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Unpredictability in physical systems: basin entropy and testing for Wada basins

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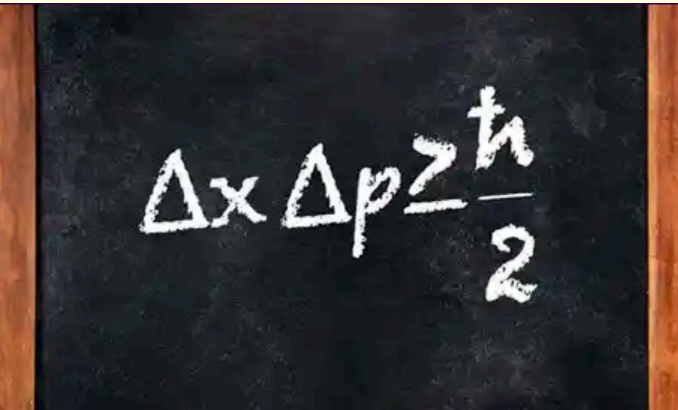
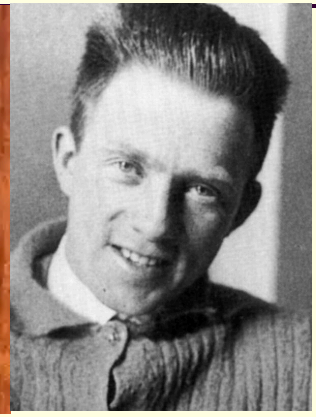
Introduction

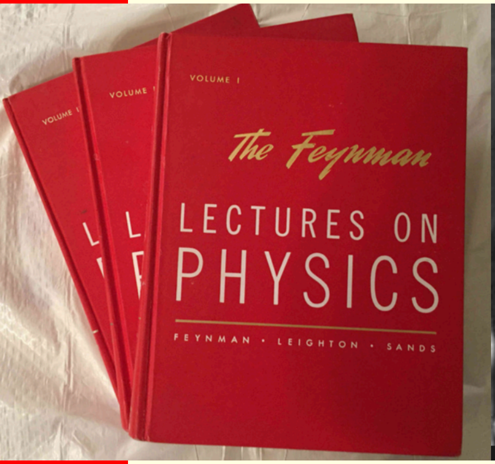
- In Physics we have **laws that determine the time evolution** of a given physical system, depending on its parameters and its initial conditions.
- In **multi-stable systems** with many basins of attraction possessing fractal or even Wada boundaries the prediction becomes harder depending on the initial conditions.
- **Chaotic systems** typically present **fractal basins.**
- A small **uncertainty** in the initial conditions gives rise to a certain **unpredictability** of the final state.
- The new notion of **Basin Entropy** provides a new quantitative way to measure the unpredictability of the final states in basins of attraction.

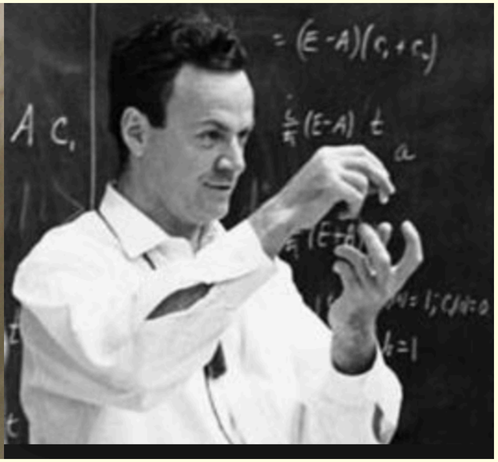
Quantum and Classical Uncertainty





Uncertainty principle of Heisenberg





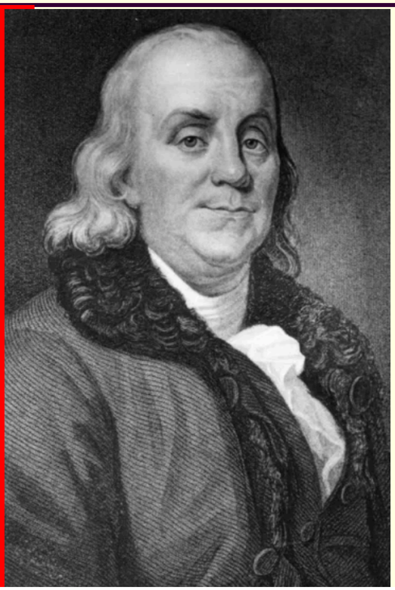
Of course we must emphasize that classical physics is also indeterminate, in a sense.

For already in classical mechanics there was indeterminability from a practical point of view.

Volume 1

38–6 Philosophical implications

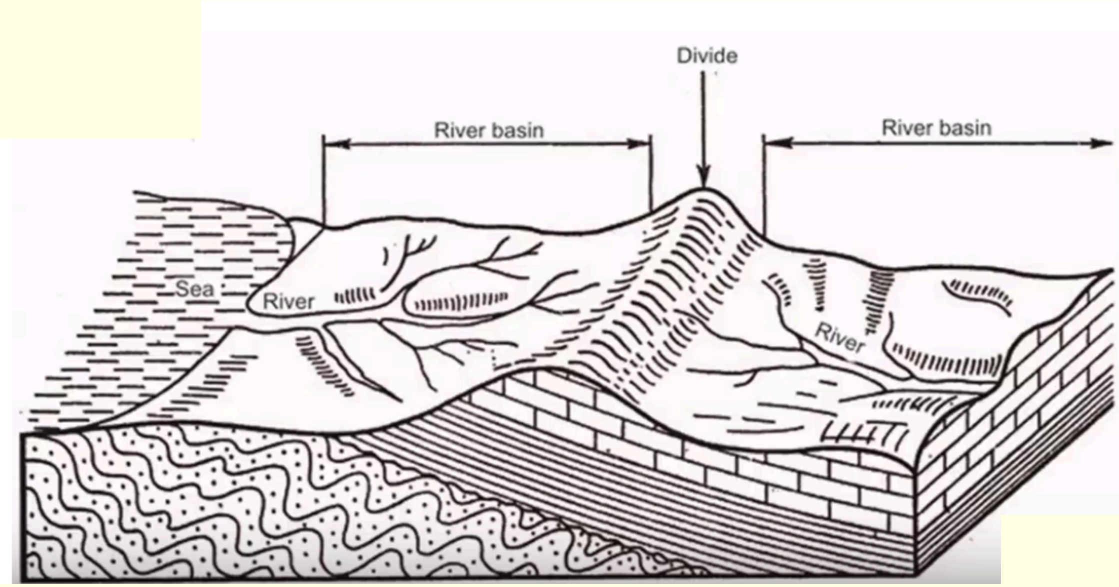
Sensitivity to initial conditions…
One of the sources of uncertainty



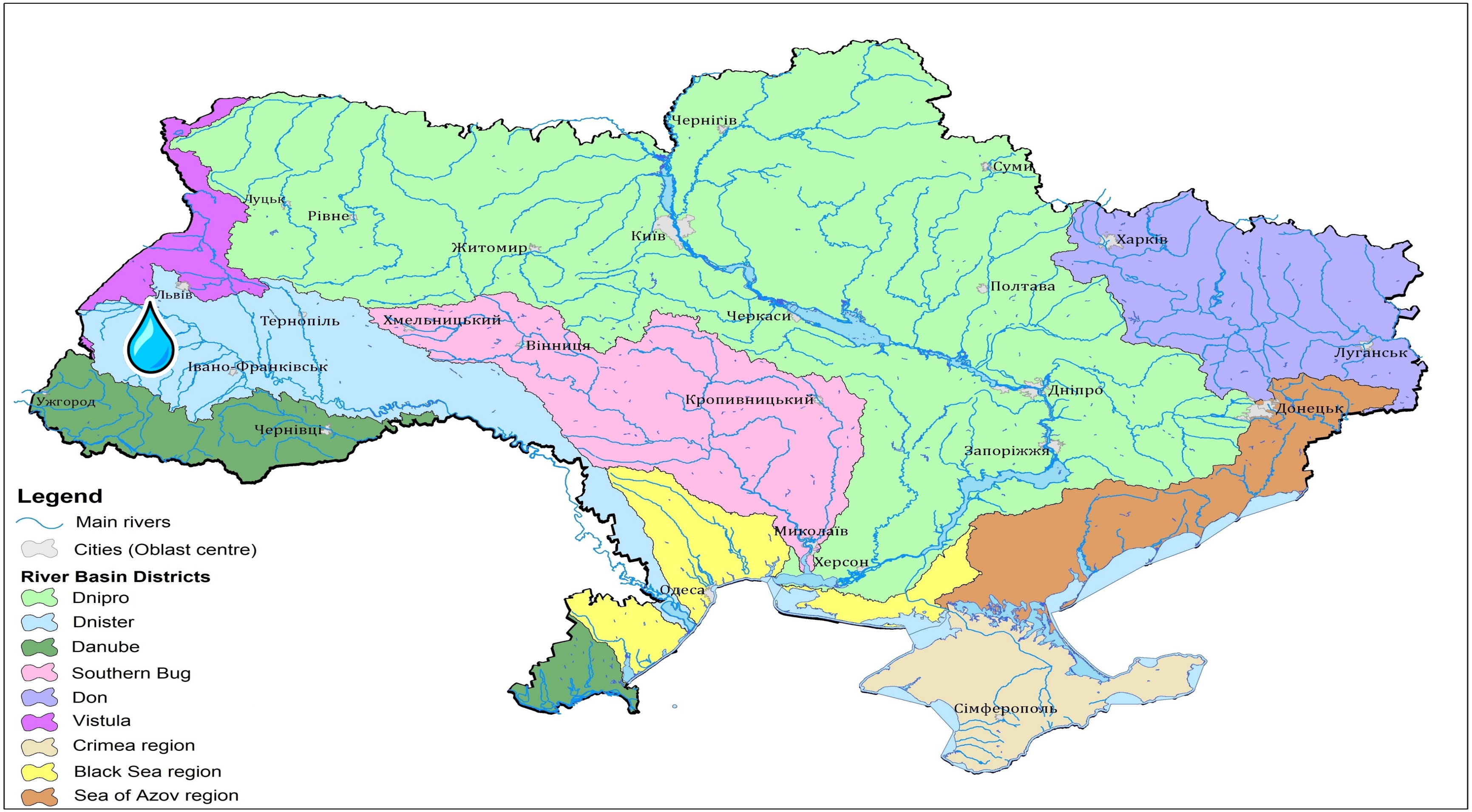
For want of a nail

For want of a nail the shoe was lost.
For want of a shoe the horse was lost.
For want of a horse the rider was lost.
For want of a rider the battle was lost.
For want of a battle the kingdom was lost.
And all for the want of a horseshoe nail.

Another source of uncertainty: fractal basins
Analogy: a river basin

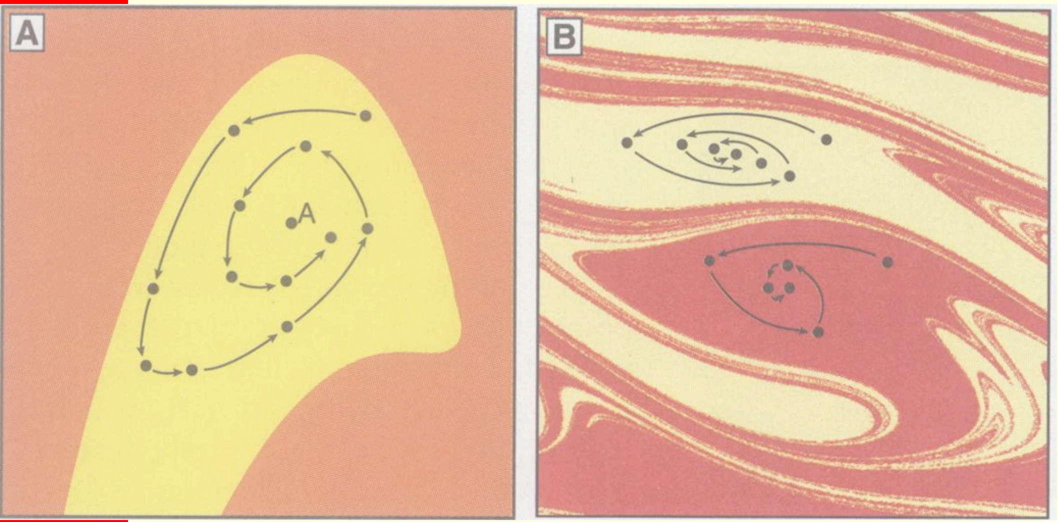


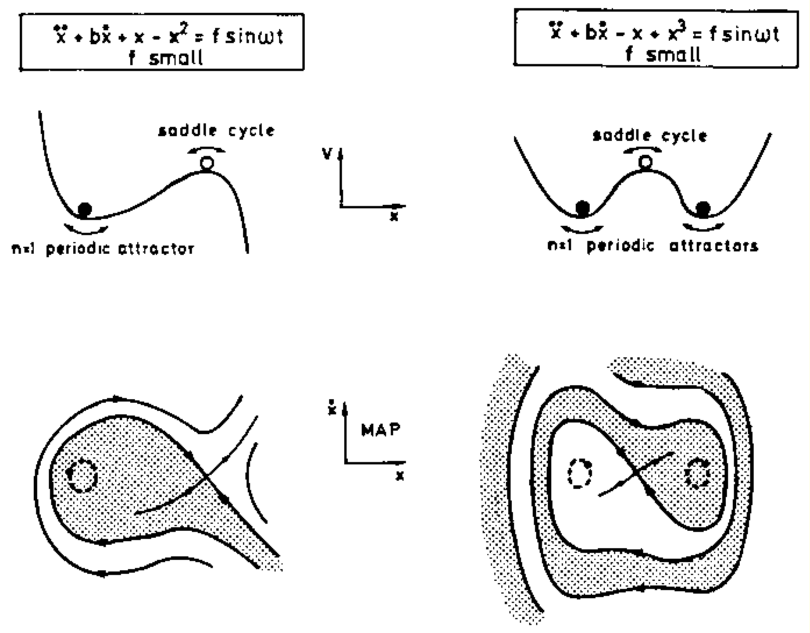
River Basins in Ukraine



A basin: if a drop falls in the region, it will go to the river

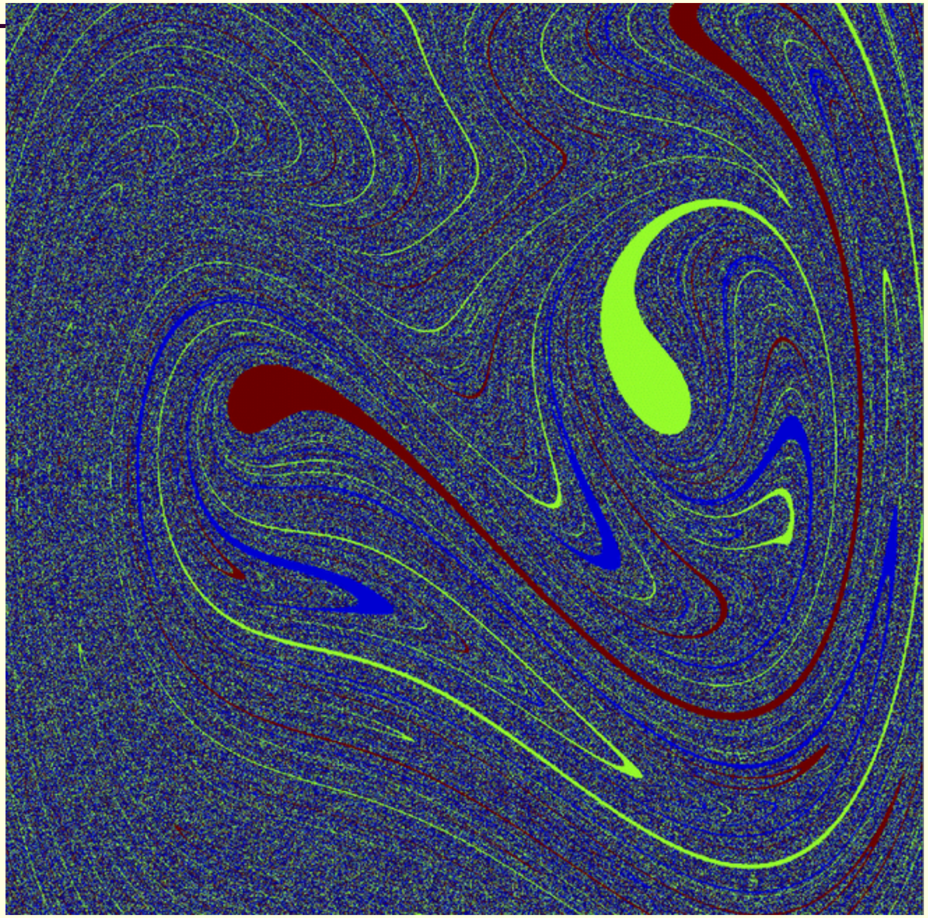
Attractors and Basins of Attraction





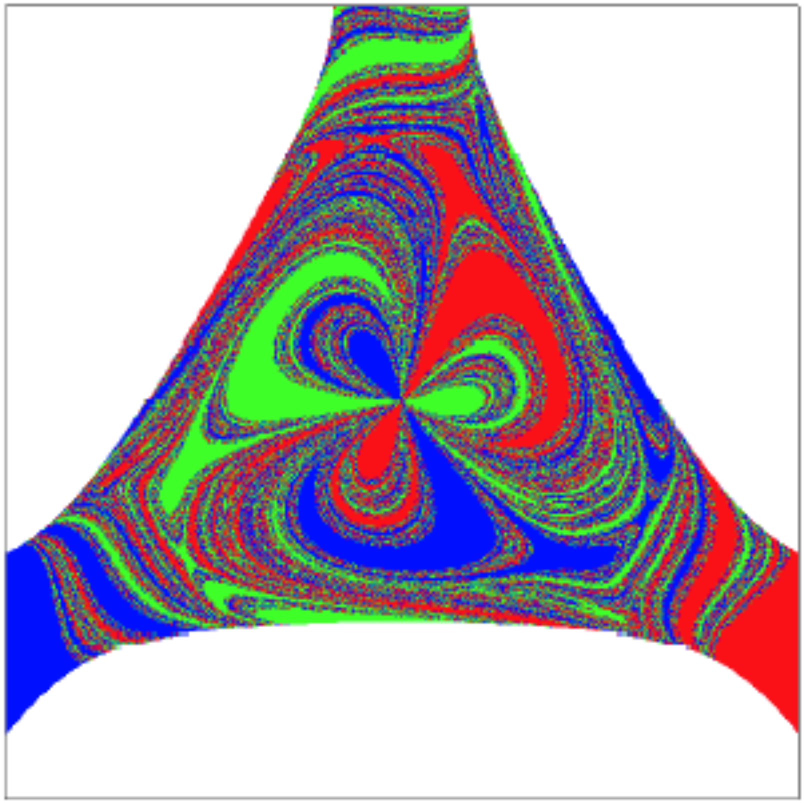
A **basin of attraction** is the set of initial conditions whose trajectories go to a specific attractor.

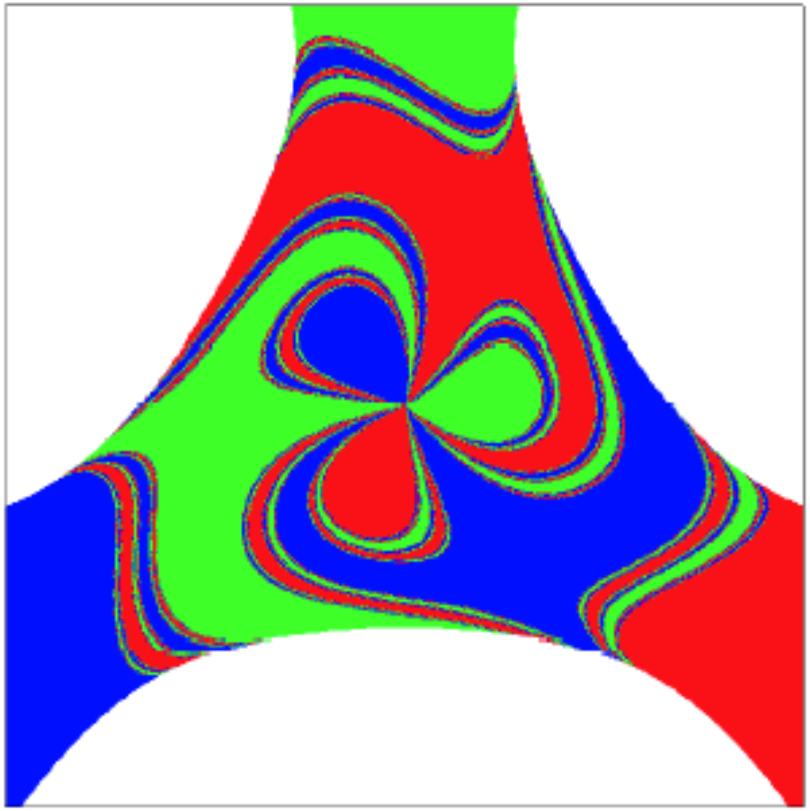
Smooth basins and fractal basins



Fractality implies Unpredictability and Uncertainty

A fundamental question





Which basin is *more unpredictable*?

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

Unpredictability and fractal boundaries: uncertainty dimension

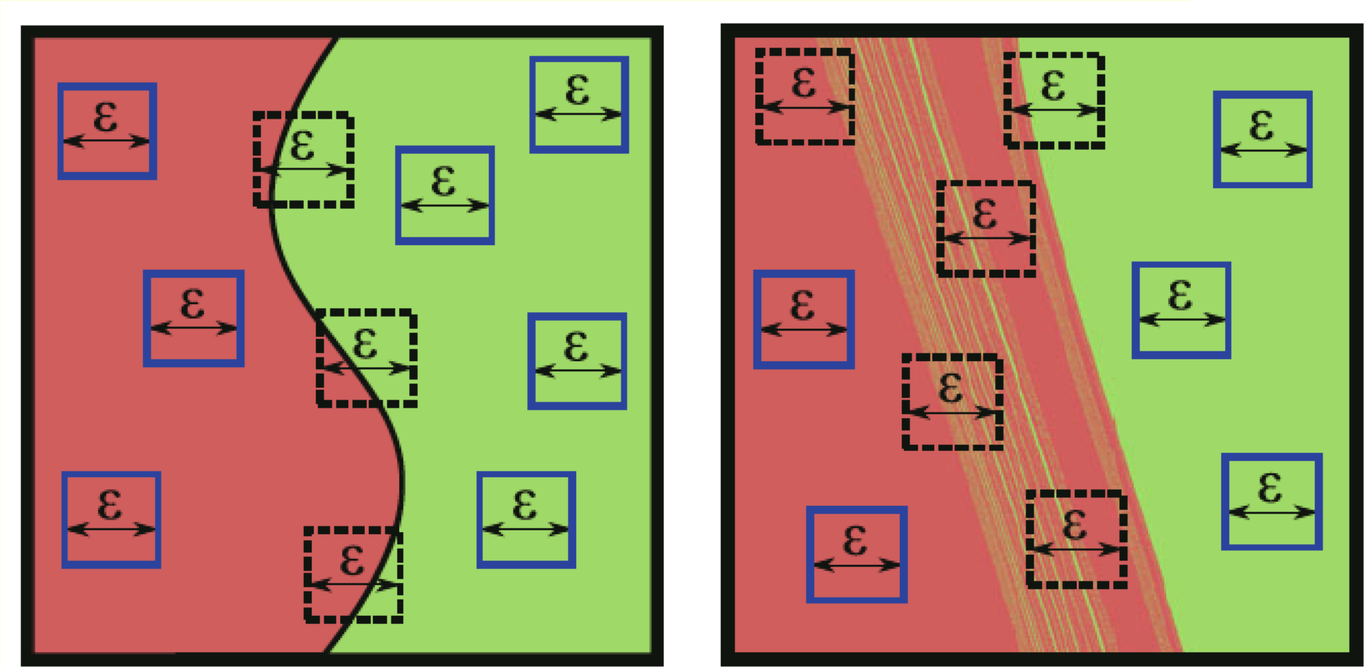
$\alpha < 1$

Smooth Boundary

$$f \sim \varepsilon \qquad \alpha = 1$$

Fractal Boundary

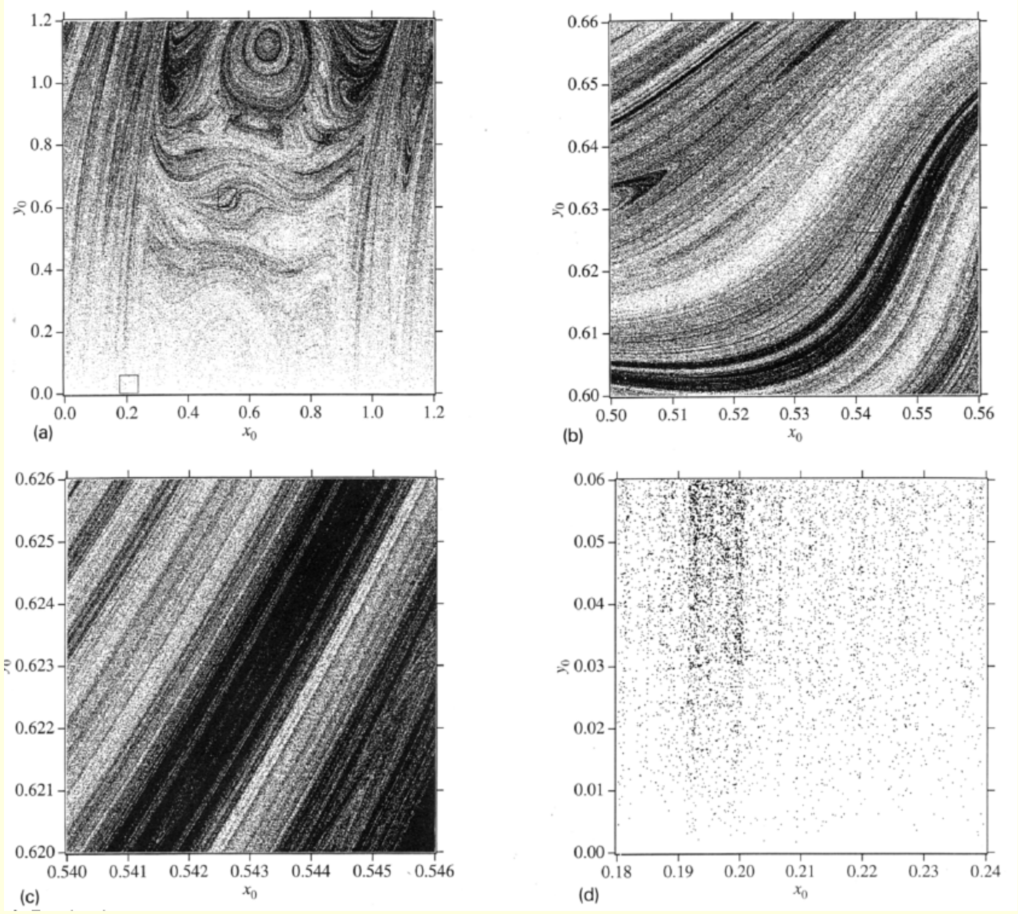
$$f \sim \varepsilon^{\alpha} \qquad \alpha = D - d$$



f: Fraction of uncertain initial conditions ε: Resolution
α: Uncertainty dimension

C. Grebogi, S. W. McDonald, E. Ott and J. A. Yorke, Final state sensitivity: An obstruction to predictability, Phys. Letters 99A: 415-418 (1983).

Riddled Basins

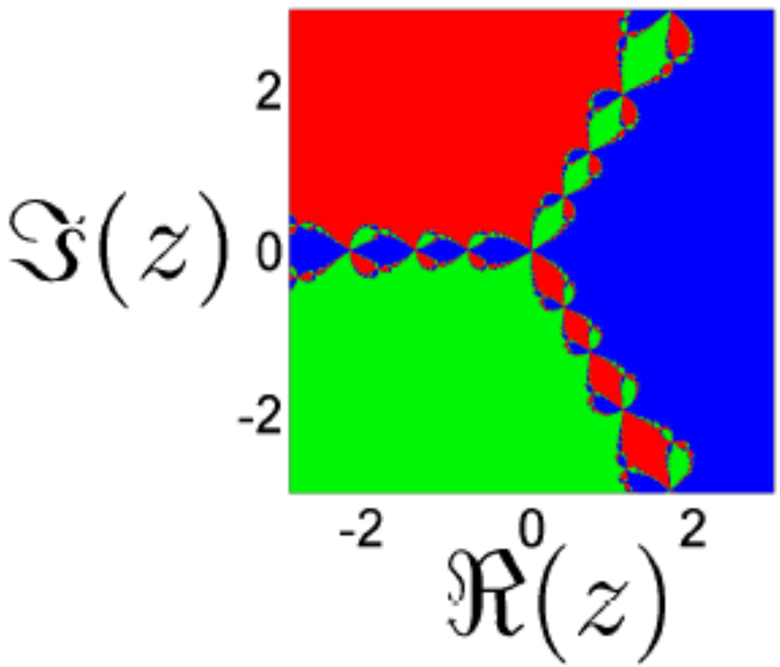


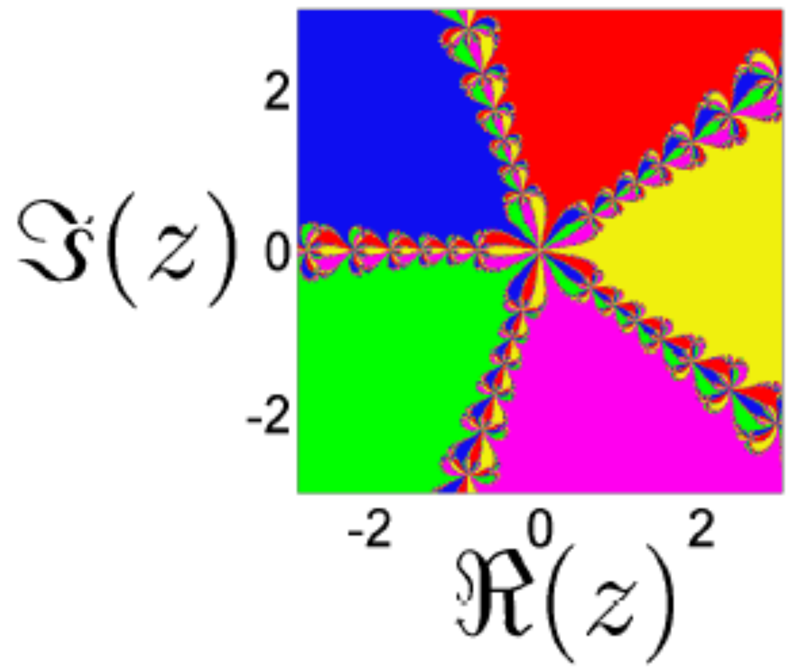
A basin **A** is riddled by **B**, if for every point of **A** is possible to find arbitrarily close points of **B**.

$\alpha = 0$ → Randomness of a deterministic system

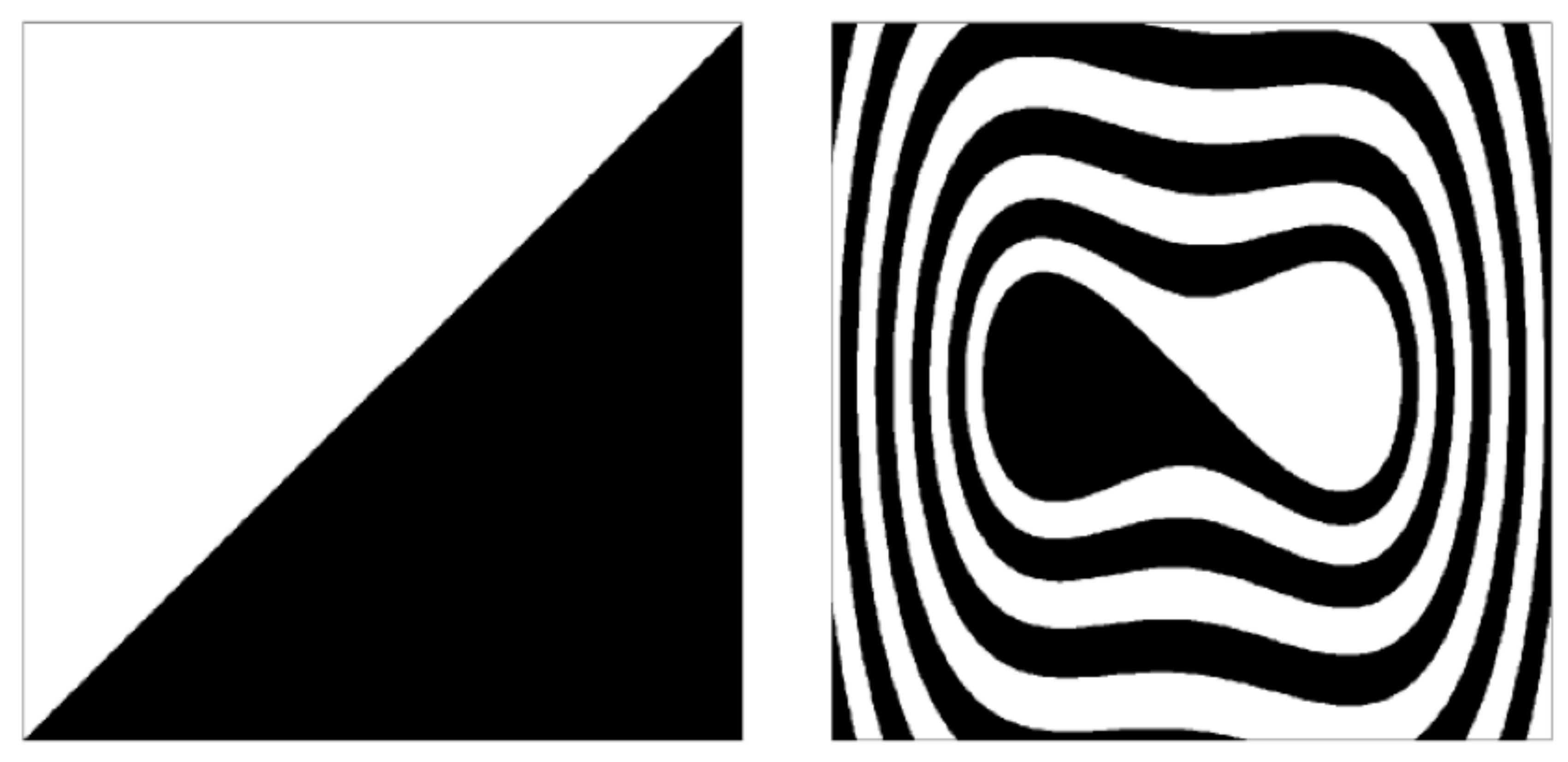
Problems and limitations 1

Problem 1: the uncertainty dimension does not take into account **how many attractors** you have.





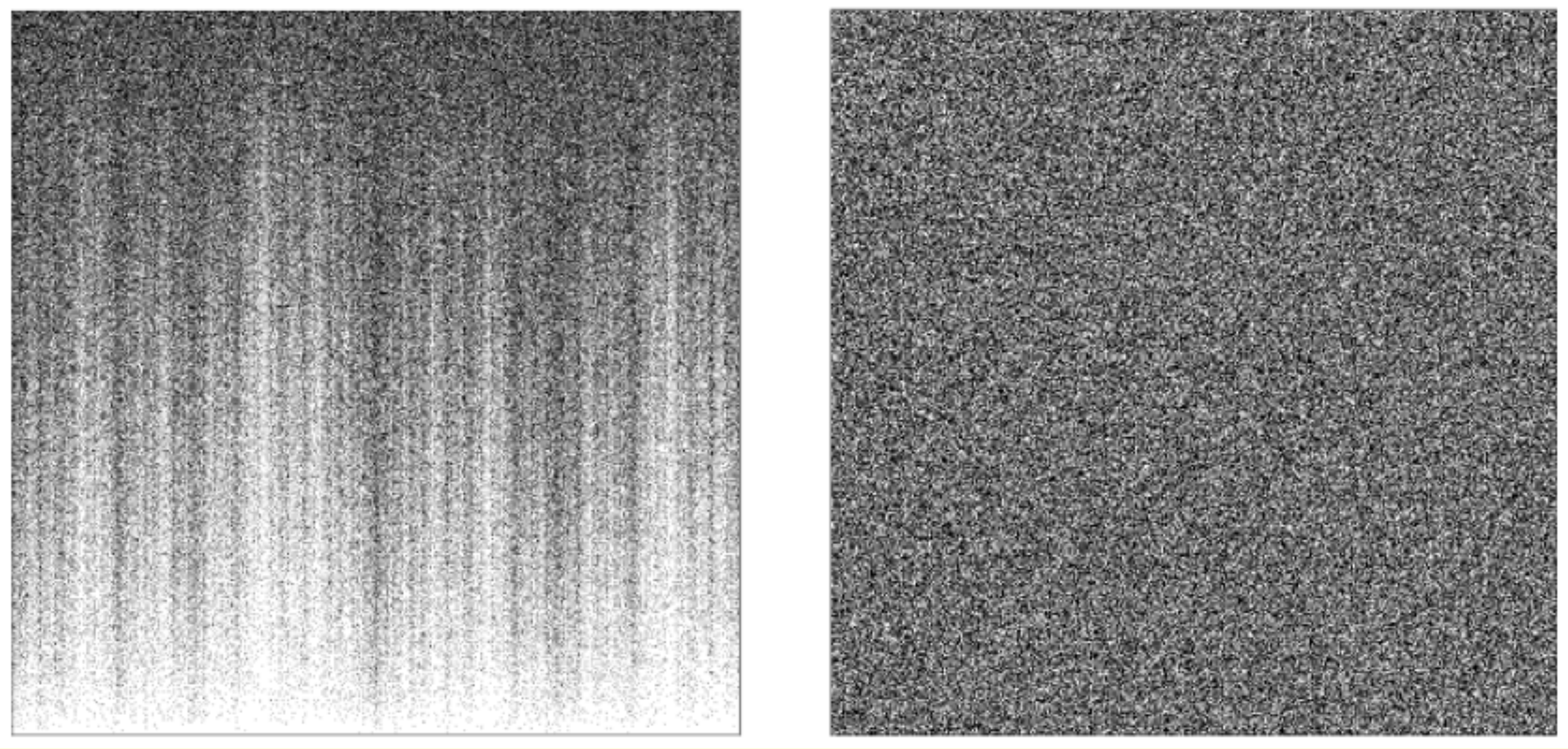
Problem 2: the uncertainty dimension does not take into account the portion of the phase space occupied by the boundary.



Both pictures have the same uncertainty dimension (α=1).

Problems and limitations 3: Riddled basins

Problem 3: the uncertainty dimension does not distinguish among riddled basins.



$\alpha \approx 0$ **…but they have different structure!!**

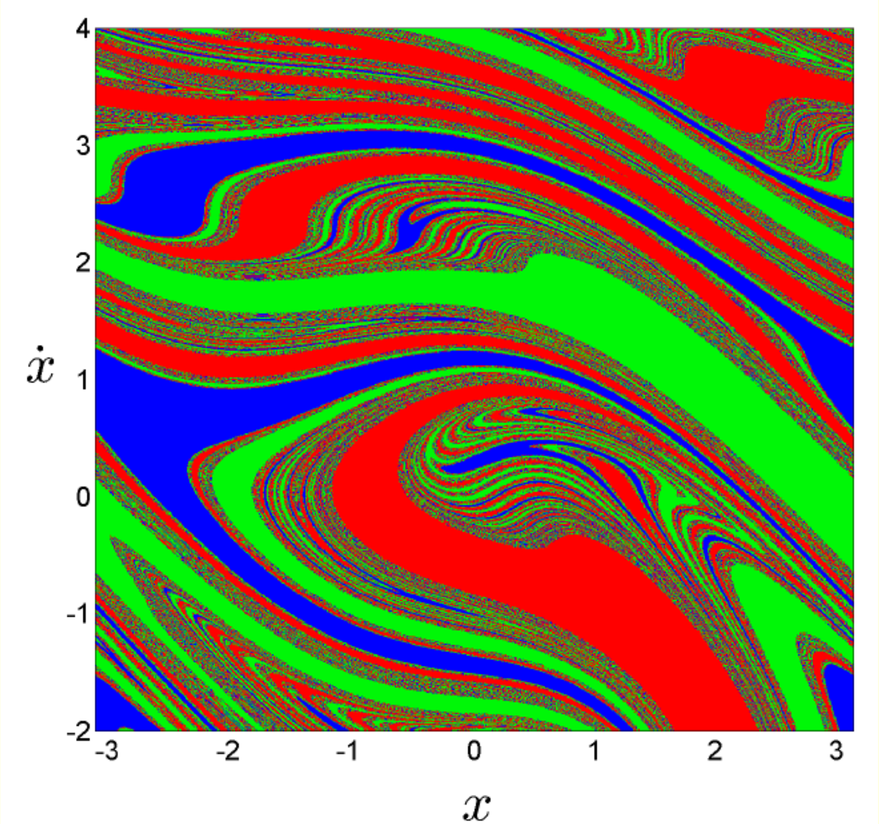
Basins of Wada*

Physica D 51 (1991) 213–225

Judy Kennedy[a] and James A. Yorke[b]

The Wada property: three or more basins have a common boundary

Wada basin boundaries: it implies more **unpredictability**



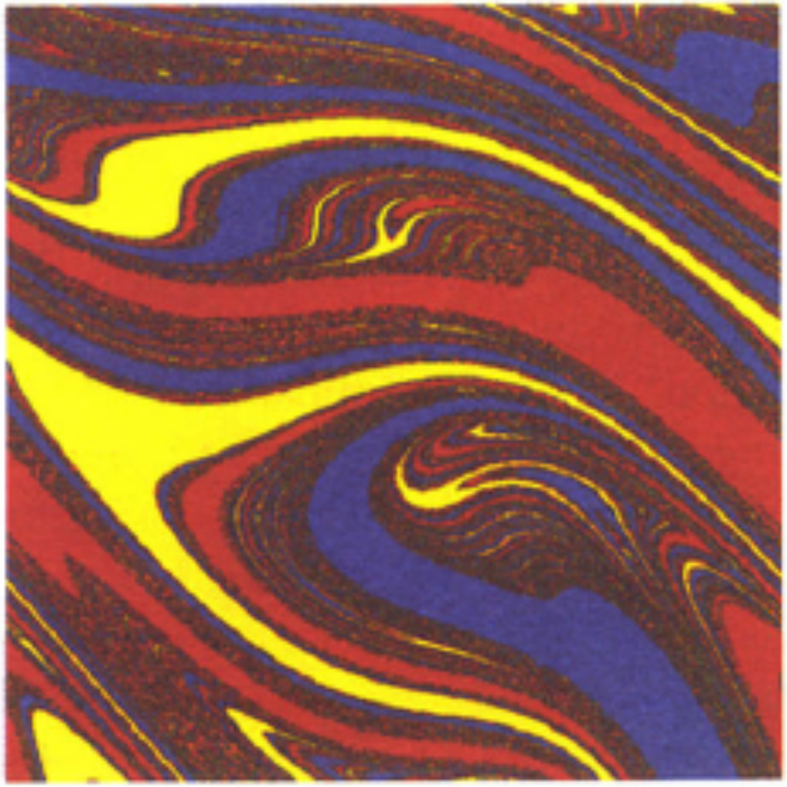
$$\ddot{x} + \delta\dot{x} + \sin x = F\cos\omega t$$

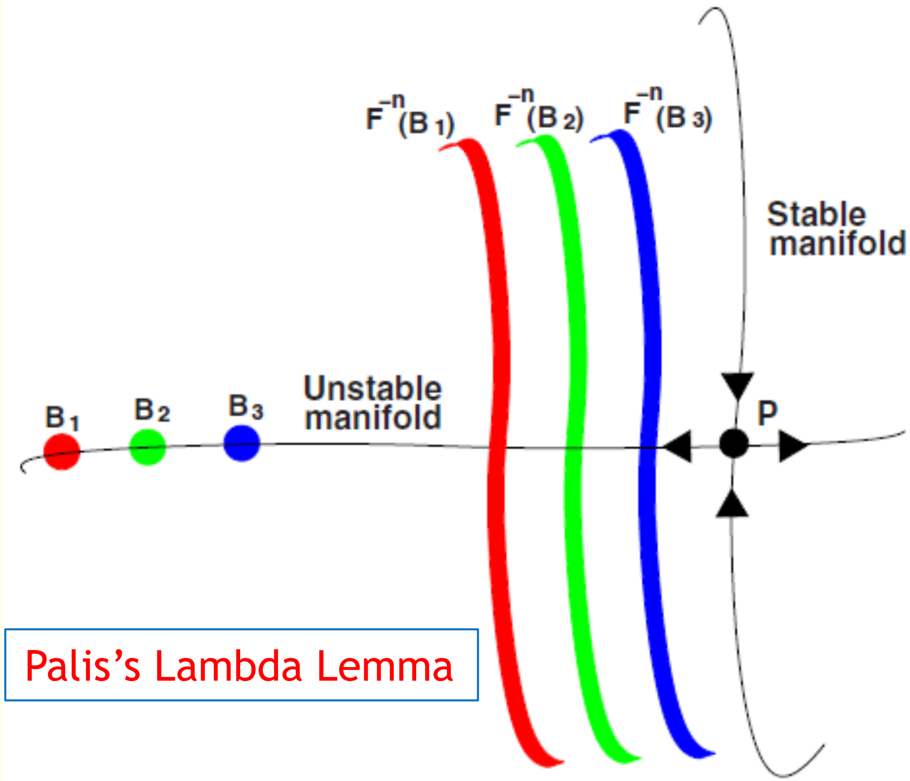
The Nusse-Yorke condition

Wada basin boundaries and basin cells ☆

Helena E. Nusse[a,b], James A. Yorke[a,c]

Physica D 90 (1996) 242–261



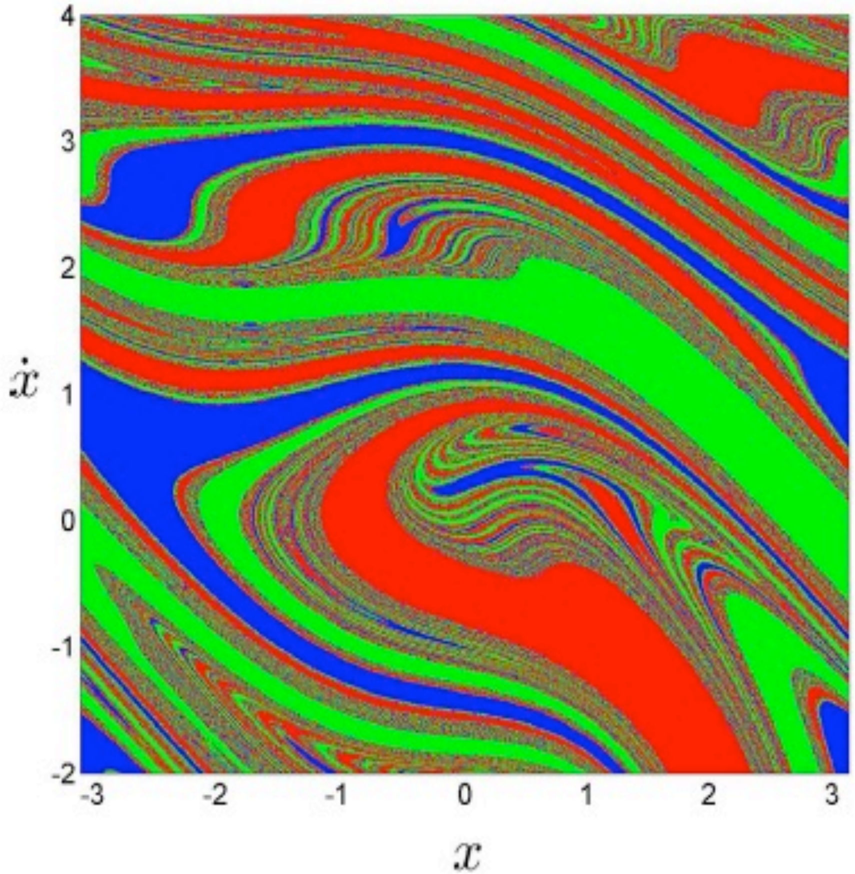


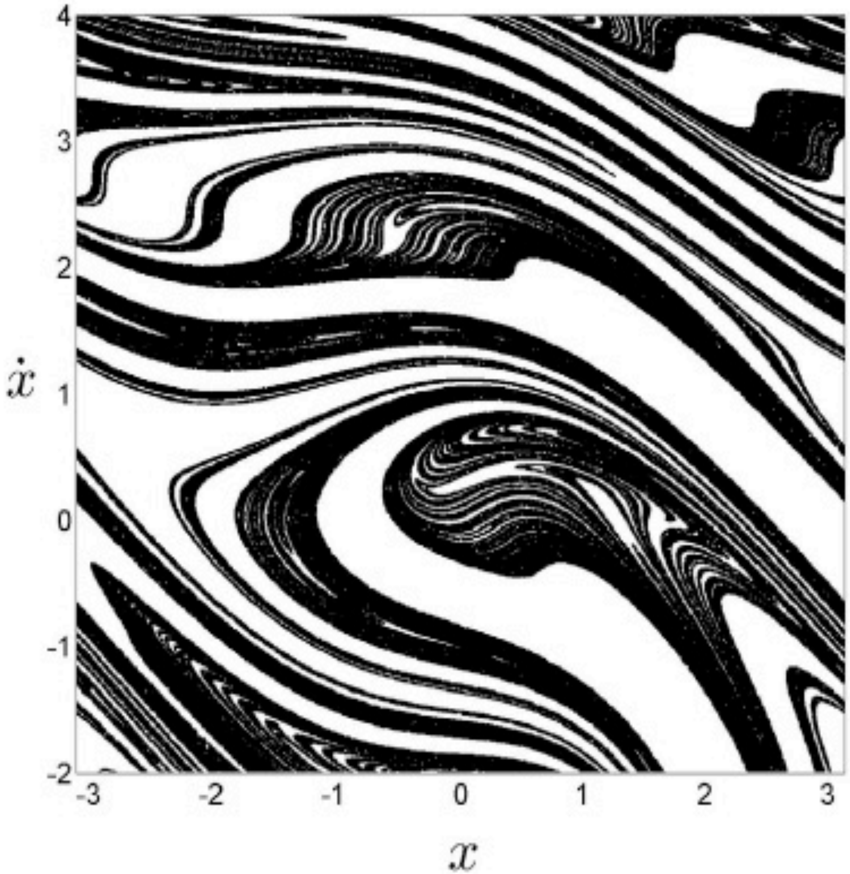
Palis's Lambda Lemma

Testing for Basins of Wada

Alvar Daza[1], Alexandre Wagemakers[1], Miguel A.F. Sanjuán[1] & James A. Yorke[2]

Scientific Reports 5, 16579 (2015)



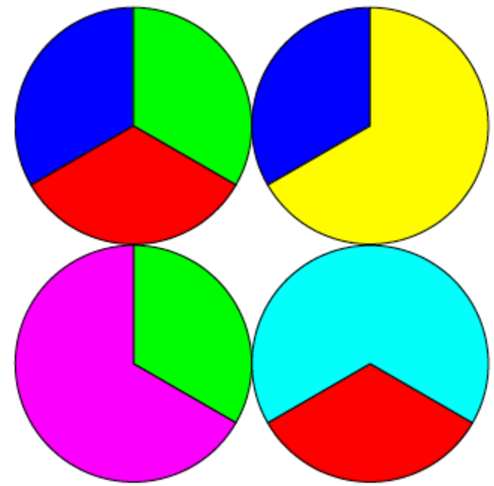


$W_3 = 1 \rightarrow$ Full Wada

Ascertaining when a basin is Wada: the merging method

Scientific Reports 8, 9954 (2018)

Alvar Daza[1], Alexandre Wagemakers[1] & Miguel A. F. Sanjuán[1,2,3]



(b)

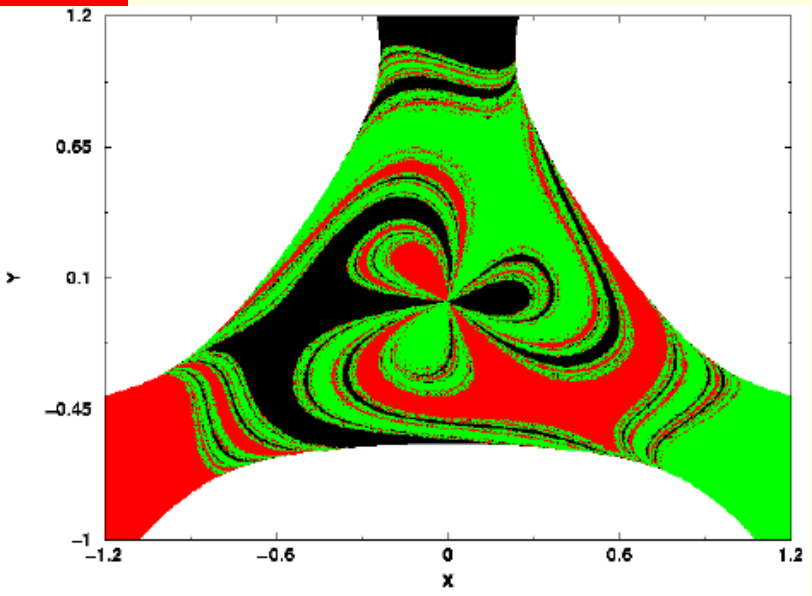
(c)

Wada boundaries remain unaltered when the basins are merged

PHYSICAL REVIEW E, VOLUME 64, 066208 (2001)

Wada basins and chaotic invariant sets in the Hénon-Heiles system

Jacobo Aguirre, Juan C. Vallejo, and Miguel A. F. Sanjuán



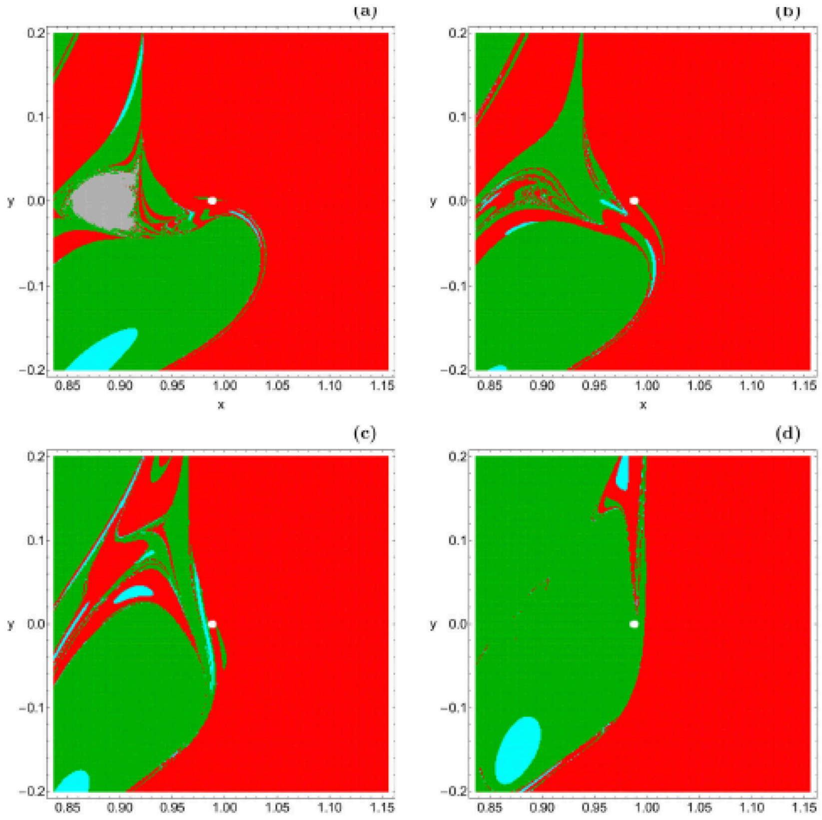
- Can we say that Wada basins are more unpredictable?
- Vague affirmations: can we measure that unpredictability?
- It seems to be a problem arising in many scientific fields…

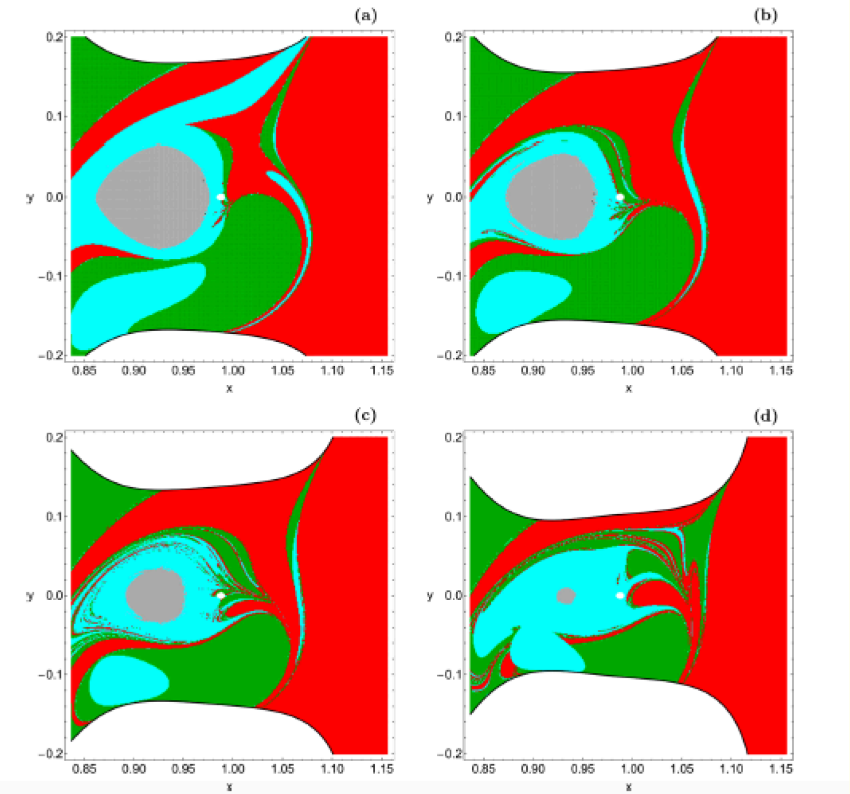
Astrophys Space Sci (2016) 361:94
DOI 10.1007/s10509-016-2683-6

ORIGINAL ARTICLE

Escape dynamics and fractal basins boundaries in the three-dimensional Earth-Moon system

Euaggelos E. Zotos[1]





THE JOURNAL OF PHYSICAL CHEMISTRY B

Article

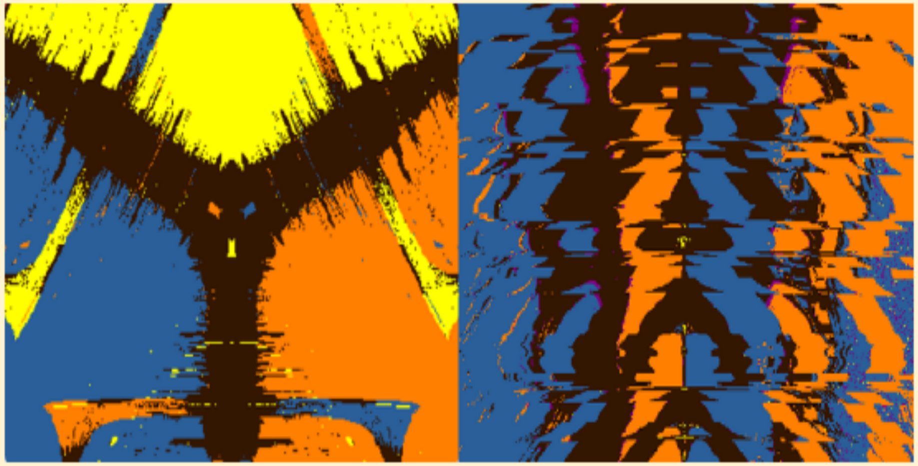
pubs.acs.org/JPCB

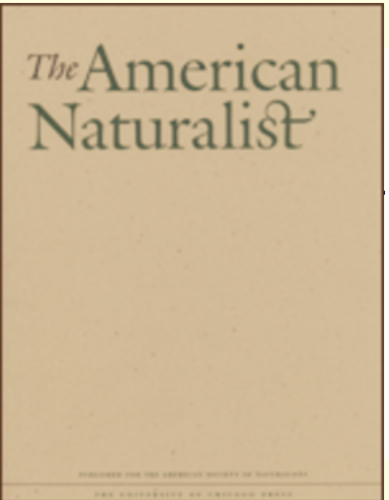
Visualizing Basins of Attraction for Different Minimization Algorithms

Daniel Asenjo, Jacob D. Stevenson,* David J. Wales, and Daan Frenkel

Department of Chemistry, University of Cambridge, Lensfield Road, Cambridge, CB2 1WE, United Kingdom

ABSTRACT: We report a study of the basins of attraction for potential energy minima defined by different minimization algorithms for an atomic system. We find that whereas some minimization algorithms produce compact basins, others produce basins with complex boundaries or basins consisting of disconnected parts. Such basins deviate from the "correct" basin of attraction defined by steepest-descent pathways, and the differences can be controlled to some extent by adjustment of the maximum step size. The choice of the most convenient minimization algorithm depends on the problem in hand. We show that while L-BFGS is the fastest minimizer, the FIRE algorithm is also quite fast and can lead to less fragmented basins of attraction.

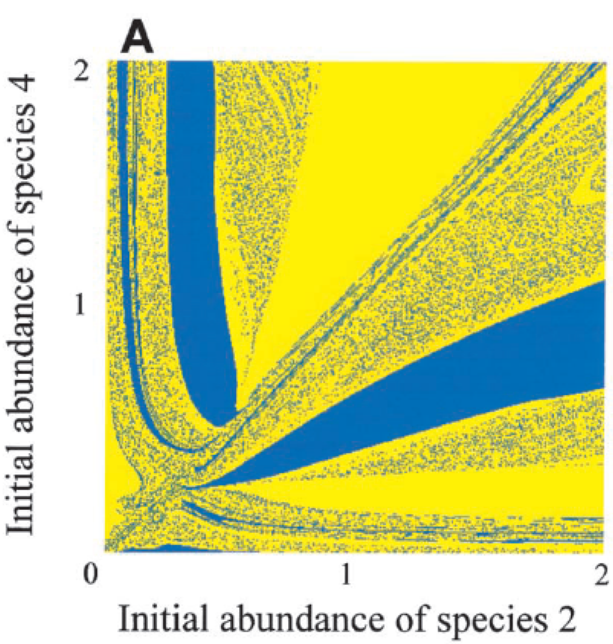


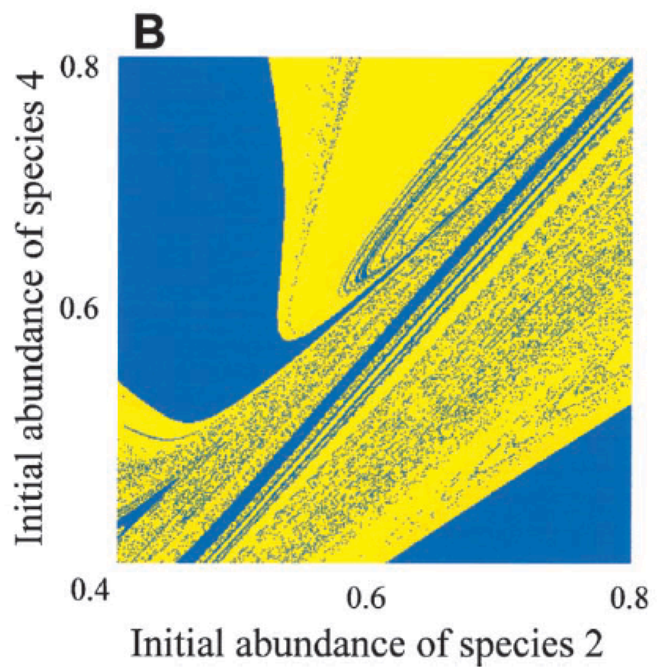


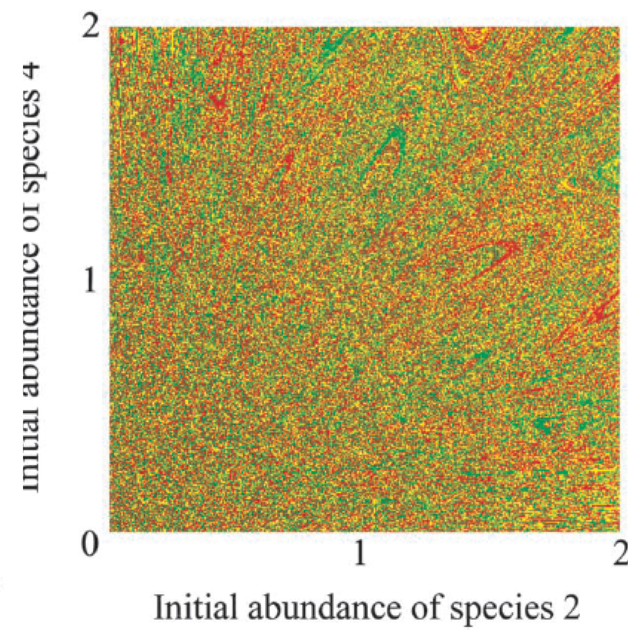
VOL. 157, NO. 5 THE AMERICAN NATURALIST MAY 2001

Fundamental Unpredictability in Multispecies Competition

Jef Huisman[1,*] and Franz J. Weissing[2,†]







PRL 107, 254104 (2011) PHYSICAL REVIEW LETTERS week ending 16 DECEMBER 2011

Exploring Classically Chaotic Potentials with a Matter Wave Quantum Probe

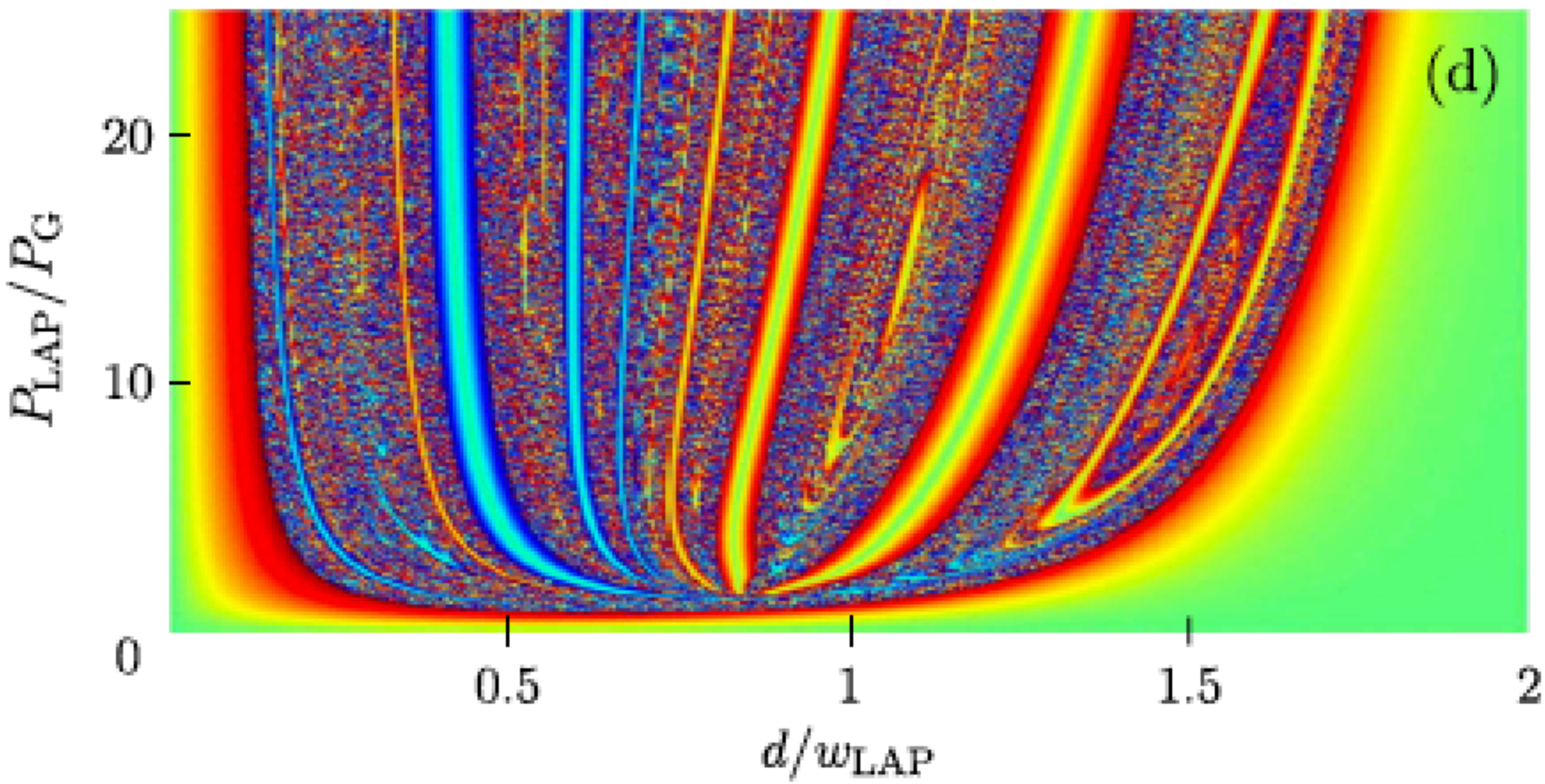
G. L. Gattobigio,[1,2] A. Couvert,[2] B. Georgeot,[3,4] and D. Guéry-Odelin[1]

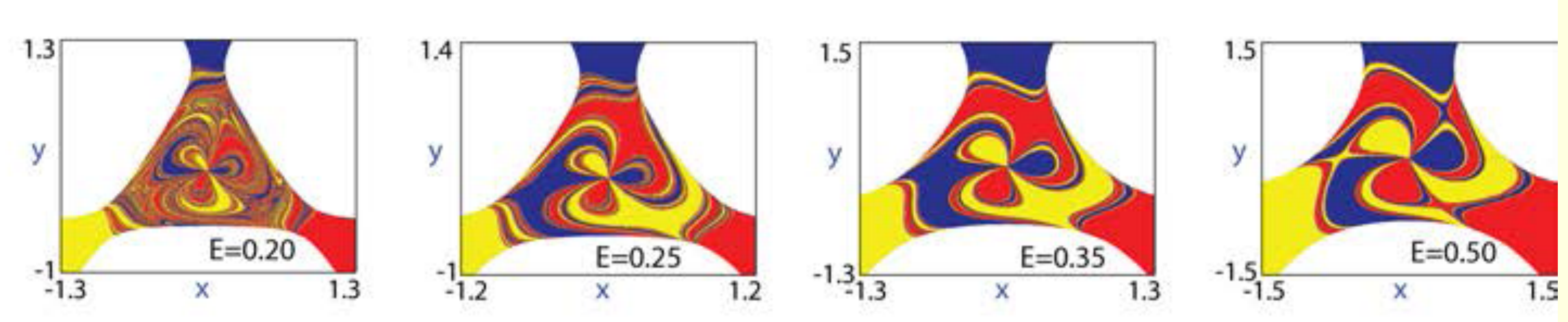
[1]*Laboratoire de Collisions Agrégats Réactivité, CNRS UMR 5589, IRSAMC, Université de Toulouse (UPS), 118 Route de Narbonne, 31062 Toulouse CEDEX 4, France*

[2]*Laboratoire Kastler Brossel, Ecole Normale Supérieure, 24 rue Lhomond, 75005 Paris, France*

[3]*Laboratoire de Physique Théorique (IRSAMC), Université de Toulouse (UPS), 31062 Toulouse, France*

[4]*CNRS, LPT UMR5152 (IRSAMC), 31062 Toulouse, France*

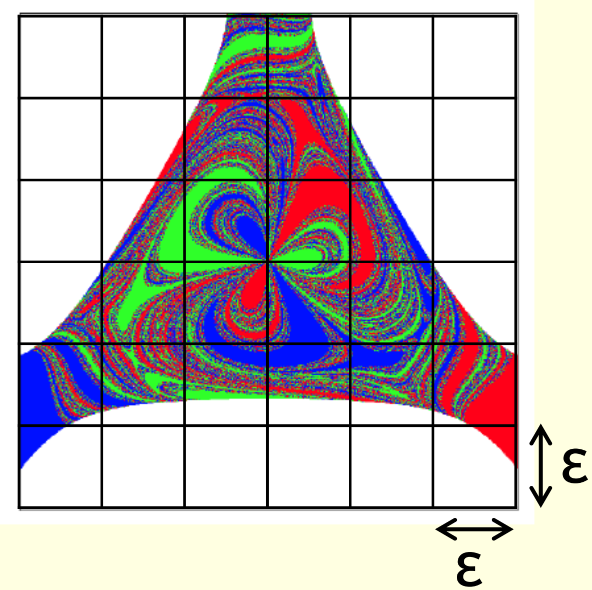




- Which basin is “more fractal”?
- Which basin has a larger uncertainty?
- How can we measure the uncertainty of a basin?

Basin entropy definition

A. Daza, A. Wagemakers, B. Georgeot, D. Guéry-Odelin, and M. A. F. Sanjuán. Basin entropy: a new tool to analyze uncertainty in dynamical systems. *Scientific Reports* 6:31416, 2016.



probability(color)=
proportion(color)
inside the box

Entropy for a box: i→box, j→color

$$S_i = -\sum_{j=1}^{N_A} p_{ij} \log p_{ij}$$

Total entropy for N boxes:

$$S = \sum_{i=1}^{N} S_i$$

Basin entropy definition:

$$S_b = \frac{S}{N}$$

Quantification of unpredictability:

$$S_b \in [0, \log N_A]$$

Number of colors or attractors

The three ingredients of the Basin Entropy

$$S_b = \sum_{k=1}^{k_{max}} \frac{n_k}{\tilde{n}} \varepsilon^{\alpha_k} \log(m_k).$$

$k \in [1, k_{max}]$ is the label for the different boundaries.

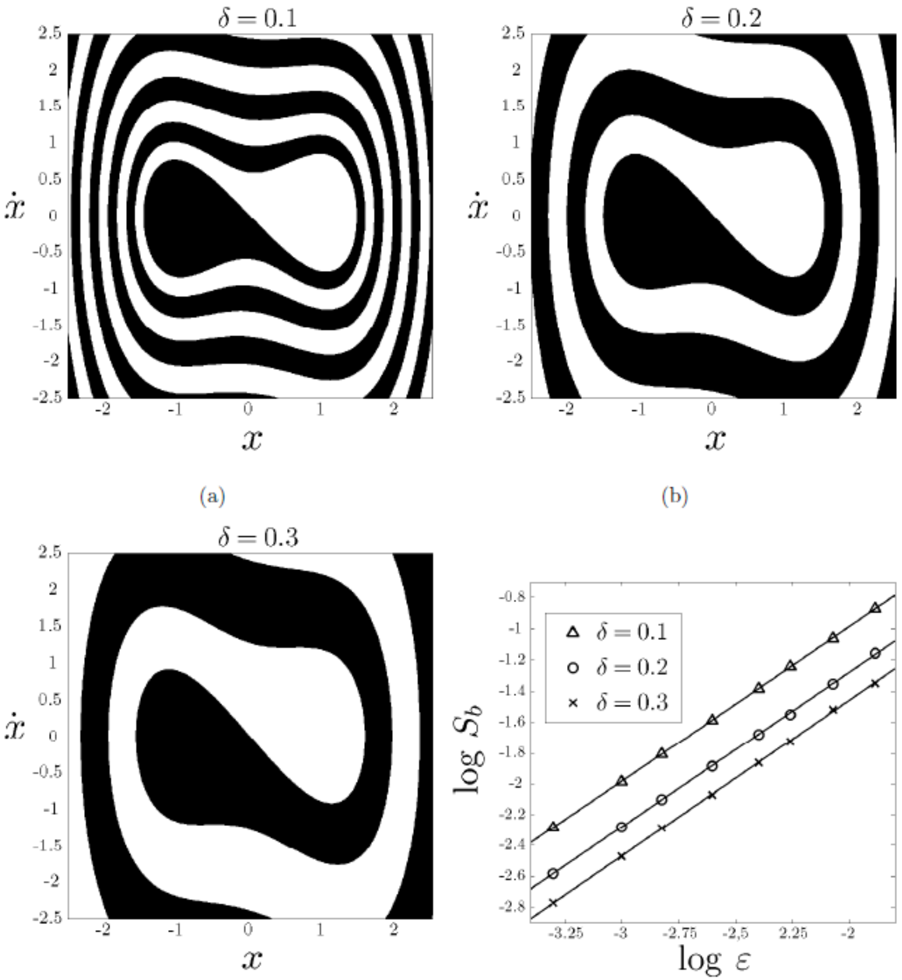
- **Boundary Size** $n_k/\tilde{n}$ independent of ε
- **Uncertainty exponent** α_k
- **Number of attractors (colors)** m_k

All these terms depend on the dynamics and are independent of the scaling box size

Dependence with the *size of the boundary*

$$\ddot{x} + \delta\dot{x} - x + x^3 = 0$$

Duffing oscillator



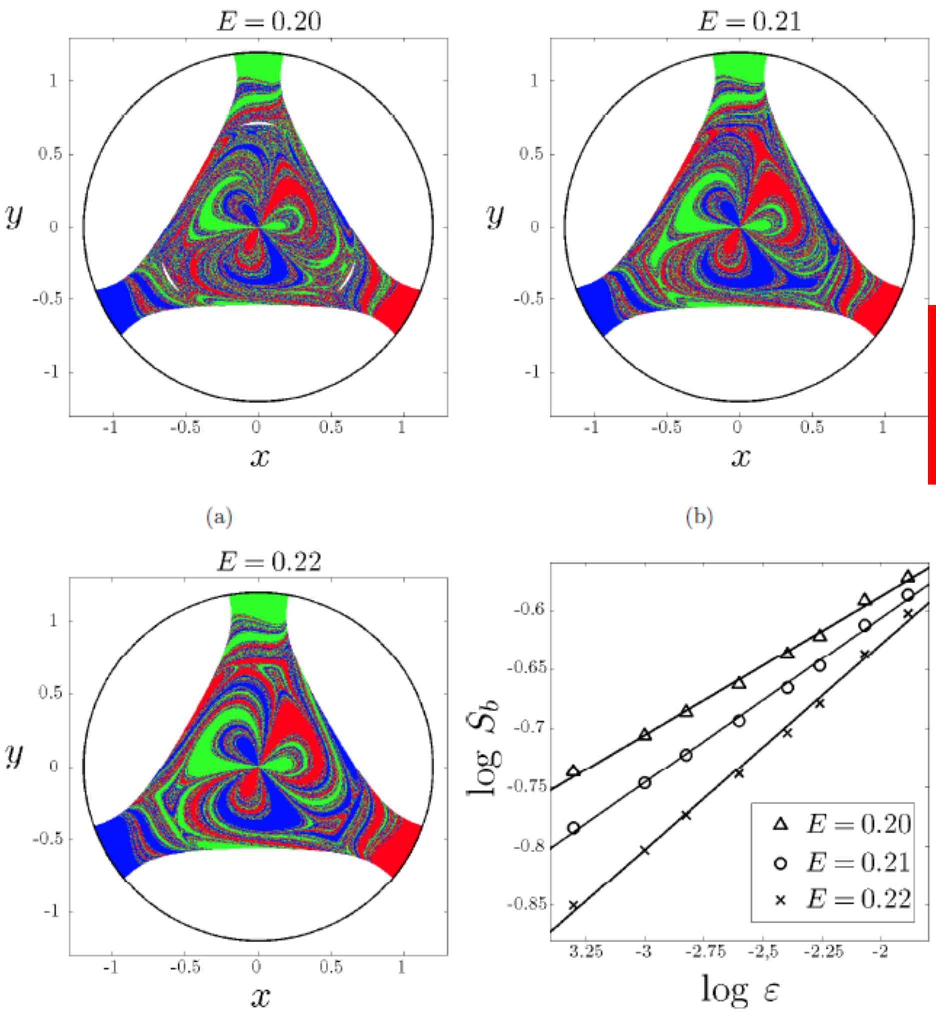
Boundary size

$$\log(S_b) = \alpha \log(\varepsilon) + \log\left(\log(N_A)\frac{n}{\tilde{n}}\right).$$

Smooth boundaries (α=1) but different basin entropy

Dependence with the uncertainty exponent

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$



Hénon-Heiles System

Uncertainty exponent

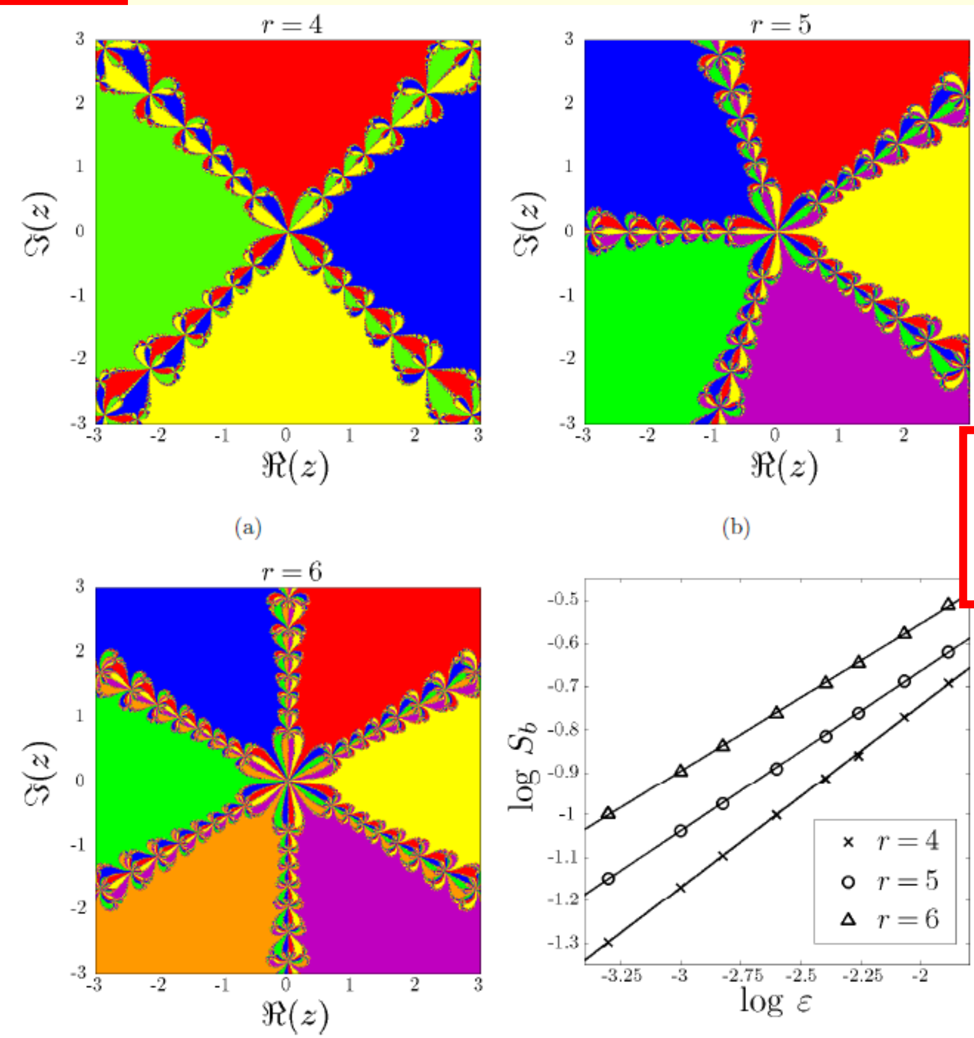
$$\log(S_b) = \alpha \log(\varepsilon) + \log\left(\log(N_A)\frac{n}{\tilde{n}}\right).$$

- Same proportion for each color 1/3, (same basin stability) but different basin entropy
- Useful also for conservative systems (escape basins).

Dependence with the number of attractors

Newton fractal:

$$z_{n+1} = z_n - \frac{z^r - 1}{r\, z^{r-1}}$$



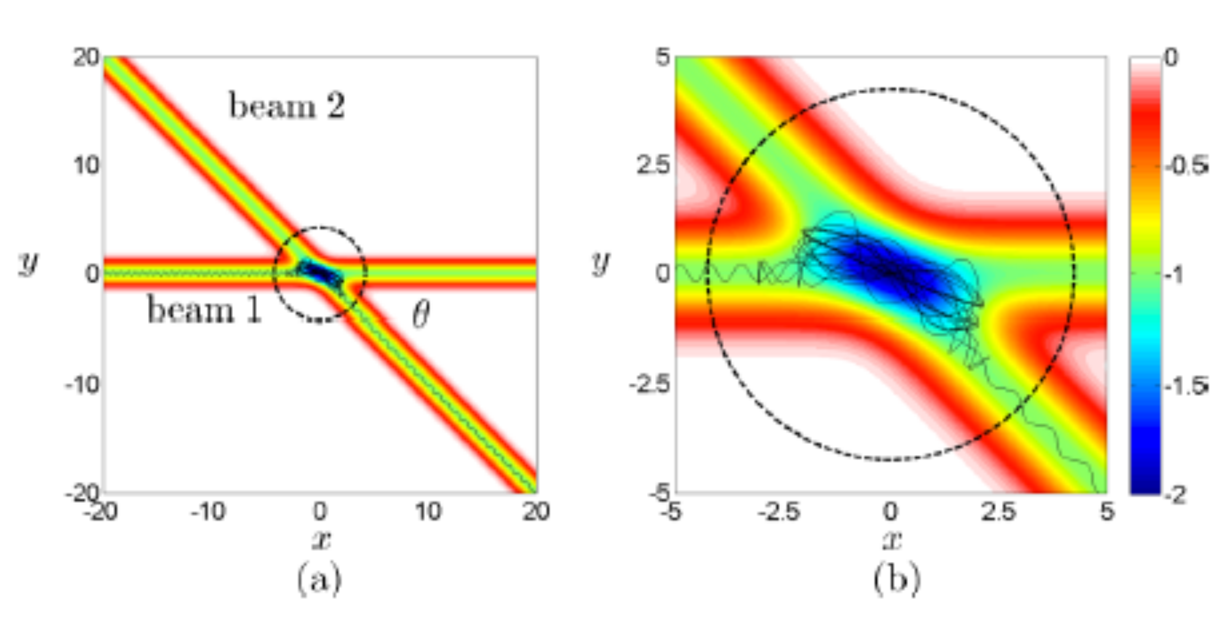
Number of attractors

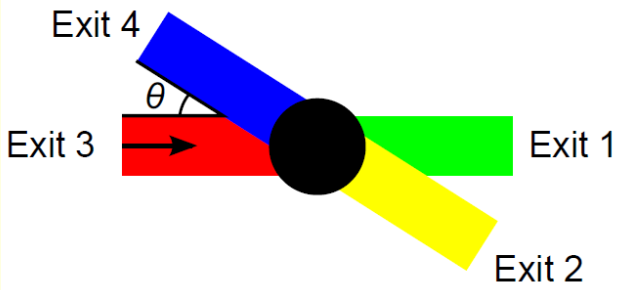
$$\log(S_b) = \alpha \log(\varepsilon) + \log\left(\log(N_A)\frac{n}{\tilde{n}}\right).$$

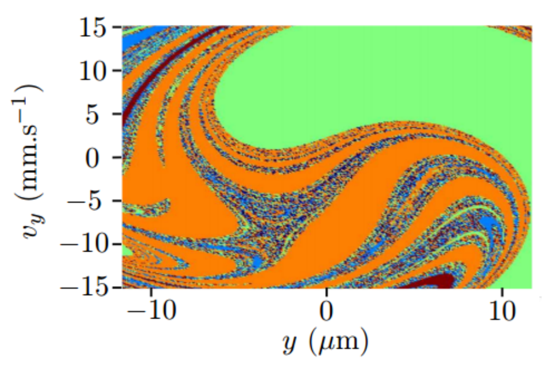
The more attractors the more unpredictable (in general)

Crossing beam model for cold atoms scattering

Gattobigio, G. L., Couvert, A., Reinaudi, G., Georgeot, B., and Guéry-Odelin, D. **Optically guided beam splitter for propagating matter waves**. *Physical Review Letters*, *109*, 030403 (2012).







$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \alpha_1 e^{-\beta_1 y^2} - \alpha_2 e^{-\beta_2 (x \sin\theta + y \cos\theta)^2}$$

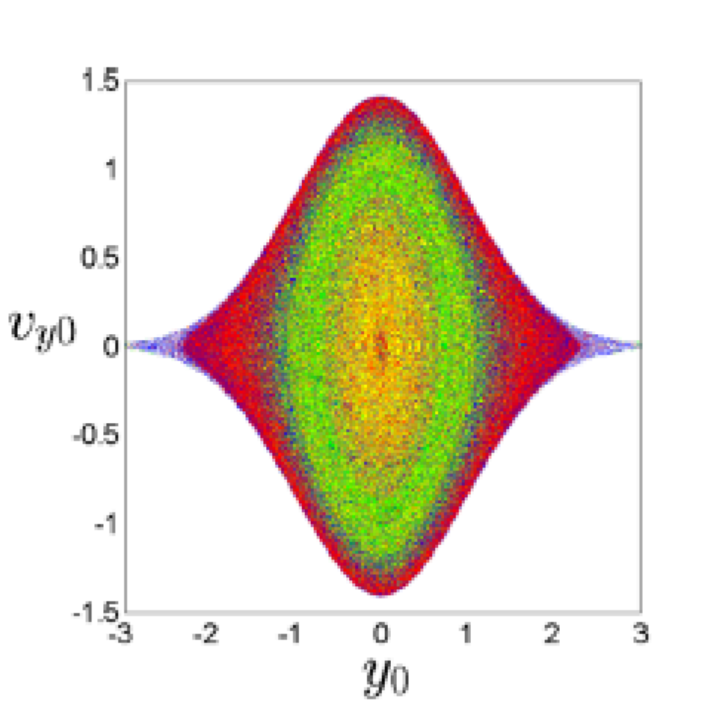
beam 1 beam 2

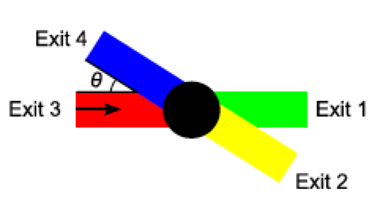
PHYSICAL REVIEW A 95, 013629 (2017)

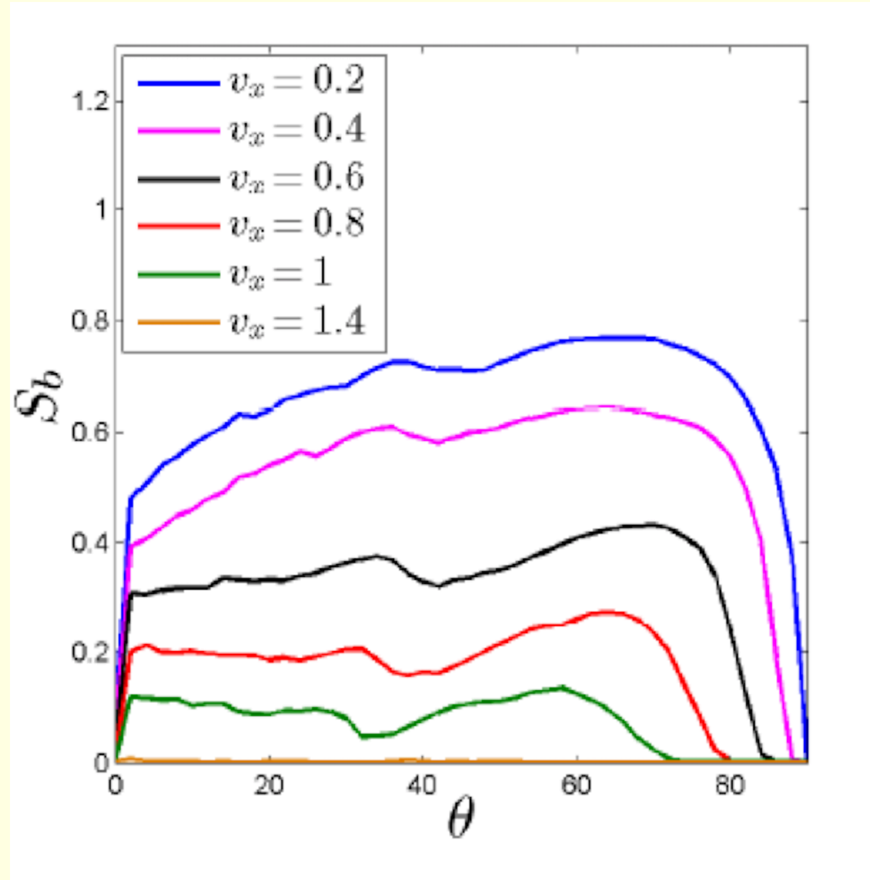
Chaotic dynamics and fractal structures in experiments with cold atoms

Alvar Daza,[1] Bertrand Georgeot,[2] David Guéry-Odelin,[3] Alexandre Wagemakers,[1] and Miguel A. F. Sanjuán[1,*]

Escape basin for low speeds







The basin entropy indicates that for low speed the unpredictability is higher.

Escape Basins and Basin Entropy versus Energy in the conservative Hénon-Heiles Hamiltonian

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

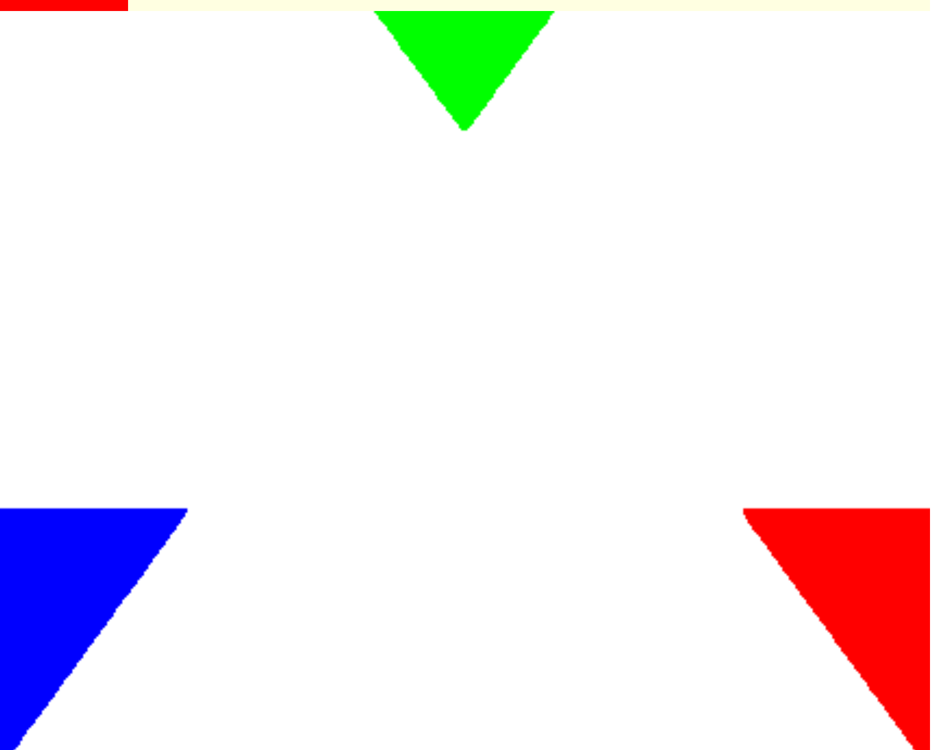
$$E \in [0.1665, 0.45]$$

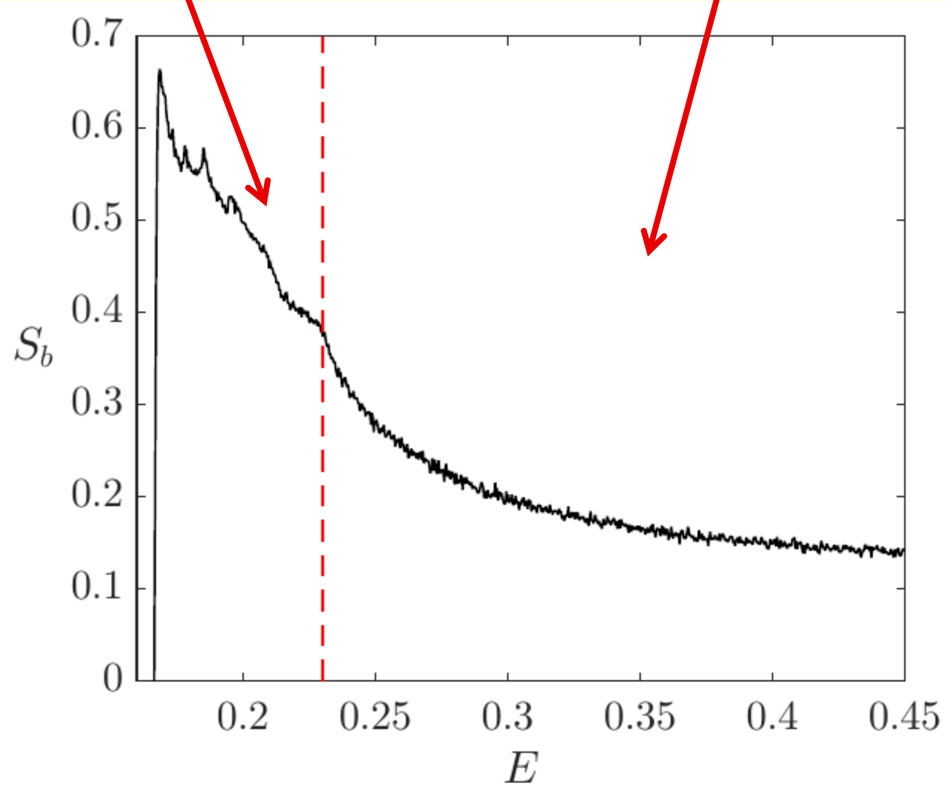
Nonhyperbolic regime

- Fluctuations in Sb
- Presence of KAM islands

Hyperbolic regime

- Monotonous decrease in S_b
- Absence of KAM islands





Basin Entropy vs β in the relativistic Hénon-Heiles system

PHYSICAL REVIEW E **95**, 032205 (2017)

Global relativistic effects in chaotic scattering

Juan D. Bernal,[1,*] Jesús M. Seoane,[1] and Miguel A. F. Sanjuán[1,2]

PHYSICAL REVIEW E **97**, 042214 (2018)

Uncertainty dimension and basin entropy in relativistic chaotic scattering

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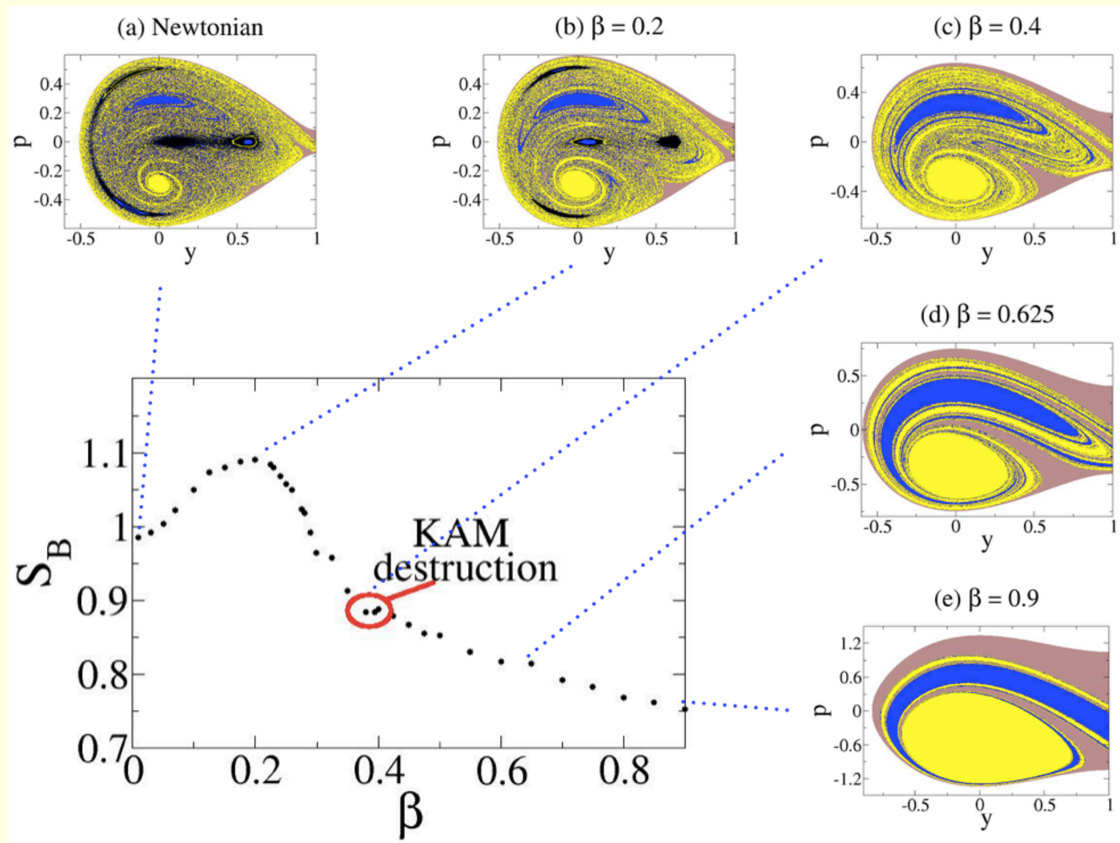
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\dot{x} = \frac{p}{\gamma},$$

$$\dot{y} = \frac{q}{\gamma},$$

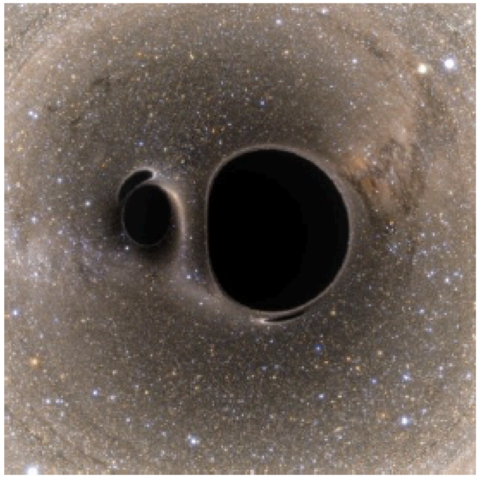
$$\dot{p} = -x - 2xy,$$

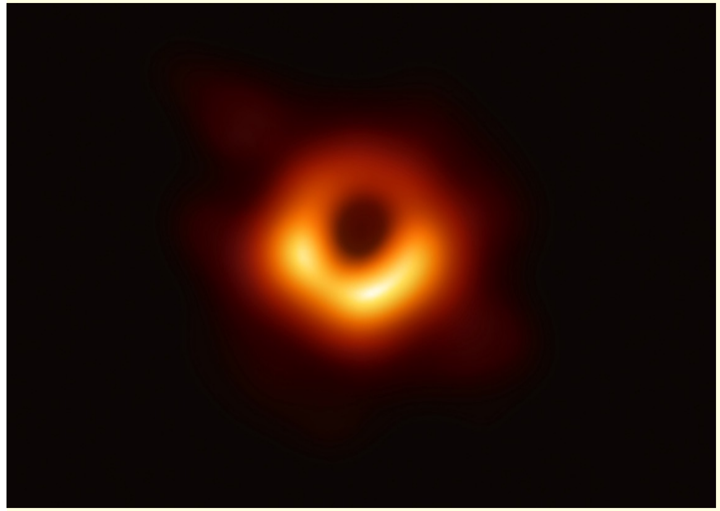
$$\dot{q} = -y - x^2 + y^2,$$



BLACK HOLE SHADOWS

Binary BH and LIGO

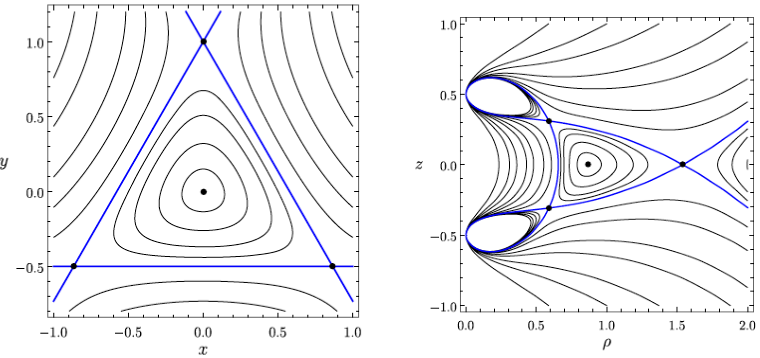




The Event Horizon Telescope collaboration made the first image of a black hole (BH shadow) at the center of the M87 galaxy **(April 10, 2019).**

- We can simulate BH shadows.
- **Ray-tracing**: trace photons away from camera's lens, backwards in time.
- **Shadow** is region of image where rays are traced back to a BH.
- Rays that escape to ∞ are bright regions of image.

The Majundar-Papapetrou Binary Black Hole



(a) Hénon–Heiles (b) Majumdar–Papapetrou

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Wada structures in a binary black hole system

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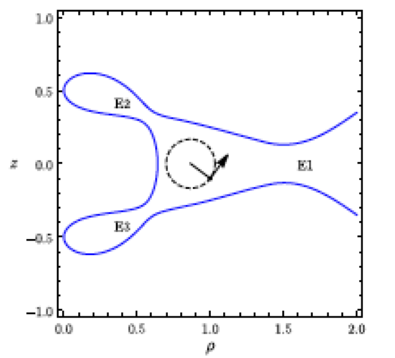
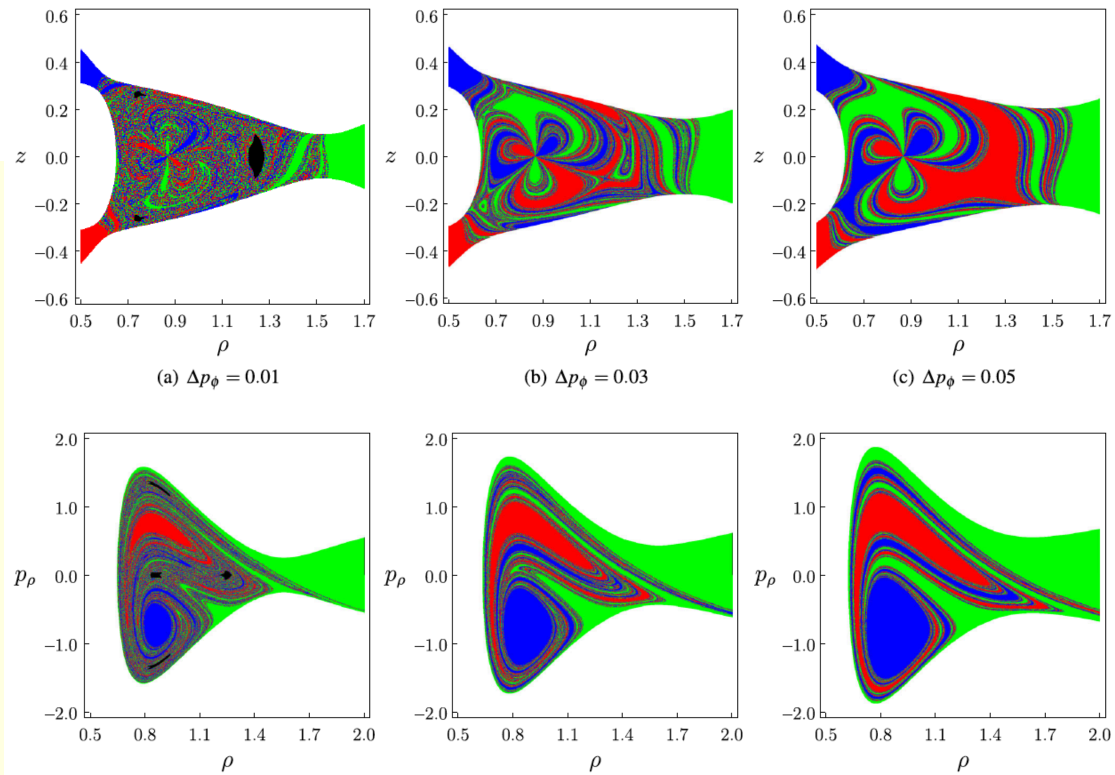


(a) (ρ, z)-space

(b) (ρ, p_ρ)-space

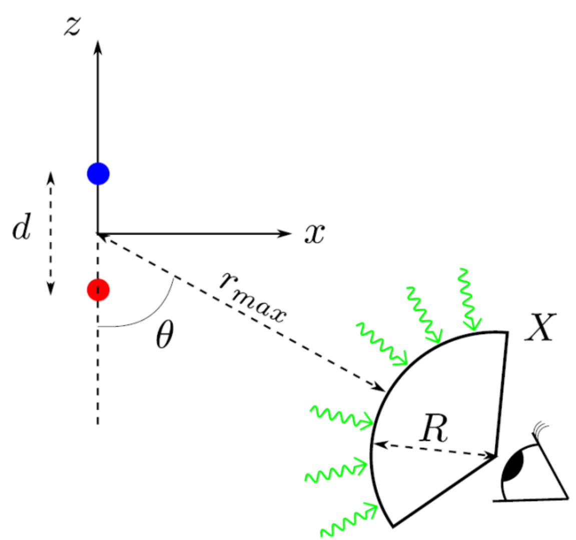


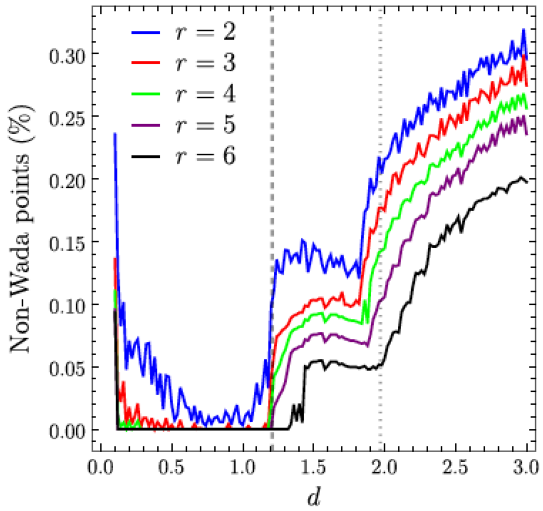
(a) $\Delta p_\phi = 0.01$ (b) $\Delta p_\phi = 0.03$ (c) $\Delta p_\phi = 0.05$

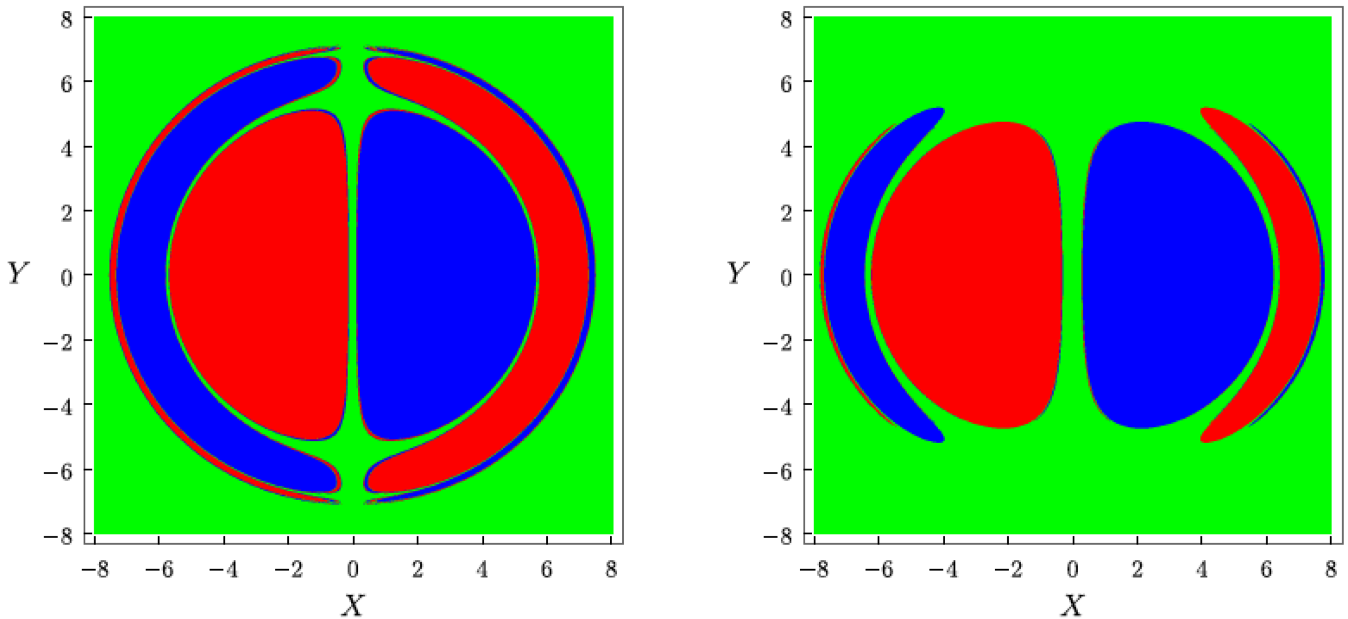
(d) $\Delta p_\phi = 0.01$ (e) $\Delta p_\phi = 0.03$ (f) $\Delta p_\phi = 0.05$

General Relativity leads to Nonlinear Dynamics

Shadows of the MP binary BH







(a) $d = 1$ (b) $d = 2$

FIG. 5. Shadows cast by the static MP binary BH for different values of the separation d. The photons which escape to spatial infinity are plotted in green; the shadow cast by the upper (lower) BH is plotted in blue (red). These three open sets can be viewed as exit basins, defined on the image plane of a distant observer.

Summary and Conclusions

- We have developed new methods for testing Wada basins: The **Grid Method**, the **Merging Method** and the **Saddle-straddle method**

- The **basin entropy quantifies** the final state **unpredictability** of dynamical systems. It constitutes a new tool for the **exploration of the uncertainity** in nonlinear dynamics

- We have applied these methods to different domains in Physics, such as **cold atoms**, shadows of **binary black holes**, and classical and relativistic chaotic scattering in **astrophysics**.

- We believe that the concept of **Basin Entropy** will become an **important tool in complex systems studies** with applications in multiple scientific fields especially those with multistability and other scientific areas as well.

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