} at $p > p_c(a)$. Let us denote

$$P_{\bar{n}} v = -\frac{p}{2} \bar{n} + \ln \sum_{m=0}^{\infty} \frac{v^m}{m!} e^{-\frac{ap}{2} m^2} e^{p \bar{n} m} e^{\beta \mu(p, \bar{n}) m}, \quad (3.16)$$

where $\mu(p, \bar{n})$ is the solution of the equation (3.3). There are some fixed values of the density $\bar{n} = \bar{y}/p$ at $T < T_c$. The first one appear in the region of the first maximum of $\bar{\mu}(y)$ (Figure 4)

$$n_G = y_G/p \quad (3.17)$$

It correspond to the value \bar{y} in the interval $\bar{y} \in (0, y_1)$ for the chemical potential μ_c (3.14). At $p = 6$ we have $n_G = 0.070146$.

The second fixed value is

$$n_L = y_L/p \quad (3.18)$$

($n_L = 0.936975$ at $p = 6$). This one correspond to the value \bar{y} on the interval $\bar{y} \in (y_2, y_{11})$ at $\mu = \mu_c$. Moreover $n_L \geq n_G$ for all $T \leq T_c$.

There is a fixed value of the chemical potential in the region of the second maximum

$$n'_{21} = \bar{y}'_{21}/p, \quad (3.19)$$

(for $p = 6$ we have $\bar{y}'_{21} = 6.163534$, $n'_{21} = 1.027256$), corresponding to the coordinate y'_{21} which is calculated at μ_{21} (Figure 4). The values (3.17), (3.18) and (3.19) can be obtained from the equation (3.3) in different intervals of the variable \bar{y} .

The equation of state of the system in the region of densities

$$n \in (0, n'_{21}) \quad (3.20)$$

including the values of \bar{y} starting from zero to y'_{21} has the following form

$$Pv = P_{\bar{n}}v\Theta(n_G - \bar{n}) + P_{n_G}v\Theta(\bar{n} - n_G)\Theta(n_L - \bar{n}) + P_{\bar{n}}v\Theta(\bar{n} - n_L)\Theta(n'_{21} - \bar{n}). \quad (3.21)$$

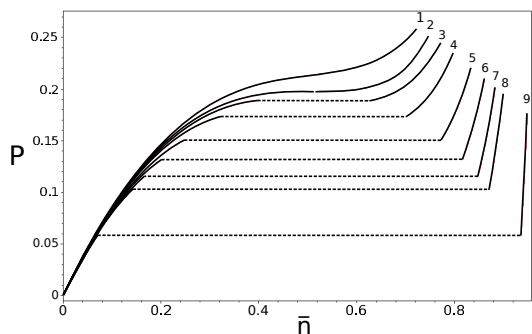


Figure 6: Isotherms of pressure as a function of density at $p > p_c(1.2)$ in the region of low densities (3.20). Curve 1 corresponds to $p = 3.8 < p_c(1.2)$. The curves 2–9 correspond to $p \geq p_c$: $p = p_c(1.2)$ (curve 2), $p = 4$ (curve 3), $p = 4.135$ (curve 4), $p = 4.3647$ (curve 5), $p = 4.5824$ (curve 6), $p = 4.8$ (curve 7), $p = 5$ (curve 8), $p = 6$ (curve 9).

Moreover the first term in (3.21) describes the behavior of the phase I with the lowest density, the third term – the phase II, which has larger density than the phase I at all $T < T_c$. At $T = T_c$ both phases has equal densities and at $T > T_c$ there exist only one density which is a monotonic increasing function of pressure (Figure 3).

The second term in (3.21) indicate that pressure remains constant and equal to

$$P_{n_G}v = -\frac{p}{2}\bar{n}_G + \ln \sum_{m=0}^{\infty} \frac{v^m}{m!} e^{-\frac{ap}{2}m^2} e^{p\bar{n}_G m} e^{\beta\mu_c m}.$$

in the interval

$$\bar{n} \in (n_G, n_L) \quad (3.22)$$

Note that the condition (3.14) gives $P_{n_G} = P_{n_L}$, where P_{n_L} is as follows

$$P_{n_L}v = -\frac{p}{2}\bar{n}_L + \ln \sum_{m=0}^{\infty} \frac{v^m}{m!} e^{-\frac{ap}{2}m^2} e^{p\bar{n}_L m} e^{\beta\mu_c m}.$$

In case of $p > p_c(a)$ there are unattainable densities in the interval (3.22) from the point of varying the chemical potential $\bar{\mu}(y)$. When $\bar{\mu}(y)$ tends to μ_c from the left-hand side we have the density region

$$\bar{n} \in (0, n_G). \quad (3.23)$$

When $\bar{\mu}(y)$ tends to μ_c from the right-hand side

$$\bar{n} \in (n_L, n'_{21}). \quad (3.24)$$

The density region (3.22) in between does not exist. The solutions of the phase I, where the density varies in the interval (3.23), transit by jump into the solutions of the phase II, with (3.24). This is connected with change of derivative value of the chemical potential in the point μ_c .

Figure 6 shows the isotherms of pressure in the interval (3.20) at different values of the parameter p .

3.4. Transition between the phase II and the phase III

The previous section was dedicated to description of behavior of the model in the region of low densities (3.20). It can be conditionally linked to the gas-liquid phase transition, since density in this case vary starting from zero up to some fixed value. Let us describe the behavior of the system for larger values of density $\bar{n} > n'_{21}$ (Figure 7) and show that transition from the phase II to the phase III (with larger density than in the phase II) is possible here. Behavior of $\bar{\mu}(y)$ is of most importance here.

Figure 7 shows the characteristic points of the function $\bar{\mu}(y)$ in the region of the second and the third maximums of the value $\bar{\mu}(y)$ from (2.13).

In the interval of the chemical potential values

$$\bar{\mu}(y) \in (\mu_{21}, \mu_{11}) \quad (3.25)$$

($\mu_{21} = -0.417380$, $\mu_{11} = 0.431080$) we have a non-monotonic behavior of $\bar{\mu}(y)$. Let us define the function

$$E_S(\bar{y}, p, \mu) = E(\bar{y}, p, \mu)\Theta(y_{12} - \bar{y})\Theta(y_{21} - \bar{y}), \quad (3.26)$$

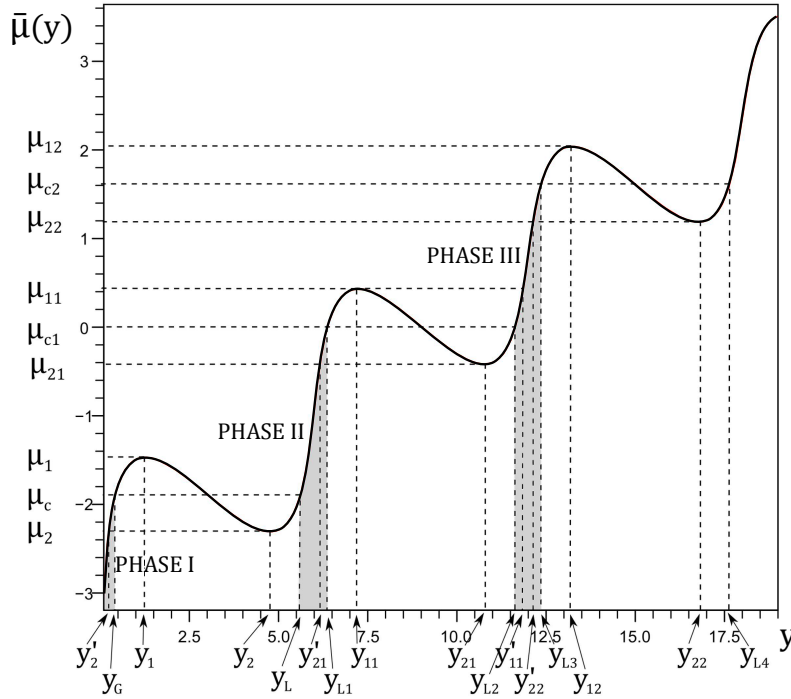


Figure 7: Plot of $\bar{\mu}(y)$ in the region of the second maximum at $p = 6$, $a = 1.2$, $v = 12$.

which is nonzero in the interval

$$\bar{y} \in (y_{21}, y_{12}).$$

where $y_{12} = 13.236413$. The function $\bar{\mu}(y)$ is monotonic in the interval

$$\bar{y} \in (y_L, y'_{21})$$

so the function $E(\bar{y}, p, \mu)$ is characterized by one value \bar{y} determining the integral (2.7), therefore it determines the expression of pressure (3.16). The phase II exists here in "pure" form.

The interval of values

$$\bar{y} \in (y'_{21}, y'_{11})$$

($y'_{21} = 6.163534$, $y'_{11} = 11.862089$) correspond to non-monotonic dependence of $E(\bar{y}, p, \mu)$ on \bar{y} . Therefore it is worthily to compare the functions

$E_L(\bar{y}, p, \mu)$ from (3.11) and $E_S(\bar{y}, p, \mu)$ from (3.26) at all values of the chemical potential from the interval (3.25). As well as in case of transition from the phase I to the phase II, there exist the chemical potential μ_{c_1} , such that

$$E_L(y_{L1}, p, \mu_{c_1}) = E_S(y_{L2}, p, \mu_{c_1}),$$

where

$$y_{L1} \in (y'_{21}, y_{11}) \quad \text{and} \quad y_{L2} \in (y_{21}, y_{12}).$$

The way of calculating μ_{c_1} is the same as represented in Section 4 for the case of μ_c . Figure 8 shows the pressure as a function of \bar{y} at $p = 4$, $p = 5$, $p = 6$ and $p = 8$ including the region of the first and the second local maximums of the curve $\bar{\mu}(y)$. Evidently, except the transition from the phase I to the phase II (at $y \in (y_G, y_L)$) there exist the transition from phase II to phase III, where the jump of the order parameter occurs in the interval of values

$$\bar{y} \in (y_{L1}, y_{L2}),$$

where $y_{L1} = 6.373312$, $y_{L2} = 11.641773$, moreover y_{L1} characterizes the maximal density of the phase II, and y_{L2} – the minimal density of the phase III at correspondent values of p .

The further growth of the chemical potential yield the cascade of phase transitions from the phase f to the phase $(f+1)$. Each subsequent cascade correspond to larger densities. Note that this model has more than a single critical point. Each of them is characterized by its critical temperature, critical density, and exist in appropriate cascade.

Sufficient growth of the attraction parameter p causes the possibility of transition of the system from the phase I to the phase III omitting the phase II. As we can see from the numerical results there exist some value of the parameter p_T , that at $p > p_T$ there occurs a competition between $E_G(\bar{y}, p, \mu)$, $E_L(\bar{y}, p, \mu)$ and $E_S(\bar{y}, p, \mu)$ aimed to find out which one of this functions is maximal at the same value of the chemical potential (see Figure 9). According to the definition (3.10) the variable \bar{y} of the function $E_G(\bar{y}, p, \mu)$ takes small values $\bar{y} \in (0, y_1)$, in $E_L(\bar{y}, p, \mu)$ this variable changes in the interval $\bar{y} \in (y_2, y_{11})$, where $y_2 > y_1$, and in $E_S(\bar{y}, p, \mu)$ – $\bar{y} \in (y_{21}, y_{12})$, moreover $y_{21} > y_{11}$. Note that in the region of values $p_c(1.2) < p < p_T$ correspondent to the temperature interval

$$T_T < T < T_c \tag{3.27}$$

there occur only the sequence of phase transitions phase I – phase II, phase II – phase III etc. This is because each $\bar{\mu}(y)$ has not more than three extremum points (two of them correspond to $\max E(\bar{y}, p, \mu)$).

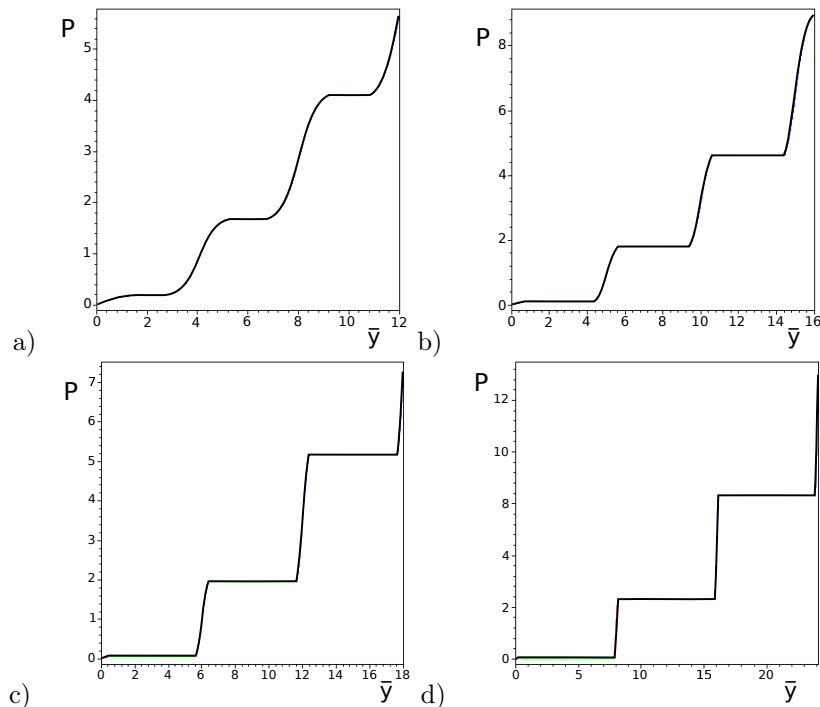


Figure 8: Plot of the pressure as a function of density in the region of first three maximums of dependence $\bar{\mu}(y)$ at $a = 1.2$, $v = 12$: a) $p = 4$, b) $p = 5$, c) $p = 6$, d) $p = 8$.

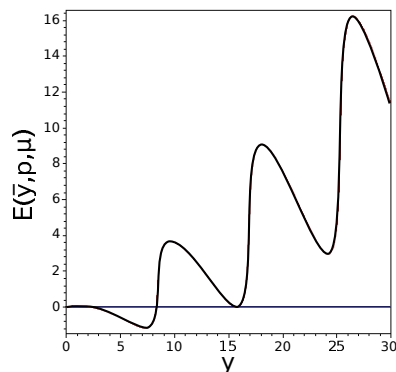


Figure 9: Comparison of values of the function $E(\bar{y}, p, \mu)$ at large attraction parameter $p = p_T$.

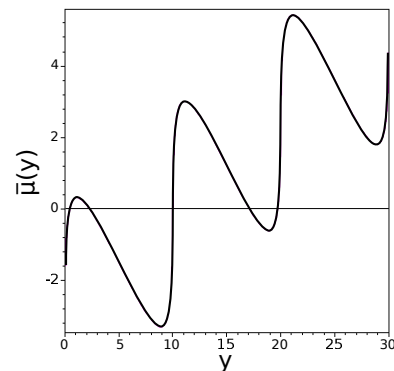


Figure 10: Plot of the function $\bar{\mu}(y)$ at $p = 10$.

At $p > p_T$ there is a situation where five extremum points of the function $E(\bar{y}, p, \mu)$ (three of them are maximums of $E(\bar{y}, p, \mu)$) correspond to single value of $\bar{\mu}(y)$ (see Figure 10). In this case it is worthily to find out which one of them is the largest. To do this we have to compare the functions $E_G(\bar{y}, p, \mu)$, $E_L(\bar{y}, p, \mu)$ and $E_s(\bar{y}, p, \mu)$ at some values of μ . There are two possible cases. In the former $E_G(\bar{y}, p, \mu)$ transits to $E_L(\bar{y}, p, \mu)$ at $\mu = \mu_c$, and $E_L(\bar{y}, p, \mu)$ transits to $E_s(\bar{y}, p, \mu)$ at $\mu = \mu_{1c}$. Moreover $\mu_{1c} > \mu_c$. This situation occurs in the region of temperatures (3.27). In the latter case $E_G(\bar{y}, p, \mu)$ transits to $E_s(\bar{y}, p, \mu)$ at once at $\mu = \mu_s$, namely there occur the transition of the phase I to the phase III, omitting the phase II. This situation can happen only in the region of $p > p_T$ which match the temperature region $T < T_T < T_c$. There are important quantitative results which show that first of all for (3.1) we have at $a = 1.2$

$$p_T = 8.440525.$$

and secondly the direct transition between the phases I and III is impossible. Therefore we have the sequence of phase transitions between neighboring phases.

3.5. Investigation of dependence on the parameter a

This parameter characterizes the relation between repulsive J_1 and attractive J_2 components of the interaction potential. The represented above calculations were executed at $a = 1.2$ and $v = 12$ from (3.1). Let us see how the results change when the parameter a vary in the interval $1 < a < 10$. In accordance to the condition of stability (2.2) $a > 1$. The parameter v will be the same as previously $v = 12$.

The quantitative results show that changing the value of a has no influence on existence of the phase transition in the model. It only lead to slightly different absolute value of the critical temperature which is defined in (3.2). The latter fact is natural, since T_c depends on the parameter p which is the repulsive component of the interaction potential and on the value $p_c(a)$ separating monotonic behavior of the extremum value of the chemical potential from non-monotonic one.

However note that the critical value of the chemical potential increases sufficiently when a grows ($v = 12$):

$$\begin{aligned} a = 0.0001 & \quad \mu_{c1}^1 = -2.516, & p_c(1) = 3.8255, \\ a = 1.2 & \quad \mu_{c1}^{1.2} = -2.105, & p_c(1.2) = 3.9282, \\ a = 2 & \quad \mu_{c1}^2 = -0.4866, & p_c(2) = 3.9973, \\ a = 10 & \quad \mu_{c1}^{10} = 15.5196, & p_c(10) = 4.0000. \end{aligned}$$

It means that with growth of repulsive part of the interaction potential (with respect to the attractive one) the phase transition occur at more larger value of the chemical potential.

Numerical way of calculating $p_c(a)$ is the following. The function $\bar{\mu}(y)$ has an inflection point in the critical point. Namely the following equalities are held

$$\frac{\partial \bar{\mu}(y)(y)}{\partial \bar{y}} = 0.$$

$$\frac{\partial^2 \bar{\mu}(y)(y)}{\partial \bar{y}^2} = 0,$$

If these equalities are held simultaneously we obtain $\bar{y} = \bar{y}_c$ and $p = p_c(a)$.

Figure 11a represent the isotherms of the equation of state at $v = 12$, $a = 1.0001$, and different p . Easy to see that curves of isotherms have no intersection at any p

The case of $a = 1.0001$ is shown on Figure 11b. At low p this curves also don't intersect but the isotherms where $p = 6$ and $p = 8$ cross in the phase transition in the third cascade. The same situation occur at $a = 1.02$ (Figure 11c). Here we have a crossing in the third cascade of the phase transition (particularly in case of $p = 5$, $p = 6$ and $p = 8$). Besides the values of pressure coincide in cases of $p = 6$ and $p = 8$ in the phase transition point of the second phase transition cascade.

In case of $a = 1.2$ and different p isotherms in the first cascade has no crossing, but there are numerous intersections in the second and the third cascades.

4. Conclusions

We made the accurate calculation of the grand partition function of single-sort cell model with Curie-Weiss potential. Consequently it is found that this model has a sequence of first order phase transitions at temperatures below the critical one T_c . The critical temperature value depends on two parameters: attractive component of the interaction potential J_1 and the parameter a . We proved the existence of multiple thermodynamic phases at the same values of the extensive model parameters – temperature and chemical potential. In contrast to the approach of [1], we deal directly with thermodynamic phases in the grand canonical setting. We provided detailed quantitative analysis at fixed values of parameters of the model in wide region of densities and temperature. Behavior of the pressure as a function of density is explored for first

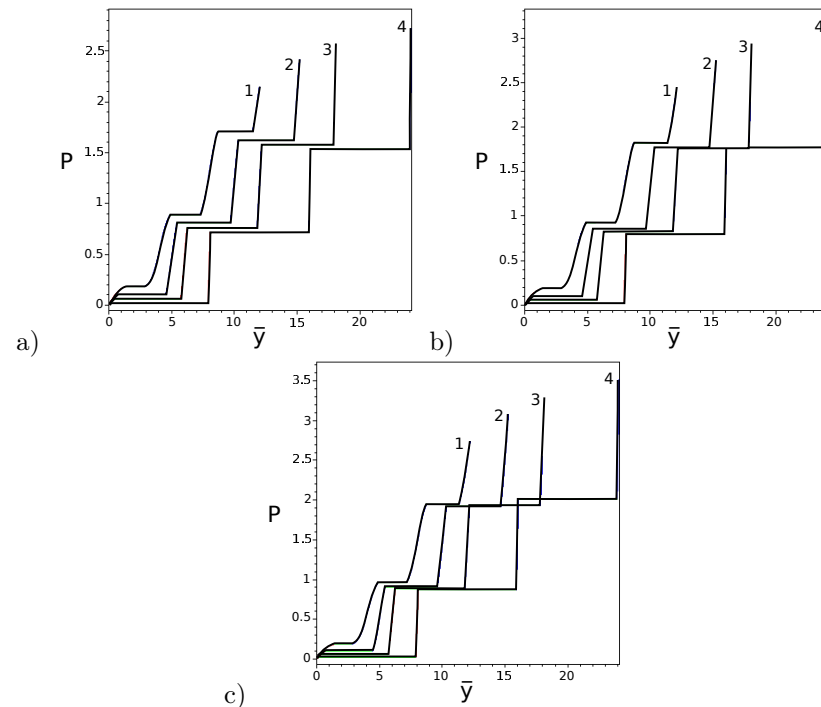


Figure 11: Isotherms of pressure as a function of density at $v = 12$, a) correspond to $a = 1.0001$, b) – $a = 1.01$ and c) – $a = 1.02$. Curve 1 is for $p = 4$, curve 2 – $p = 5$, curve 3 – $p = 6$ and curve 4 – $p = 8$.

three cascades of first-order phase transitions at temperatures below the critical one T_c . To the best of our knowledge, this is the first result of this kind.

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