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TIME–ASYMMETRIC SCALAR AND VECTOR INTERACTIONS IN THE TWO–DIMENSIONAL MODEL OF THE FRONT FORM OF DYNAMICS

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Часоасиметричні скалярна та векторна взаємодії у двовимірній моделі фронтової форми динаміки

В.Шпитко

Анотація. В рамках фронтальної форми динаміки у двовимірному просторі–часі розглянуто релятивістичну двочастинкову систему із часоасиметричними скалярною та векторною взаємодіями. Гамільтонів опис дозволяє продовжити рух поза критичні точки і отримати гладкі світові лінії, які приводять до квазікласичного наближення, яке добре узгоджується із квантовими результатами.

Time-asymmetric scalar and vector interactions in the two-dimensional model of the front form of dynamics

V. Shpytko

Abstract. Relativistic two-particle system with the time-asymmetric scalar and vector interactions in the two–dimensional space–time is considered within the framework of the front form of dynamics. Hamiltonian formalism permits to extend the motion beyond singular points and to obtain smooth world lines which lead to semiclassical approximation, which is in accord with the quantum results.

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Introduction

The relativistic direct interactions theory (RDIT) [1]–[3] describes particle systems in Poincaré–invariant way, using finite number of degrees of freedom. Poincaré–invariance means that one can formally apply this theory in the case of arbitrary velocities ($v < c$). Such a theory is physically meaningful only in the region of relatively little velocities when there are not creation and annihilation processes. One immediately asks the questions: what is the point in having a Poincaré–invariant theory whose validity does not extend on the all region $v < c$? Could an ability of RDIT to describe in a formal way Poincaré–invariant particle systems mean that it possible to construct on the base of RDIT a more general relativistic particle description which also deals with finite (maybe changeable) number degrees of freedom?

This article does not answer none of these questions. It is not our aim and we do not know at the moment how to construct the description which permits to describe physical phenomena on the boundary between relativistic mechanics and field theory. We investigate only the behaviour of sufficiently simple relativistic two–particle models with field type interactions in the essential relativistic region where this boundary should exist. It turns out that the basic Fokker–action integral does not contain the information about the whole evolution of the system. Using the Hamiltonian description within the framework of the front form of dynamics we construct smooth world lines of particles in $\mathbb{M}_2$. Particles reach the speed of light and do not destroy smoothness of world lines. Moreover we demonstrate that physically sensible mass spectra in semiclassical approximation need information about the whole evolution of the system and obtained in this article smooth world lines.

The RDIT allows a wide class of exactly solvable classical and quantum two-particle phenomenologic models [4]–[9] as well as models connected with the field theory [10]–[14] via Fokker-type action integrals [15]–[21]. The most interesting models are those, which may be interpreted in terms of massless field of integer spin [23,24]. The simplest of them are so–called time-asymmetric models, when one particle responds only to retarded field and the other particle responds only to advanced field. These models have been considered in the four-dimensional Minkowski space $\mathbb{M}_4$ in Refs [8], [14], [13], [18] – [21] and in two-dimensional one in Refs [10], [12], [25]. The time-asymmetric case leads to ordinary differential equations of motion in contrast to the time-symmetric one. For the choice of the time-symmetric Green's function the equations of motion of the 2-body problem are differential-difference equations

[23,26,27]. This makes the problem hard to handle and requires consideration of approximate solutions [27] or very special exact one [28] only.

In Refs [10], [12] vector and scalar time–asymmetric interactions on the line (2-dimensional Minkowski space $\mathbb{M}_2$) were investigated only in the repulsion case for $M > m_1 + m_2$. In contrast to above-mentioned references and Ref. [25], which deals with the Lagrangian description, we consider here the Hamiltonian description of 2-body system on the line with scalar and vector interactions for repulsive and attractive cases for all values of total mass $M > 0$ in the framework of the front form of dynamics [4], [22].

For vector and scalar time–asymmetric interactions [8] there are in $\mathbb{M}_4$ both the motions which have a good nonrelativistic limit and have not singular points ($dx/dx^0 = c$) as well as the motions without nonrelativistic limit and with singular points. In $\mathbb{M}_2$ the one-dimensional case almost all possible motions (even such, that have nonrelativistic limit) have singular points. The case ($M > m_1 + m_2, g_1 g_2 > 0$) which have been investigated for scalar and vector interactions in Refs [12], [10], respectively, is an exception.

The system is not defined at singular points. How do particles move after the passing through singular points? There is not, of course, a unique way to prolong the motion after singular points. We shall demonstrate that Hamiltonian formalism which is considered in the section 2 suggests the possible solution of this problem. In the sections 3 and 4 we consider the prolongation method for the motion (for scalar and vector interactions respectively) beyond singular points which permits to construct smooth world lines in the two–dimensional Minkowski space. This prolongation allows to construct semiclassical approximation (section 5), which coordinates with the quantum results [29,30].

1. Lagrangian description of time–asymmetric Fokker–type action integrals in the two–dimensional variant of the front form

Among various forms of relativistic dynamics [31] the front form [4] takes special place. In the four-dimensional space-time $\mathbb{M}_4$ the front form is determined by the family Σ_F of simultaneity hypersurfaces: $n_\mu x^\mu = \tau$, $\tau \in \mathbb{R}$ ($n_\mu n^\mu = 0$). This form of dynamics is characterized by the largest set of generators of Poincaré group $\mathcal{P}(1,3)$ which map Σ_F onto itself. The stability group has seven generators [4].

In the two-dimensional space-time $\mathbb{M}_2$ the front form of dynamics corresponds to the foliation $\mathbb{M}_2$ by isotropic hyperplanes

$$x^0 + x = \tau. \tag{1.1}$$

The Poincaré group $\mathcal{P}(1,1)$ has three generators and is an automorphism group of Σ_F. The quantity τ is the evolution parameter of the system [4,22]. The motion of particles is described by functions $x_a(\tau)$, and the parametric equations of world lines have the form $x = x_a(\tau)$, $x^0 = \tau - x_a(\tau)$. The functions $x_a(\tau)$ are defined as solutions of the Hamilton principle $\delta S = 0$ with an action integral

$$S = \int d\tau \mathcal{L}. \tag{1.2}$$

The general structure of the Lagrange function $\mathcal{L}$ is determined by the Poincaré-invariance conditions. The invariance of the family of simultaneity hypersurfaces (1.1) with respect to transformations of the Poincaré group $\mathcal{P}(1,1)$ permits the solutions, which do not contain derivatives higher than first order. The Lagrangian function for N–particle system in this case has the form [22]

$$\mathcal{L} = -\sum_{a=1}^{N} m_a k_a + \sum_{a<b} r_{ab} V_{ab}(r_{ab}k_a, r_{ab}k_b), \tag{1.3}$$

where $k_a = \sqrt{1-2v_a}, v_a = dx_a/d\tau, r_{ab} = x_a - x_b, a,b = \overline{1,N}$, and V_{ab} – arbitrary functions of indicated arguments. As a result of the Poincaré invariance of the Lagrangian function (1.3) there exist three conserved quantities: the energy E, the momentum P, and the center-of-inertia integral of motion K. They have the form [22]

$$E = \sum_{a=1}^{N} v_a \frac{\partial \mathcal{L}}{\partial v_a} - \mathcal{L}, \quad P = \sum_{a=1}^{N} \frac{\partial \mathcal{L}}{\partial v_a} + E,$$
$$K = -\tau P - \sum_{a=1}^{N} x_a \frac{\partial \mathcal{L}}{\partial v_a} . \tag{1.4}$$

The formalism of Fokker–type action integrals [15,16,23] is one of the most meaningful branches of the classical RDIT. It gives a possibility to connect classical relativistic mechanics and classical field–like description. We are interested in the Fokker–type action integral which has the following form [16]

$$S = S_f + S_{int} = -\sum_{a=1}^{N} \int \sqrt{\dot{x}_a^2} d\tau_a -$$
$$- \sum_{a<b} g_a g_b \int d\tau_a \int d\tau_b \sqrt{\dot{x}_a^2}\sqrt{\dot{x}_b^2} F(\omega_{ab}) G(\rho_{ab}), \tag{1.5}$$

where function F describes a particle interaction and

$$\rho_{ab} = \epsilon_{ab}\sqrt{\eta_{\mu\nu}(x_a^\mu - x_b^\mu)(x_a^\nu - x_b^\nu)} \equiv \epsilon_{ab}|(x_a - x_b)^2| \ ,$$
$$\omega_{ab} = \frac{\eta_{\mu\nu}\dot{x}_a^\mu \dot{x}_b^\nu}{\sqrt{\dot{x}_a^2 \dot{x}_b^2}} \equiv \frac{(\dot{x}_a \dot{x}_b)}{\sqrt{\dot{x}_a^2 \dot{x}_b^2}} \ , \tag{1.6}$$

$\eta_{\mu\nu}$ is metric tensor of Minkowski space $\mathbb{M}_4$, $\epsilon_{ab} = \mathrm{sign}(x_a^0 - x_b^0)$ and $g_a,\ g_b \in \mathbb{R}$ correspond to particle "charges". $G(\rho_{ab})$ is a Green's function of d'Alembert or Klein–Gordon equation. If $G(\rho_{ab})$ is a Green's function of d'Alembert equation and $F(0) = 1$ then in the nonrelativistic limit ($c \to \infty$) action integral (1.5) becomes the action integral of nonrelativistic Coulomb problem. The choice of time–symmetric Green's function of d'Alembert equation $G(\rho_{ab}) = \delta(\rho_{ab}^2)$ leads to the Wheeler–Feynman type field theories. In the case $F = \omega_{ab}$ we obtain the Wheeler–Feynman electrodynamics. With the time–symmetric Green's function we can eliminate the field from the consideration. But in this case we obtain nonlocal (in time) Lagrangians and as a result the difference–differential equation of motion [16,27].

In the case of time–asymmetric Green's function of d'Alembert equation

$$G(\rho_{ab}) = (1 + \rho_{ab}/|\rho_{ab}|)\delta(\rho_{ab}^2) = 2\Theta(x_a^0 - x_b^0)\delta(\rho_{ab}^2). \tag{1.7}$$

we get usual local single–time Lagrangians in the four–dimensional Minkowski space in the light–cone form of dynamics for two–particle system [20] and in the two–dimensional space–time in the front form of dynamics for N–particle system [22].

Let us substitute the time–asymmetric Green's function (1.7) into (1.5). Then in the two-dimensional space-time in the front form for the two–particle system the action integral (1.5) leads to the Lagrangian [22]

$$\mathcal{L} = -\sum_{a=1}^{N} m_a k_a - \frac{g_1 g_2 k_1 k_2}{|r|} F(\delta), r > 0, \tag{1.8}$$

where

$$\delta = \left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)/2 \tag{1.9}$$

and $r = r_{12}$. The case

$$F(\delta) = T_\ell(\delta) \tag{1.10}$$

where $T_\ell(\delta)$ are Tchebyshev polynomial, corresponds to the particle interaction through a local relativistic massless field of rank ℓ [24], in such a way that a–th particle responds only to retarded field and the b–th particle responds to advanced field. We are going to consider the case of two–particle system with time–asymmetric scalar ($\ell = 0$) and vector ($\ell = 1$) interactions when the function (1.10) has the form

$$F(\delta) = \delta^\ell \ ; \quad \ell = 0, 1 \ . \tag{1.11}$$

Using more convenient in the front form quantities $P_\pm = E \pm P$, we obtain for the Lagrangian (1.8) with the function $F(\delta)$ in the form (1.11) with arbitrary integer $\ell \geq 0$ following integrals of motion

$$P_+ = m_1/k_1 + m_2/k_2 - \frac{2\alpha\delta^{\ell-1}}{|r|}[\delta^2(\ell-1) - \ell] \ , \tag{1.12}$$

$$P_- = m_1 k_1 + m_2 k_2, \tag{1.13}$$

$$K = -\frac{t(P_+ + P_-)}{2} - \frac{x_1 m_1}{k_1} - \frac{x_2 m_2}{k_2} - \frac{2\alpha\delta^{\ell-1}}{|r|}\left[(1+\ell)(x_1+x_2) + (1-\ell)\left(\frac{x_1 k_2^2}{k_1^2} + \frac{x_2 k_1^2}{k_2^2}\right)\right] . \tag{1.14}$$

In the classical mechanics, the Lagrangian function is determined on the tangent bundle $T\mathcal{M}$ [32]. If the configuration space $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^N$, then tangent bundle is a trivial one: $T\mathcal{M} \approx \mathbb{R}^N \times \mathbb{R}^N$. This means, that a single chart with coordinates $(x_1, ..., x_N, v_1, ..., v_N)$ covers the whole $T\mathcal{M}$.

Our configuration space coincides with $\mathbb{R}^2$: $\mathcal{M} = \{(x_1, x_2) \in \mathbb{R}^2 \backslash \{r = 0\} | r > 0\} \approx \mathbb{R}^2$. Hence one can expect that it should not be any complications connected with a global structure. But the relativistic Lagrangian functions and particularly the Lagrangian (1.8) are not determined on the whole $T\mathcal{M}$. The Lagrangian (1.8) is determined on the submanifold defined by conditions

$$v_a \leq 1/2 \ , \quad a = 1, 2. \tag{1.15}$$

Inequalities (1.15) mean that world lines in $\mathbb{M}_2$ are time-like: $|dx_a/dx_a^0| < 1$. This submanifold has not a structure of the vector bundle. We can do not pay attention to this fact if a system does not reach boundaries ($v_a = 1/2$), or at least there is a domain of the initial data, starting from which a system does not reach boundary region. We call such a domain as a "good domain". There is more difficult case when a "good domain" does not exist and for arbitrary initial data a system reaches the points of the boundary region (singular points). System is not defined in singular points. Theorem of existence and uniqueness for Euler-Lagrange differential equations breaks at the singular points. There is not unique way to continue the motion after them. Just the same case takes place for the Lagrangian (1.8). It has been shown in [25] that Lagrangian description for the systems with interaction function (1.11) does not lead to the continuous world lines. We are going to demonstrate that Hamiltonian formalism suggests some solution of this problem.

2. Hamiltonian formalism for two-particle system with scalar and vector time–asymmetric interactions

It is well known that Legendre transformation is a differentiable mapping $\Lambda : T\mathcal{M} \rightarrow T^*\mathcal{M}$. The Legendre transformation associated with the Lagrangian (1.8) with $F(\delta) = \delta^\ell$ has the form

$$p_a = \frac{\partial L}{\partial v_a} = \frac{m_a}{k_a} + \frac{\alpha}{2|r|}\left(1 + \ell + (1-\ell)\frac{k_{\bar{a}}^2}{k_a^2}\right)\delta^{\ell-1}. \tag{2.1}$$

Here $a = 1, 2,\ \bar{a} = 3 - a$. This transformation is a diffeomorphism in the region $\Omega \subset T\mathcal{M} \approx \mathbb{R}^4$, where

$$h_\ell = \det\left\|\frac{\partial^2 L}{\partial v_1 \partial v_2}\right\| = \frac{m_1 m_2}{k_1^3 k_2^3} - \frac{\alpha(m_2 k_1 + m_1 k_2)\delta^{(\ell-2)}(\delta^2(\ell+1) - \ell)}{|r| k_1^3 k_2^3} \text{ exists and } \neq 0, \tag{2.2}$$

and maps the open region $\Omega \subset T\mathcal{M}$

$$\Lambda : \Omega \rightarrow \Lambda\Omega \tag{2.3}$$

into open one $\Lambda\Omega \subset T^*\mathcal{M} \approx \mathbb{R}^4$.

The Hamiltonian case is equivalent to the Lagrangian one only in the region $\Lambda\Omega$ [32]. In the strict sense a motion in the Hamiltonian case is defined on $\Lambda\Omega$ only. In the other words we should consider $\Lambda\Omega$ as a whole phase space of the system. But in this case as we mentioned above one can not obtain continuous world lines [25] excepting the repulsion case ($\alpha > 0$) if the total mass M of the system is great then the sum of particle masses $m = m_1 + m_2$. This is a consequence of the non–existence of a "good domain" for other values of quantities M^2, α. We shall demonstrate that the system describing by the Lagrangian (1.8),(1.11) reaches singular points for which $h_\ell = 0$ or $h_\ell \to \infty$. To obtain smooth world lines we shall consider whole $\mathbb{R}^4$ as a phase space and define the motion in the domain $\mathbb{R}^4 \setminus \Lambda\Omega$.

Let us consider scalar ($\ell = 0$) and vector ($\ell = 1$) interactions. It is possible to solve Eq.(2.1) with respect to velocities in these cases. Solving the system (2.1) with respect to velocities v_a and substituting them in the expressions for conserved quantities, we obtain from (1.4) the generators of the Lie algebra of the Poincaré group $\mathcal{P}(1,1)$.

$$P_+ = p_1 + p_2, \;\; K = x_1 p_1 + x_2 p_2, \tag{2.4}$$

$$P_- = \frac{m_1^2 p_2 + m_2^2 p_1 + A_\ell \alpha/|r|}{p_1 p_2 + (-1)^\ell \alpha^2/|r|^2 + B_\ell \alpha P_+/|r|}, \tag{2.5}$$

where

$$A_\ell = (1-\ell)(2m_1 m_2) - \ell(m_1^2 + m_2^2); \quad B_\ell = -\ell. \tag{2.6}$$

Quantities (2.4), (2.5) satisfy the following Poisson brackets relations

$$\{P_+, P_-\} = 0, \qquad \{K, P_\pm\} = \pm P_\pm. \tag{2.7}$$

The classical total mass squared function $M^2 = P_+ P_-$ has vanishing Poisson brackets with all generators (2.4),(2.5). The separation of external and internal motion is carried out by the choice

$$P_+ = p_1 + p_2 \;, \quad Q = K/P_+ \;; \quad \{Q, P_+\} = 1 \tag{2.8}$$

as new external canonical variables. As internal variables we choose

$$\xi = \frac{m_2 p_1 - m_1 p_2}{P_+}, \quad q = r\frac{P_+}{m}; \quad \{q, \xi\} = 1, \tag{2.9}$$

where $m = m_1 + m_2$. The sign of coordinates difference sign(r) is an integral of motion [22] and $r > 0$ in the region Ω. Therefore we can neglect the module sign because such a Hamiltonian system will be equivalent in the region $\Lambda\Omega$ to the basic Lagrangian one. Then, in terms of variables (2.9) the function M^2 which determines the inner motion of the system has the form

$$M^2 = X/Y, \tag{2.10}$$

where

$$X = m\big(m m_1 m_2 q + m(m_2 - m_1) q\xi + \alpha A_\ell\big),$$
$$Y = m_1 m_2 q + (m_2 - m_1) q\xi - \Big(q\xi^2 + (-1)^\ell \frac{\alpha^2}{q}\Big) + \alpha m B_\ell. \tag{2.11}$$

One can represent phase line equation $M^2 = M^2(q,\xi)$ which describes inner motion as a quadratic equation with respect to momenta ξ

$$(\xi - \xi_M)^2 = \frac{(\mu^2-1)m^2 m_1^2 m_2^2 q^2 - 2\alpha M^2 m_1 m_2 m \mu^\ell q + (-1)^{\ell+1} M^4 \alpha^2}{M^4 q^2}, \tag{2.12}$$

where

$$\xi_M = \frac{(M^2 - m^2)(m_2 - m_2)}{2M^2}, \quad \mu = \frac{M^2 - m_1^2 - m_2^2}{2m_1 m_2} \;. \tag{2.13}$$

The motion is possible in the region where quadratic form

$$\mathcal{D}_\ell = (\mu^2 - 1)m^2 m_1^2 m_2^2 q^2 - 2\alpha M^2 m_1 m_2 m \mu^\ell q + (-1)^{\ell+1} M^4 \alpha^2 \tag{2.14}$$

is non–negative. Then we obtain that for the finite motion $q \in [q_1, q_2]$, where q_1, q_2 – real solutions of quadratic equation $\mathcal{D}_\ell = 0$:

$$q_1 = \frac{2\alpha M^2 (-1)^{\ell+1}}{(M^2 - (m_1 - m_2)^2)m}, \quad q_2 = \frac{2\alpha M^2}{(M^2 - m^2)m}. \tag{2.15}$$

We have to remember that to obtain in the Hamiltonian case Lagrangian picture we must restrict a motion of the system to the region $\Lambda\Omega$, where Hessian h_ℓ (2.2) exists and is positive. For scalar and vector interaction it has the form

$$h_\ell = \Big(m_1 m_2 - \frac{\alpha}{r}(m_1 k_1 + m_2 k_2)(\ell - 1)\Big)\, k_1^{-3} k_2^{-3} \;. \tag{2.16}$$

Solving the system (2.1) with respect to quantities k_a and taking into account Eq.(2.9) we get expressions for k_a in terms of canonical variables P_+, q, ξ. They have the form

$$\kappa_1 = \frac{m(-m_1\xi + m_1 m_2 + m_2 \alpha/q)}{P_+(-\xi^2 + \xi(m_2 - m_1) + m_1 m_2 - \alpha^2/q^2)} \equiv$$

$$\frac{my_1}{P_+y_3}\ ; \tag{2.17}$$

$$\kappa_2 = \frac{m(m_2\xi + m_1m_2 + m_1\alpha/q)}{P_+(-\xi^2 + \xi(m_2 - m_1) + m_1m_2 - \alpha^2/q^2)} \equiv \frac{my_2}{P_+y_3}$$

for the scalar case and the form

$$\kappa_1^{-1} = \frac{P_+}{m_1m}\left(m_1 + \xi - \frac{\alpha}{q}\right) \equiv \frac{P_+}{m_1m}\tilde{y}_1\ ;$$
$$\kappa_2^{-1} = \frac{P_+}{m_2m}\left(m_2 - \xi - \frac{\alpha}{q}\right) \equiv \frac{P_+}{m_2m}\tilde{y}_2 \tag{2.18}$$

for the vector case.

From Eqs.(2.7)–(2.9) and relation

$$H = \frac{M^2 + P_+^2}{2P_+} \tag{2.19}$$

we obtain the following Hamiltonian equation of motion

$$\dot{Q} = 1/2 - \frac{M^2}{2P_+}, \quad \dot{P}_+ = 0, \tag{2.20}$$

$$\dot{q} = \frac{1}{2P_+}\frac{\partial M^2}{\partial \xi}, \quad \dot{\xi} = -\frac{1}{2P_+}\frac{\partial M^2}{\partial q}. \tag{2.21}$$

Using phase trajectory equation (2.12) and solving Eqs(2.21) we obtain law of the particle motion in parametric form

$$t^{\pm} - t_0^{\pm} = \frac{2P_+m^2}{M^2}\left[\frac{m_1 - m_2}{2M^2}q \pm \right.$$
$$\left. \pm \frac{(m_1^2 + m_2^2)M^2 - (m_1^2 - m_2^2)^2}{4M^2m^2m_1^2m_2^2(\mu^2 - 1)}\mathcal{D}_\ell^{1/2} \pm \frac{\alpha M^2\mu^{1-\ell}}{m(\mu^2 - 1)}\mathcal{I}_0\right], \tag{2.22}$$

where $t_0^{\pm}$ are integration constants,

$$\mathcal{I}_0 = \begin{cases} -\dfrac{1}{2mm_1m_2\sqrt{1-\mu^2}} \times \arcsin\dfrac{mm_1m_2(\mu^2-1)q - \alpha M^2\mu^\ell}{M^2\alpha\mu^{1-\ell}} & \mu^2 < 1 \\ \dfrac{1}{2mm_1m_2\sqrt{\mu^2-1}} \times \ln\left|\dfrac{2(mm_1m_2(\mu^2-1)q - \alpha M^2\mu^\ell)}{\sqrt{\mu^2-1}} + 2\sqrt{\mathcal{D}_\ell}\right| & \mu^2 > 1 \end{cases} \tag{2.23}$$

and signs $\pm$ corresponds to the different solutions of the quadratic equation (2.12). Inequality $\mu^2 < 1$ means that $(m_1 - m_2)^2 < M^2 < m^2$ and corresponds to finite motion. Inequality $\mu^2 > 1$ means that $0 < M^2 < (m_1 - m_2)^2$ or $M^2 > m^2$ and is connected with infinite motions. Definition of the front form of relativistic dynamics (1.1) and relations (2.8), (2.9) of particle coordinates x_1, x_2 and canonical coordinates Q, q together with Eq.(2.22) give us equations for world lines in $\mathbb{M}_2$ in parametric form

$$x_1^0(q) = t(q) - x_1(q)\ , \quad x_2^0(q) = t(q) - x_2(q)\ . \tag{2.24}$$

$$x_1(q) = \frac{K}{P_+} + \frac{m_2 - \xi(M^2, q)}{P_+}q\ ;$$
$$x_2(q) = \frac{K}{P_+} - \frac{m_1 + \xi(M^2, q)}{P_+}q\ . \tag{2.25}$$

The world lines in $\mathbb{M}_2$ for time asymmetric scalar case have been obtained by P.Stephas in Ref.[10] and for time–asymmetric vector interaction by R.A.Rudd, R.N.Hill in Ref.[12]. In both articles world lines have been obtained for the the repulsion case for $M > m_1 + m_2$ only by immediate integrating of equation of motion in the two–dimensional space–time that is equivalent to the Lagrangian description. Lagrangian description in the front form of dynamics in $\mathbb{M}_2$ for the field–like time–asymmetric interaction has been investigated by A.A.Mayorov, S.N.Sokolov, V.I.Tretyak in Ref.[25].

The behaviour of the system in the scalar and vector cases is quite different. Therefore we consider the scalar and vector case separately.

3. Scalar interaction

In the scalar case if $\alpha > 0$ the region $\Lambda\Omega$ of the phase plane corresponds to the region $q > 0$ restricted by curves $y_1 = 0$, $y_2 = 0$. If $\alpha < 0$ then indicated region lies between the curves $y_1 = 0$, $y_2 = 0$ to the right of their intersection point. As we can see on the phase portraits (Figs 1, 3, 5, 7, 9, 11) only the phase curves corresponding to $\alpha > 0$, $M^2 > m^2$, lie completely in $\Lambda\Omega$. Just the same case have been considered in Ref.[12]. Phase trajectories which corresponds to other values of parameters α, M^2 pass the boundaries of this region or lie outside. It will be noted that even the Lagrangian description permits the motions for which $h_s < 0$. That corresponds to the parts of phase trajectories which lie between curves $y_1 = 0$, $y_2 = 0$, $q > 0$ if $\alpha < 0$, $\mu^2 < 1$, $\mu < 0$ (Fig. 5) and if $\alpha < 0$, $0 < M^2 < (m_1 - m_2)^2$ (Fig. 11). Then the coordinate q belongs to the interval $\left[q_2, \frac{-\alpha M^2}{m m_1 m_2}\right]$. If we restrict ourself by the region $\Lambda\Omega$ then we obtain continuous world lines only for $M^2 > m^2$, $\alpha > 0$ (Fig. 1). For other values of parameters world lines are not continuous or do not exist (the whole phase curves lie outside the region $\Lambda\Omega$). The same takes place in the Lagrangian case. If we want to obtain continuous (or even smooth) world lines we have to prolong in some way the motion after the system reaches singular points. On the boundary of the region Ω the Lagrangian system is not defined. Therefore there not exists unique method of such a prolongation. The Hamiltonian description suggests possible solution of this problem. We shall regard the whole plane $\mathbb{R}^2$ as the inner phase space of the system. Then the whole curves (not only their parts) shown in the Figs 1, 3, 5, 7, 9, 11 will be phase curves. Taking this into account we obtain from Eqs (2.22), (2.24), (2.25) smooth world lines for every values of quantities M^2, α which are shown in the Figs 2, 4, 6, 8, 10, 12. We put $t_o^+ = 0$ and following values of integration constant

$$\left\{ \begin{array}{l} t_o^- = \dfrac{4\alpha P_+ m\mu^{1-\ell}}{(\mu^2-1)} J_o^{(\mu^2>1)}(q_2), \ \alpha > 0 \\[2ex] t_o^- = \dfrac{4\alpha P_+ m\mu^{1-\ell}}{(\mu^2-1)} J_o^{(\mu^2>1)}(q_1), \ \alpha < 0 \end{array} \right| m^2 < M^2 , \qquad (3.1)$$

$$\left\{ \begin{array}{l} t_o^- = \dfrac{4\alpha P_+ m\mu}{(\mu^2-1)} J_o^{(\mu^2>1)}(q_1), \alpha > 0 \\[2ex] t_o^- = \dfrac{4\alpha P_+ m\mu}{(\mu^2-1)} J_o^{(\mu^2>1)}(q_2), \alpha < 0 \end{array} \right| 0 < M^2 < (m_1 - m^2)^2, \qquad (3.2)$$

$$t_o^- = \frac{\alpha P_+ \pi \mu^{1-\ell}}{m_1 m_2 (1-\mu^2)^{3/2}} , \quad \alpha < 0 , \quad \mu^2 < 1 . \qquad (3.3)$$

In the equation (3.1), (3.3) it is necessary to put $\ell = 0$. The intersection points of the phase trajectories and curves $y_1 = 0$, $y_2 = 0$ correspond the case when one of particles reaches the speed of light. In the region $\mathbb{R}^2 \backslash \overline{\Lambda\Omega}$ (as well as in the region $\Lambda\Omega$) velocities of both particles are less than the speed of light. Here $\overline{\Lambda\Omega}$ is the closure of $\Lambda\Omega$ in $\mathbb{R}^2$ ($\Lambda\Omega \bigcup$ {curves $y_1 = 0$, $y_2 = 0$}).

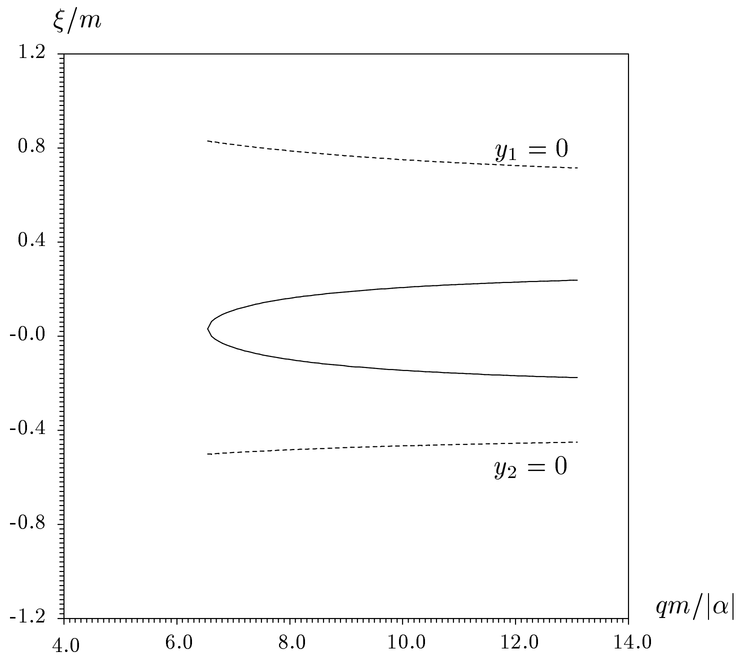

Figure 1. Scalar interaction. Phase trajectories (continuous curves) for Stephas case: $(m_2 - m_1)/m = 0.2$; $M/m = 1.2; \alpha > 0$. Dashed curves $y_1 = 0$, $y_2 = 0$ corresponds to the singularities of Hessian.

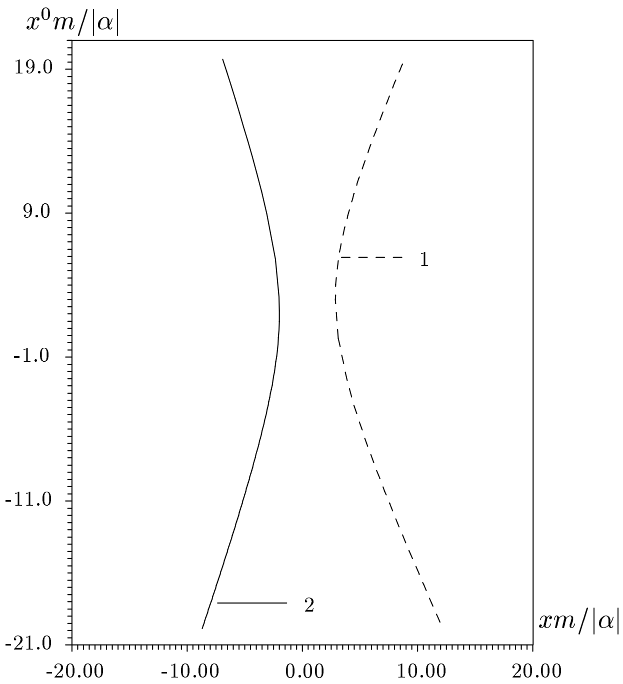

Figure 2. Scalar interaction. World lines in $\mathbb{M}_2$. Stephas case: $(m_2 - m_1)/m = 0.2;\ M/m = 1.2; \alpha > 0$.

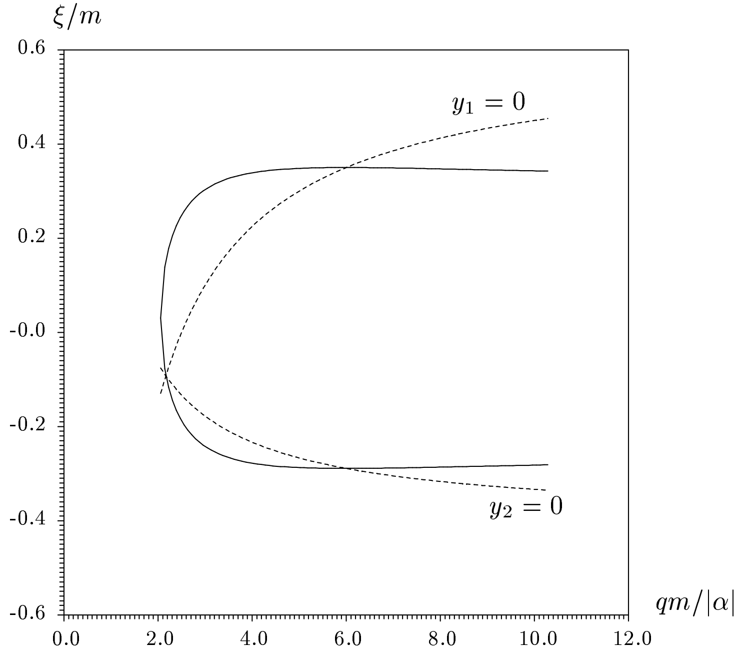

Figure 3. Scalar interaction. Phase trajectories (continuous curves) for following values of parameters: $(m_2 - m_1)/m = 0.2;\ M/m = 1.2; \alpha < 0$. Dashed curves $y_1 = 0,\ \ y_2 = 0$ corresponds to the singularities of Hessian.

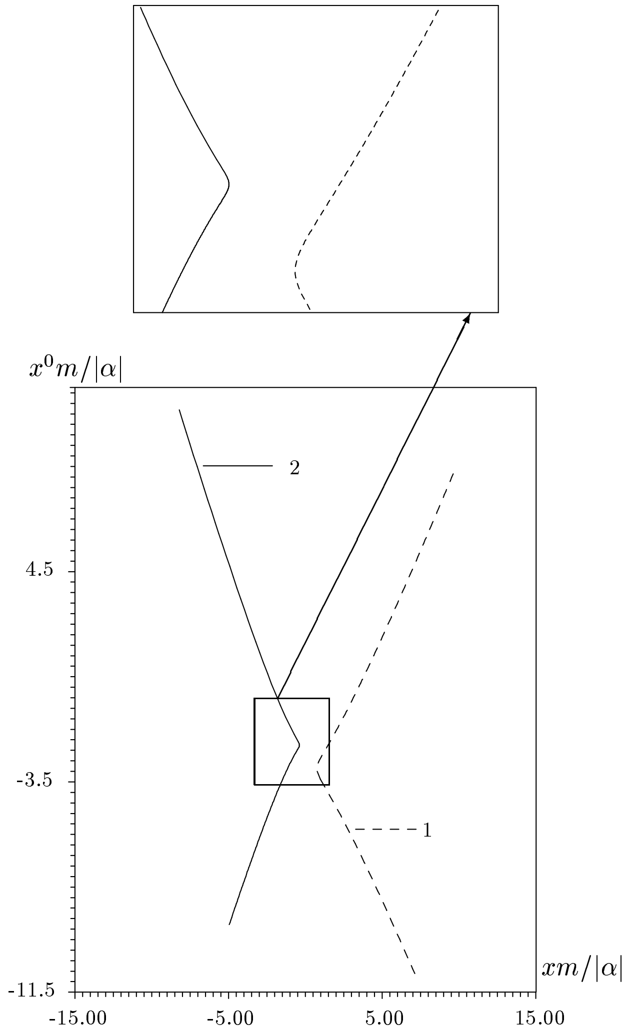

Figure 4. Scalar interaction. World lines in $\mathbb{M}_2$ for following values of parameters: $(m_2 - m_1)/m = 0.2$; $M/m = 1.2; \alpha < 0$.

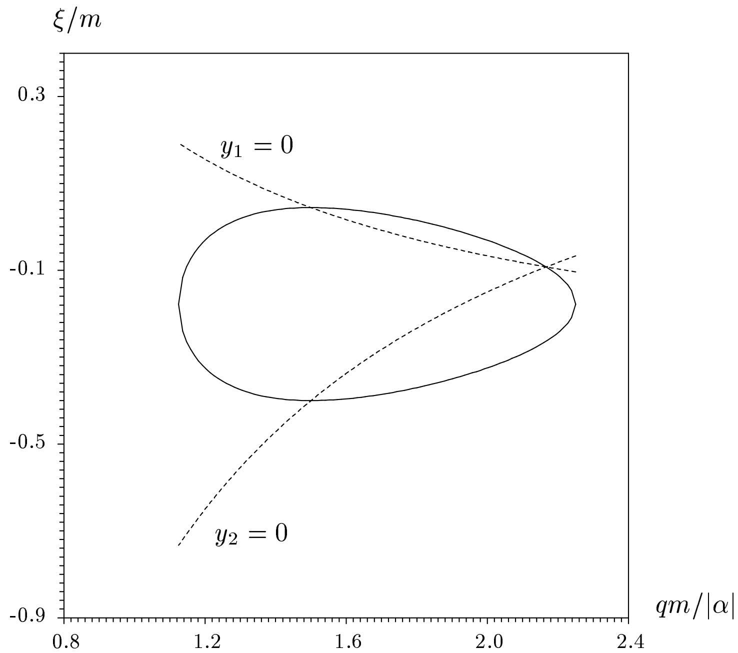

Figure 5. Scalar interaction. Phase trajectories (continuous curves) for finite motion: $(m_2 - m_1)/m = 0.2$; $M/m = 0.6$ $(\mu^2 < 1, \mu < 0)$; $\alpha < 0$. Dashed curves $y_1 = 0$, $y_2 = 0$ corresponds to the singularities of Hessian.

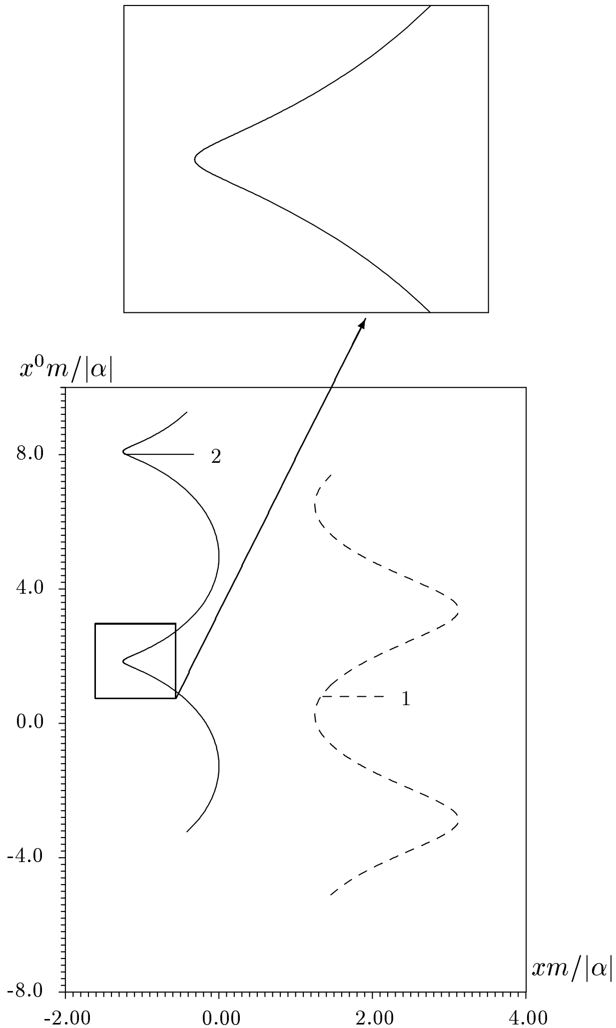

Figure 6. Scalar interaction. World lines in $\mathbb{M}_2$ for finite motion: $(m_2 - m_1)/m = 0.2$; $M/m = 0.6$ $(\mu^2 < 1, \mu < 0)$; $\alpha < 0$.

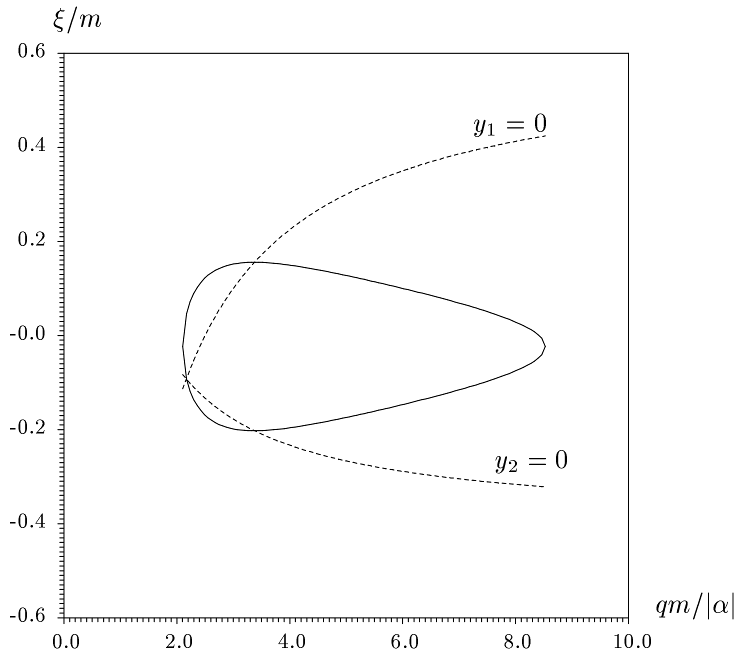

Figure 7. Scalar interaction. Phase trajectories (continuous curves) for finite motion: $(m_2 - m_1)/m = 0.2$; $M/m = 0.9$ $(\mu^2 < 1, \mu > 0)$; $\alpha < 0$. Dashed curves $y_1 = 0$, $y_2 = 0$ corresponds to the singularities of Hessian.

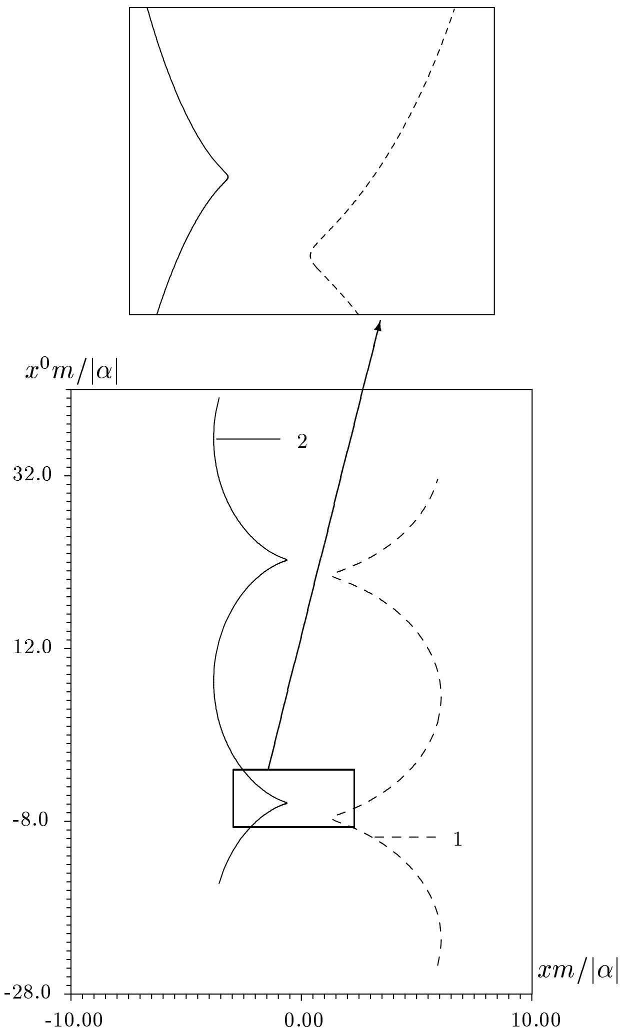

Figure 8. Scalar interaction. World lines in $\mathbb{M}_2$ for finite motion: $(m_2 - m_1)/m = 0.2$; $M/m = 0.9$ $(\mu^2 < 1, \mu > 0)$; $\alpha < 0$.

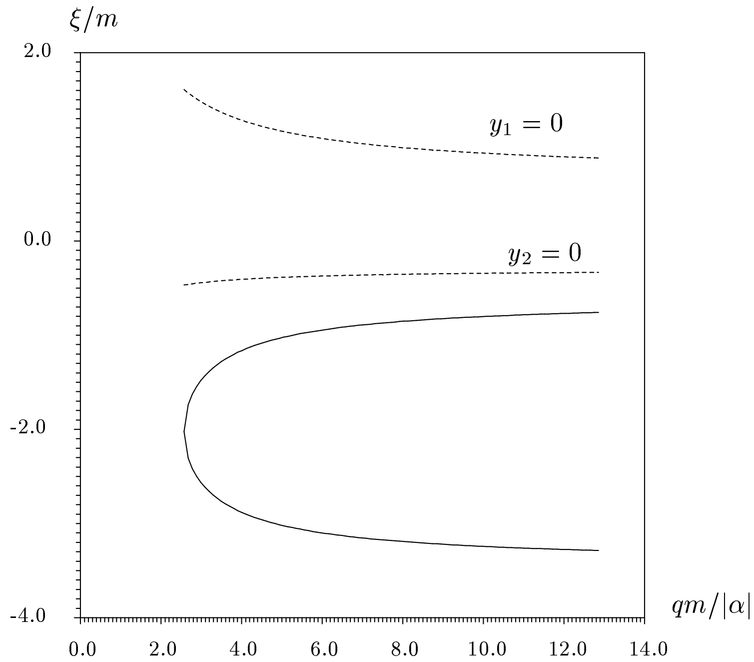

Figure 9. Scalar interaction. Phase trajectories (continuous curves) for following values of parameters: $(m_2 - m_1)/m = 0.4$; $M/m = 0.3$; $\alpha > 0$. Dashed curves $y_1 = 0$, $y_2 = 0$ corresponds to the singularities of Hessian.

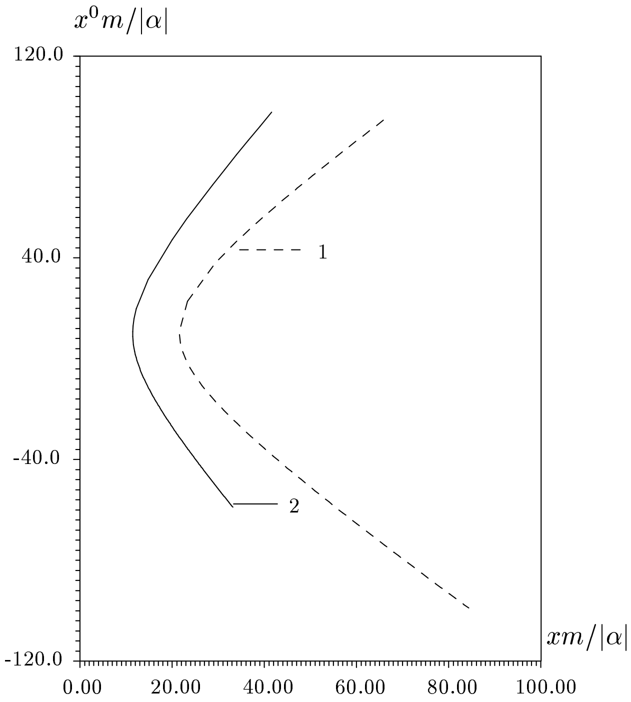

Figure 10. Scalar interaction. World lines in $\mathbb{M}_2$ for following values of parameters: $(m_2 - m_1)/m = 0.4$; $M/m = 0.3$; $\alpha > 0$.

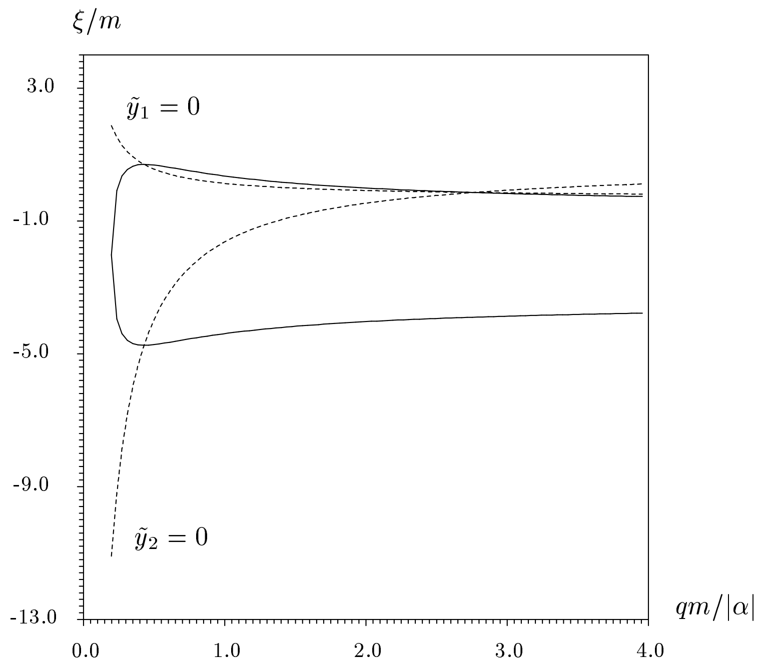

Figure 11. Scalar interaction. Phase trajectories (continuous curves) for following values of parameters: $(m_2 - m_1)/m = 0.4$; $M/m = 0.3$; $\alpha < 0$. Dashed curves $y_1 = 0$, $y_2 = 0$ corresponds to the singularities of Hessian.

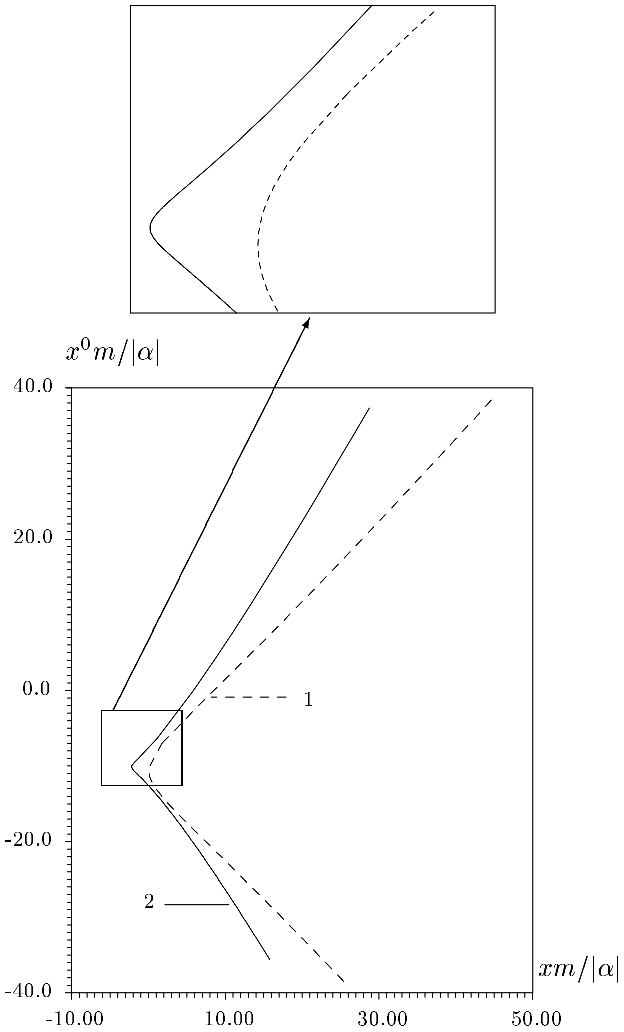

Figure 12. Scalar interaction. World lines in $\mathbb{M}_2$ for following values of parameters: $(m_2 - m_1)/m = 0.4;\ M/m = 0.3;\ \alpha < 0$.

4. Vector interaction

In the vector case if $\alpha < 0$ the region $\Lambda\Omega$ corresponds to the region bounded by curves $\tilde{y}_1 = 0,\ \tilde{y}_2 = 0,\ q = 0$. If $\alpha > 0$ then indicated region lies between the curves $\tilde{y}_1 = 0,\ \tilde{y}_2 = 0$ to the right of their intersection point. Analogously to the scalar interaction, only the phase curves corresponding to $\alpha > 0,\ M^2 > m^2$, lie completely in $\Lambda\Omega$ (Fig 13). Just the same case has been considered in Ref.[10]. Corresponding world lines are shown in the Fig. 14. Phase trajectories for other values of parameters $\alpha,\ M^2$ (excepting $\alpha > 0$, $0 < M^2 < (m_1 - m_2)^2$) describe particles collisions ($q = 0$). For $q > 0$ they lie in $\Lambda\Omega$. From now on we consider these motions in a formal way only and are not interested in their physical sense. Analogously to the scalar interaction, we regard the whole plane $\mathbb{R}^2$ as the inner phase space of the system. As was mentioned above, in the Lagrangian case $\mathrm{sgn}(r)$ is integral of motion. Moreover we obtain the time–asymmetric Lagrangian (1.8) as a consequence of condition $r > 0$. This condition (or equivalent one $q > 0$) is satisfied for the motions in the Lagrangian region $\Lambda\Omega$. If we do not restrict the Hamiltonian motions by the region $\Lambda\Omega$ where the Lagrangian and Hamiltonian systems are equivalent, then we can ignore this condition. In this case we get two possibilities: to renew the module sign in Eqs. (2.11), (2.12), (2.14) or keep these equations in the previous form. The first possibility does not lead to continuous world lines. Therefore we consider the second one. This means that we take into consideration negative solutions of quadratic equation $\mathcal{D}_1 = 0$. Just the same prolongation of phase curves is shown in Figs 15, 17, 19, 21 and leads to the smooth world lines in $\mathbb{M}_2$ (Figs 16, 18, 20, 22).

At the collision points ($q = 0$) when particles mutually change their positions (Figs. 15, 17, 19, 21) the phase trajectories break up. Taking into account equalities

$$
\begin{aligned}
k_1\,|_{q\to+0,\ \xi\to-\infty} &= k_1\,|_{q\to-0,\ \xi\to\infty} \to 0\ ,\\
k_1\,|_{q\to+0,\ \xi\to\infty} &= k_1\,|_{q\to-0,\ \xi\to-\infty} \to \mathrm{const}\ ,\\
& 0 < \mathrm{const} < \infty\ ,\\
k_2\,|_{q\to+0,\ \xi\to\infty} &= k_2\,|_{q\to-0,\ \xi\to-\infty} \to 0\ ,\\
k_2\,|_{q\to+0,\ \xi\to-\infty} &= k_2\,|_{q\to-0,\ \xi\to\infty} \to \mathrm{const}\ ,\\
& 0 < \mathrm{const} < \infty\ ,
\end{aligned}
\tag{4.1}
$$

it will be seen that the jumps in the phase trajectories $(+0, -\infty) \longleftrightarrow (-0, \infty)$; $(+0, \infty) \longleftrightarrow (-0, -\infty)$ correspond to the particles motion along the smooth world lines. This means that such jumps are not observable in the two-dimensional Minkowski space and therefore we can assume that they are not physical. Our canonical variables describe the system in the proper way only in some finite region in $\mathbb{R}^2$ and we can identify the points$(+0,-\infty) \sim (-0,\infty)$; $(+0,\infty) \sim (-0,-\infty)$ because each pair correspond to the one point on the world lines. In other words our canonical variables are only local coordinates and cannot describe the whole evolution of the system.

We put $t_o^+ = 0$. Integration constant t_o^- for the cases $M^2 > m^2$ and $\mu^2 < 1$ is determined by equations (3.1), (3.3), where we put $\ell = 1$. If $0 < M^2 < (m_1 - m_2)^2$ then

$$\left\{ \begin{array}{l} t_o^- = \dfrac{4P_+\alpha m}{(\mu^2-1)} J_o^{(\mu^2>1)}(q_2), \ \alpha>0 \\ \\ t_o^- = \dfrac{4\alpha P_+ m}{(\mu^2-1)} J_o^{(\mu^2>1)}(q_1), \ \alpha<0 \end{array} \right| 0<M^2<(m_1-m_2)^2 \ . \tag{4.2}$$

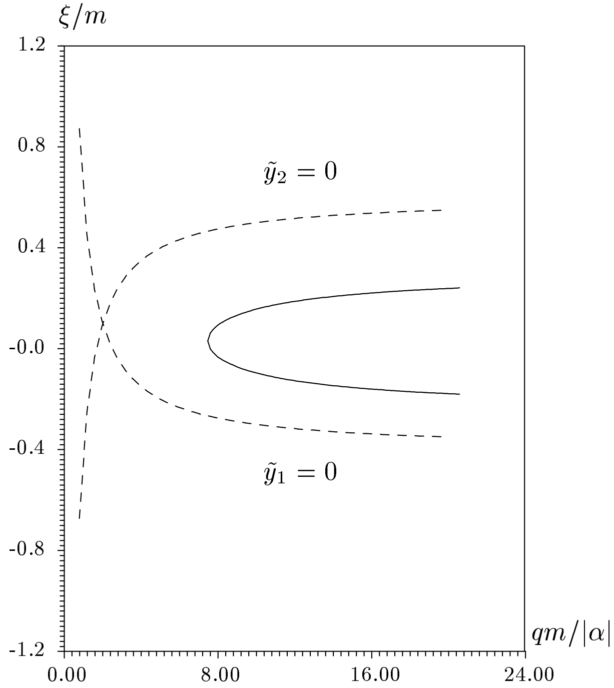

Figure 13. Vector interaction. Phase trajectories (continuous curves) for Rudd–Hill case: $(m_2-m_1)/m = 0.2$; $M/m = 1.2$; $\alpha > 0$. Dashed curves $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$ corresponds to the singularities of Hessian.

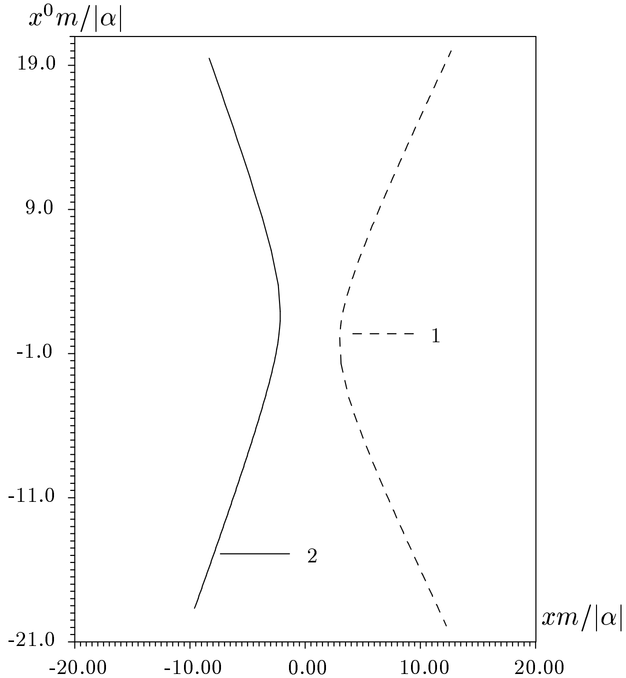

Figure 14. Vector interaction. World lines in $\mathbb{M}_2$. Rudd–Hill case: $(m_2 - m_1)/m = 0.2$; $M/m = 1.2$; $\alpha > 0$.

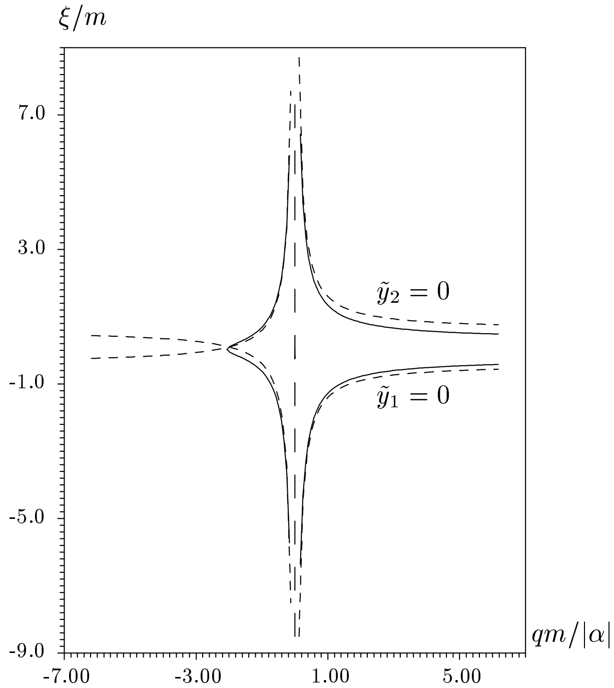

Figure 15. Vector interaction. Phase trajectories (continuous curves) for following values of parameters: $(m_2 - m_1)/m = 0.2$; $M/m = 1.2$; $\alpha < 0$. Dashed curves $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$ corresponds to the singularities of Hessian.

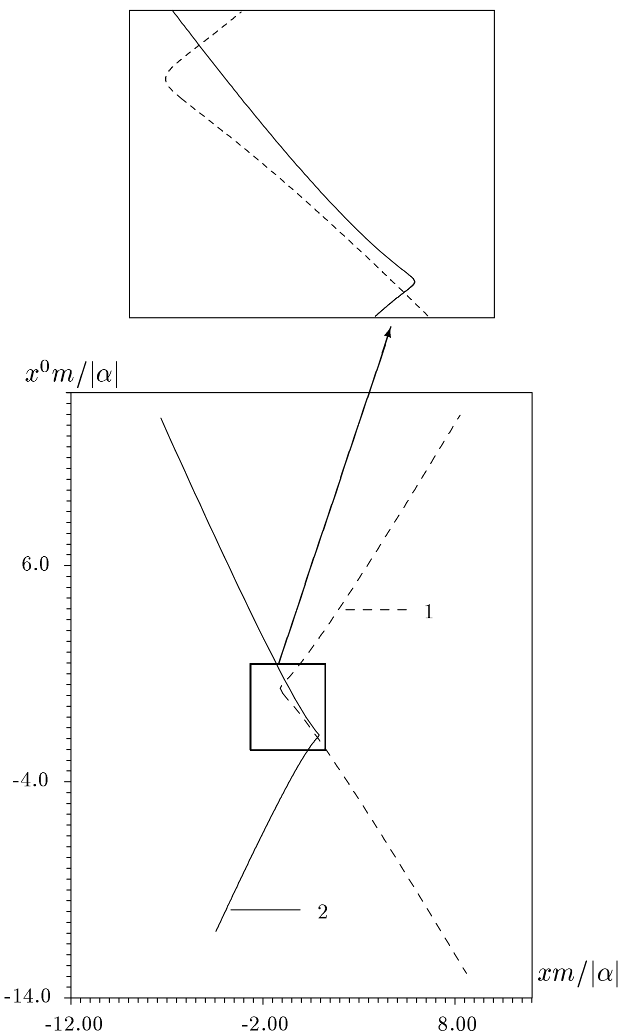

Figure 16. Vector interaction. World lines in $\mathbb{M}_2$ for following values of parameters: $(m_2 - m_1)/m = 0.2$; $M/m = 1.2$; $\alpha < 0$.

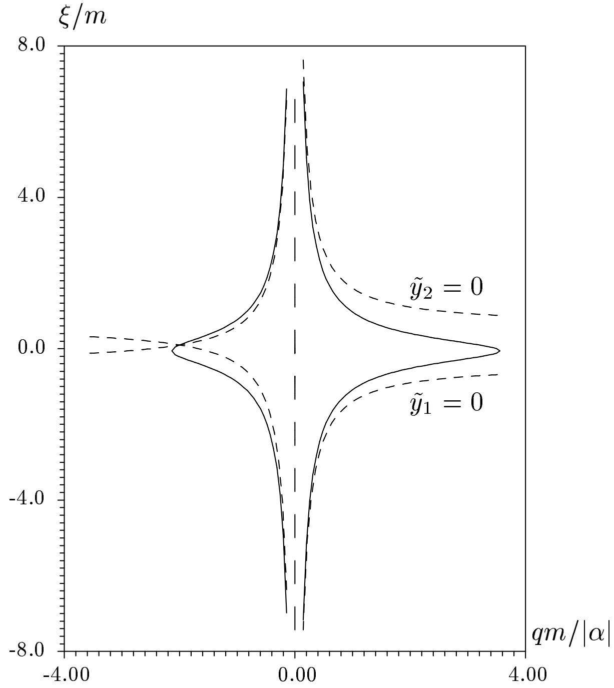

Figure 17. Vector interaction. Phase trajectories (continuous curves) for finite motion: $(m_2 - m_1)/m = 0.2$; $M/m = 0.8$; $\alpha < 0$. Dashed curves $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$ corresponds to the singularities of Hessian.

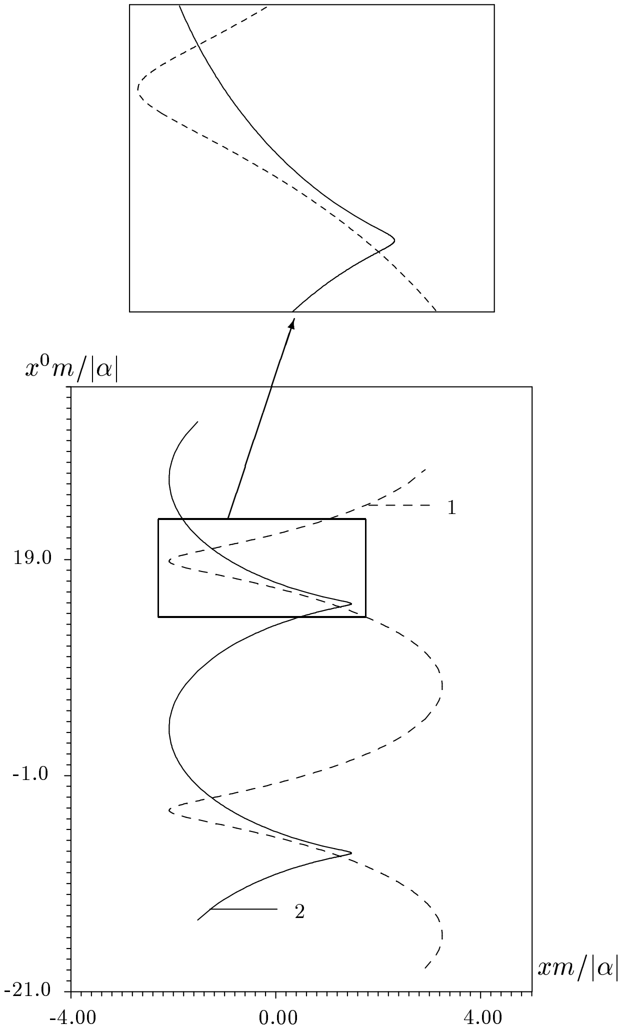

Figure 18. Vector interaction. World lines in $\mathbb{M}_2$ for finite motion: $(m_2-m_1)/m = 0.2$; $M/m = 0.8$; $\alpha < 0$.

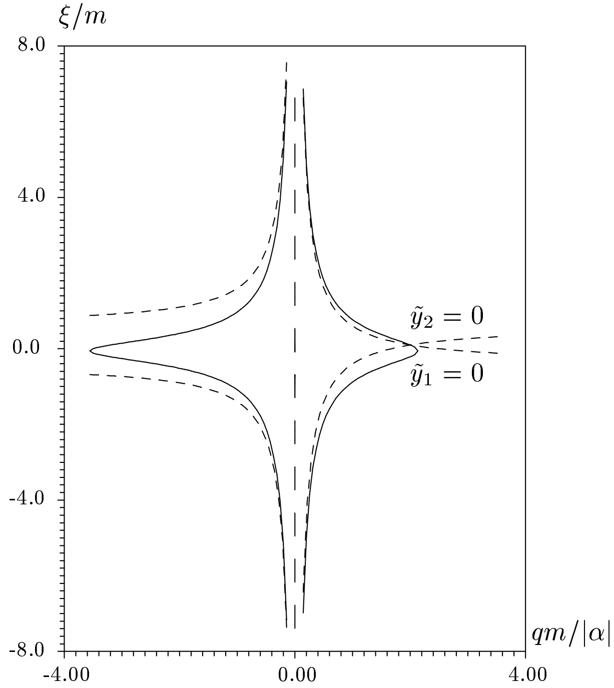

Figure 19. Vector interaction. Phase trajectories (continuous curves) for finite motion: $(m_2-m_1)/m = 0.2$; $M/m = 0.8$; $\alpha > 0$. Dashed curves $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$ corresponds to the singularities of Hessian.

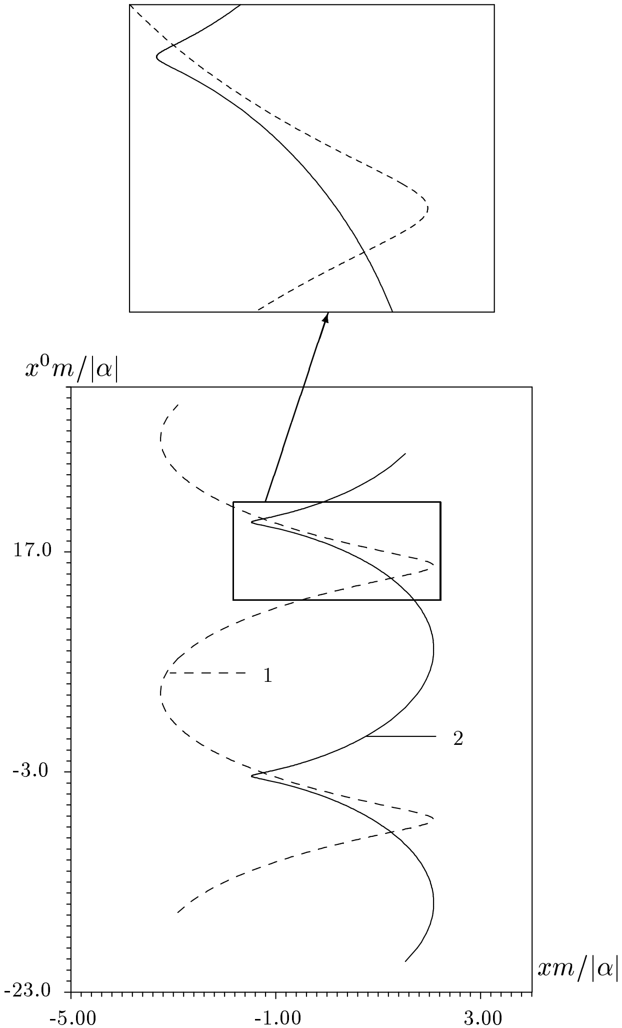

Figure 20. Vector interaction. World lines in $\mathbb{M}_2$ for finite motion: $(m_2-m_1)/m = 0.2$; $M/m = 0.8$; $\alpha > 0$.

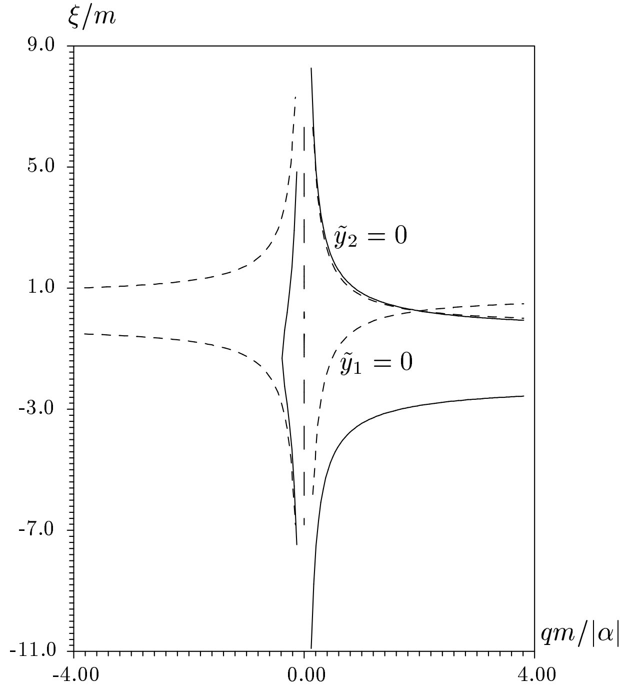

Figure 21. Vector interaction. Phase trajectories (continuous curves) for following values of parameters: $(m_2-m_1)/m = 0.5$; $M/m = 0.4$; $\alpha > 0$. Dashed curves $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$ corresponds to the singularities of Hessian.

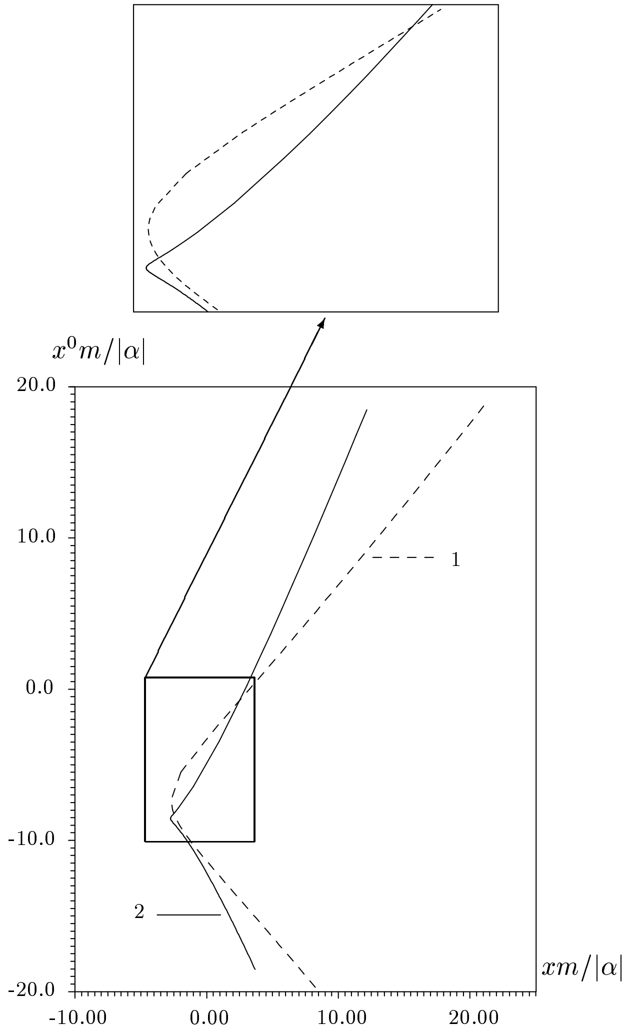

Figure 22. Vector interaction. World lines in $\mathbb{M}_2$ for following values of parameters: $(m_2 - m_1)/m = 0.5$; $M/m = 0.4$; $\alpha > 0$.

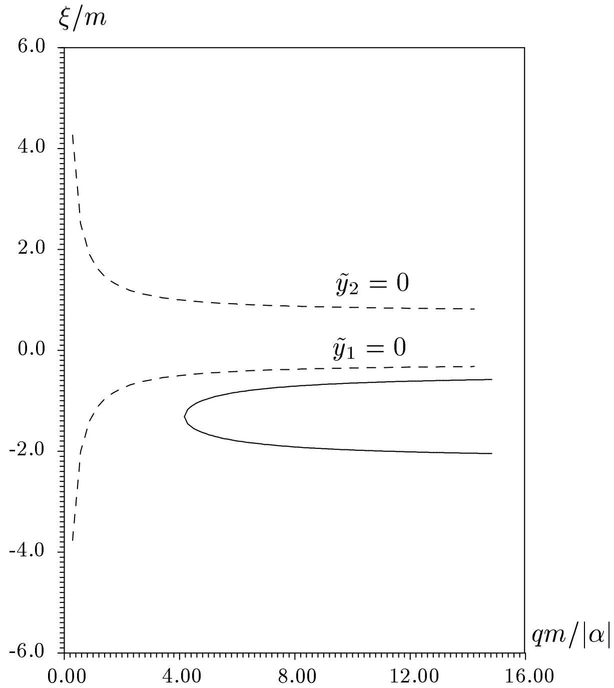

Figure 23. Vector interaction. Phase trajectories (continuous curves) for following values of parameters: $(m_2 - m_1)/m = 0.5$; $M/m = 0.4$; $\alpha < 0$. Dashed curves $\tilde{y}_1 = 0$, $\tilde{y}_2 = 0$ corresponds to the singularities of Hessian.

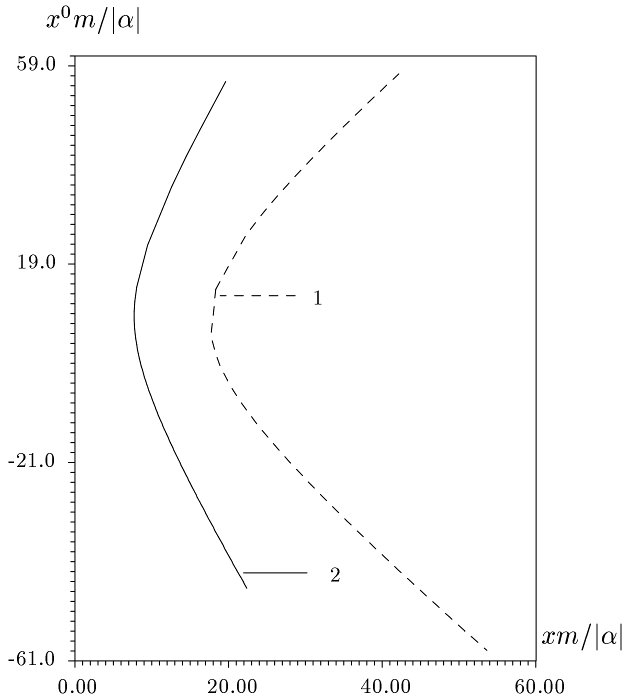

Figure 24. Vector interaction. World lines in $\mathbb{M}_2$ for following values of parameters: $(m_2 - m_1)/m = 0.5$; $M/m = 0.4$; $\alpha < 0$.

5. Semiclassical approximation

In the previous sections we have demonstrated that investigated method of prolongation of world lines permits the smooth periodic finite motions in the two–dimensional Minkowski space. As it follows from (2.22), (3.3), the period of finite motion is determined by the equation

$$T_\ell = t^-(q_1) - t^-(q_2) + t^+(q_2) - t^+(q_1) = \\ -\frac{2\alpha\pi\mu^{1-\ell}P_+}{(1-\mu^2)^{3/2}m_1m_2} \ , \tag{5.1}$$

if $\alpha\mu^{1-\ell} < 0$, and by the equation

$$T_l = t^+(q_2) - t^+(q_1) + t^-(q_1) - t^-(q_2) = \\ \frac{2\alpha\pi\mu^{1-\ell}P_+}{(1-\mu^2)^{3/2}m_1m_2} \ , \tag{5.2}$$

if $\alpha\mu^{1-\ell} > 0$. For the finite motion in the case of scalar interaction we get usual closed phase curves (Figs 5, 7). The vector case is quite different because phase trajectories of finite motion (in terms of variables q, ξ) are not like any closed curves (Figs 17, 19). Moreover, in this case the area enclosed by phase curve (or even one of its parts and the line $q = 0$) is infinite. As we pointed out above the pairs of points $(+0, -\infty) \sim (-0, \infty)$; $(+0, \infty) \sim (-0, -\infty)$ describe the same physical reality because they correspond to the same points on the world lines. Our canonical variables describe the system in the proper way only in some finite region in $\mathbb{R}^2$ and we cannot apply correctly our description for the whole evolution of the system.

For both cases the period of finite motion is finite. In the vector case we have obtained the smooth periodic world lines for finite motion in the same manner as for scalar interaction. We may assume that there exists some symplectic manifold on which phase trajectories are smooth or at least continuous and phase curves for finite motion enclose a finite area Π_1 which is connected with the period T_1. We also assume that the jumps in the phase trajectories and the necessity of consideration of infinite points in q–ξ plane are a consequence of bad mapping from some this manifold in $\mathbb{R}^2$.

To obtain mass spectra for both cases we yield as follows. We use (2.19) and general relation [33]

$$\partial\Pi(E)/\partial E = T(E) \ , \tag{5.3}$$

which holds for every closed phase trajectory. Here $E = H$ is the energy of system. If $M^2 = m_1^2 + m_2^2$, then in the scalar case $q_0 = q_1 = q_2$, and the phase trajectory is a point. Therefore,

$$\Pi(m_1^2 + m_2^2) = 0 \tag{5.4}$$

that one can consider as an initial value for integration of the equation

$$\frac{\partial \Pi_\ell(M)}{\partial M} = \frac{2\pi|\alpha\mu^{1-\ell}|M}{(1-\mu^2)^{3/2} m_1 m_2} \ . \tag{5.5}$$

In such a way we obtain

$$\Pi_0 = 2\left(-\frac{\alpha\pi}{\sqrt{1-\mu^2}} + \alpha\pi\right) \ ; \quad \Pi_1 = \frac{2\pi|-\alpha\mu|}{\sqrt{1-\mu^2}} \ . \tag{5.6}$$

It will be noted that in the scalar case one can obtain Π_0 by immediate calculation of the area enclosed by phase curves (Figs 5, 7).

The mass spectrum is determined in semiclassical approximation by condition [33]

$$\Pi(M) = 2\pi\hbar(n + \nu/4), \ n = 0, 1, 2, 3, ... \tag{5.7}$$

where ν is a number of returning points or in general case may be Maslov index [33]. As a result we have semiclassical mass spectrum for the scalar interaction

$$(M_n^{\pm})_0^2 = m_1^2 + m_2^2 \pm 2m_1 m_2 \sqrt{1 - \frac{\alpha^2}{c^2\hbar^2(s + 1/2 - \alpha/(\hbar c))^2}}, \tag{5.8}$$

$s = 1, 2, 3, \dots$.

There is not such a correspondence with the second equation in (5.6) and area enclosed by phase trajectories in terms of the canonical variables q, ξ in the vector case. As we mentioned above, in this case the area enclosed by phase curves is infinite. According our assumption there exits some mechanical description on symplectic manifold which leads to the same smooth periodic world lines and gives closed inner phase curves with the area Π_1.

Then using Eq. (5.7) we get mass spectrum

$$(M_n^{\pm})_1^2 = m_1^2 + m_2^2 \pm 2m_1 m_2 \left(1 + \frac{\alpha^2}{c^2\hbar^2(s + \nu/4)^2}\right)^{-1/2} , \tag{5.9}$$

where quantity ν is undefined.

One can easily check that the difference $\Pi_{q>0} - \Pi_{q<0}$ is finite and coincides with the expression for Π_1 in (5.6). This strange fact also could mean that we have not a global description for such a system. We can assume that the last equation describes the semiclassical mass spectrum for the vector interaction. Then to determine quantity ν we must construct a global description for this system. The sign "+" in the last equation corresponds to the attraction ($\alpha < 0$) and "-" corresponds to the repulsion ($\alpha > 0$).

The semiclassical mass spectra (5.8), (5.9) correlate well with the exactly quantum results [29,30] which have been obtained in purely algebraic way [30]. It will be noted that the essential feature of the construction of the mass spectra in the semiclassical approximation is the investigated here method of smooth continuation of world lines.

We have obtained the semiclassical mass spectra from the world lines without explicit calculation of the area enclosed by phase trajectories. Such a method does not give an understanding how non–Lagrangian parts of phase trajectories form the the semiclassical mass spectra (5.8), (5.9). It is possible also to obtain mass spectra (5.8), (5.9) for both interactions considering phase trajectories in a following way. It follows from the definition of quantity δ (see Eq. (1.8)) that $\delta \in [1, \infty)$. Thus we can put

$$\sin\beta = \frac{1}{\delta} \ , \tag{5.10}$$

where $\beta \in [0, \pi/2]$. Substituting (2.4) into expression for inner momentum ξ (2.13) and using (1.15), (2.1), (5.10) we get parametric representation of mass shell equation for arbitrary ℓ

$$\xi = \xi_M + \frac{m m_1 m_2[\ell\sin\beta - (\ell-1)\mu]\cos\beta}{M^2(\ell - 1 - \ell\sin^2\beta)} \tag{5.11}$$

$$q = \frac{\alpha M^2(\ell - 1 - \ell\sin^2\beta)}{m m_1 m_2 \sin^\ell\beta(1 - \mu\sin\beta)} \tag{5.12}$$

To take into consideration different signs before $\cos\beta$ we put in (5.11), (5.12) $\beta \in [0, \pi]$. To take into account non–Lagrangian parts of phase trajectories in the scalar and vector case we have to assume that

$$\beta \in [0, 2\pi] \ . \tag{5.13}$$

For the finite motion ($\mu^2 < 1$) parameter β runs the whole interval $[0, 2\pi]$ and the points $\beta = 0, 2\pi$ correspond to the same point on the phase trajectory. This means that $\beta \in S^1$. For infinite motions β runs only part of the circle S^1. In the vector case the jumps on the phase trajectories (4.1) correspond to passage of the parameter through the points $\beta = 0, \pi$.

Let us consider the integral

$$\mathcal{J}_{\mu^2<1} = \int\limits_0^{2\pi} \xi(M^2, \beta) \frac{dq}{d\beta} d\beta \ . \tag{5.14}$$

In the scalar case the integral describes an area enclosed by phase trajectory: $\mathcal{J}_{\mu^2<1} = \Pi_0$. Thus putting

$$\mathcal{J}_{\mu^2<1} = 2\pi\hbar(n + \nu/4) \tag{5.15}$$

we obtain mass spectrum (5.8). In the vector case

$$\mathcal{J}_{\mu^2<1} = \Pi_{q>0} - \Pi_{q<0} = \Pi_1 \ . \tag{5.16}$$

Using (5.15) we get mass spectrum (5.9)

If $\beta \in (\pi, 2\pi)$ then as it follows from (5.10) $\delta < 0$. This means that for non–Lagrangian parts of phase trajectories one of the Lorentz factors is negative: k_1 or $k_2 < 0$. It is easily to check that these parts of phase trajectories one can obtain from the Lagrangian (1.8) by change of signs before the Lorentz factor of one of particles: $k_a \to -k_a$, $a = 1$ or 2.

Let us consider the Lagrangian

$$\tilde{\mathcal{L}} = -\sum_{a=1}^{N} m_a \tilde{k}_a - \frac{g_1 g_2 \tilde{k}_1 \tilde{k}_2}{|r|} F(\tilde{\delta}); \ \tilde{\delta} = \frac{1}{2}\left(\frac{\tilde{k}_1}{\tilde{k}_2} + \frac{\tilde{k}_2}{\tilde{k}_1}\right), \tag{5.17}$$

where

$$\tilde{k}_a = \frac{1}{2m_a}\left(\lambda_a^{-1} k_a^2 + \lambda_a m_a^2\right) \ . \tag{5.18}$$

This Lagrangian is obtained from (1.8) by replacement $k_a \to \tilde{k}_a$. Free particle part of the Lagrangian (5.17) is similar to that used in the string theory [34]. It permits to describe massless particles. The idea to use similar type of Lagrangians for relativistic two–particle systems with the time–asymmetric scalar and vector interaction was suggested by A. Duviryak [35]. The equations of motion for quantity λ

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{k}_a}\left(-\frac{k_a^2}{\lambda_a^2} + m_a^2\right) = 0 \tag{5.19}$$

give us two possibilities which do not break symmetry between particles:

$$\begin{aligned} \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{k}_1} &= -m_1 - \frac{\alpha \tilde{k}_2}{|r|}\left(F + \left(\frac{\tilde{k}_1}{\tilde{k}_2} - \frac{\tilde{k}_2}{\tilde{k}_1}\right) F'\right) = 0 \ , \\ \frac{\partial \tilde{\mathcal{L}}}{\partial \tilde{k}_2} &= -m_2 - \frac{\alpha \tilde{k}_1}{|r|}\left(F - \left(\frac{\tilde{k}_1}{\tilde{k}_2} - \frac{\tilde{k}_2}{\tilde{k}_1}\right) F'\right) = 0 \ ; \end{aligned} \tag{5.20}$$

or

$$\lambda_a^2 = \frac{k_a^2}{m_a^2} \ . \tag{5.21}$$

Solving the system (5.20) with respect to $\tilde{k}_a$ we get

$$k_a = f_a\left(m_1, m_2, \frac{\alpha}{|r|}\right) \ . \tag{5.22}$$

Substitution of these solutions into (5.17) gives us the Lagrangian which does not contain the velocities and therefore does not describe a dynamics:

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(m_1, m_2, \alpha/|r|) \ . \tag{5.23}$$

Substituting solutions of the system (5.21) into (5.17) we obtain the Lagrangian function (1.8) in which

$$k_a \to \pm k_a \ . \tag{5.24}$$

As we mentioned above, $k_1 < 0$ or $k_2 < 0$ for non–Lagrangian parts of phase trajectories. Thus the Lagrangian (5.17) in the scalar and vector case describes the whole evolution of the system. The quantities λ_a one can interpret as "one–dimensional metric" along the world lines. If the particle velocity tends to the speed of light then $\lambda \to 0$. This means that we obtain the smooth world lines only in affine sense but not in the metric one.

Conclusion and discussion

We have seen that Hamiltonian description of the two–particle system with the time–asymmetric scalar and vector interactions permits to construct smooth world lines in $\mathbb{M}_2$. For all values of the total mass M and the signs of interaction constant α (excepting the case $M > m, \ \alpha > 0$) one can obtain in the Lagrangian formalism only segments of particles world lines. If one of the particle velocities tends to the speed of light then Hessian $h_\ell \to 0$ or does not exists and in the framework of the Lagrangian formalism we cannot prolong the particle motion beyond the singular points. This means that basic Fokker–action integral does not describe the whole evolution of the system.

The Hamiltonian description is equivalent to the Lagrangian formalism only if the Legendre transformation Λ is a diffeomorphism. In our case this is true in the region $\Omega \subset T\mathcal{M}$, where Hessian $h_\ell > 0$. The Hamiltonian description is equivalent to the Lagrangian one in the region $\Lambda\Omega \subset T^*\mathcal{M} \approx \mathbb{R}^4$. We have extended the Hamiltonian description on the whole $\mathbb{R}^4$ and by this have suggested the natural method of smooth prolongation of world lines through the singular points. In such a way we have constructed world lines which describe the whole evolution of the system and are smooth everywhere. Moreover, they permit to obtain semiclassical expressions for mass spectra which correlate well with exact quantum results. This means that our smooth world lines as well as the method of their construction have some physical sense. However, as it follows from the consideration of vector interaction, we have not well determined description everywhere on $\mathbb{R}^4$. The necessity of consideration of infinite points (at $q = 0$) on the phase plane means that our inner canonical variables describe the system in the proper way only in some finite regions in $\mathbb{R}^2$ and the topology of the phase space (if it exists) is non–trivial.

We did not pay attention to the physical sense of time–asymmetric Fokker–type models or the physical conclusions of our prolongation method of world lines. Our purpose was the consideration in a formal way the solutions of Hamiltonian equation. The questions of physical interpretation of "non–Lagrangian" segments of smooth world lines and variation principle for the whole world lines were out of our consideration. We are going to consider these question in next works.

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