UDK 530.1;538.0

VARIATIONAL PRINCIPLE AND LOW-FREQUENCY DYNAMICS OF MAGNETIC SYSTEMS

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Received November 1, 1993

Abstract

Hamiltonian approach to the low-frequency dynamics of manysublattice magnets has been developed. The Poisson brackets for dynamical variables are derived from variational principle proceeding from transformations that leave invariant the kinematic part of the action. Nonintegral terms of the action variation are interpreted as generators of these transformations. Magnetic systems with totally broken symmetry with respect to spin rotations and antiferromagnetics are considered. Hydrodynamical asymptotics of Green functions are found. In the case of description of magnet on the base of Landau-Lifshits equations it is shown how the reduction does occur from complete set of variables (sublattice spins) to the short description variables (rotation matrix and density of total spin).

1 Introduction

Determination of the Poisson brackets (PB) for dynamic variables plays a principal role in the Hamiltonian approach to the theory of various physical systems [1]. In contrast to ordinary mechanical systems, in case of condensed media the PB have a nontrivial structure. This report shows how to obtain the PB for different physical systems, proceeding from the transformations that leave invariant the kinematic part of the action. Nonintegral terms of the action variation are interpreted as generators of these transformations. The equations for finding of admitable transformations are obtained. In terms of developed approach we have considered the magnetic systems with totally broken symmetry with respect to spin rotations, and antiferromagnetics. In each case the density of summarised spin introduced as generator of spin rotations. Low-frequency asymptote of antiferromagnet Green's functions for arbitrary dynamical quantities a and b has been found as well as the spectrum of spin waves. Futher we consider the manysublattice magnet with exchange interaction as an example of the system with total breaking of symmetry.In the hydrodynamical stage of evolution the state of manysublattice magnetic described by orthogonal matrix of rotations and density of total spin. It is shown how to get dynamical equations for these variables starting from Landau-Lifshits equations for sublattice spins. Solving the variational problem for finding equilibrium distribution of spins in the space uniform case we write the form of local-equilibrium distribution and on base of this formulate the functional hypothesis. Futher we show how the reduction does occur from complete description of magnet in terms of sublattice spins to reduced description with help of rotation matrix and density of total spin. Introducing reduced functionals, we write dynamical equations for reduced variables and compare them with equations, obtained in general case for system with totally broken symmetry.

2 Fundamentals of the formalism

First we give the formulation of formalism for physical systems with finite number of degrees of freedom and after that for continuum systems. Systems with finite number of degrees of freedom. Let us consider the Lagrange function of the system

$$L = L_k(x, \dot{x}) - H(x), \tag{2.1}$$

where x_i are dynamical variables, H is the Hamiltonian, L is a kinematical part of the Lagrangian, which we take in the from

$$L_k(x, \dot{x}) = F_i(x)\, \dot{x}_i\,. \tag{2.2}$$

Here $F_i(x)$ is an arbitrary function of dynamical variables. Now consider infinitesimal transformations

$$x_i \longrightarrow x_i' = x_i + \delta x_i. \tag{2.3}$$

Variation of action $W = \int\limits_{t_1}^{t_2} L dt$, connected with transformations (2.3) equals

$$\delta W = G(t_2) - G(t_1) + \int\limits_{t_1}^{t_2} dt \delta x_j \left(J_{ji}(x)\, \dot{x}_i - \frac{\partial H}{\partial x_j} \right), \tag{2.4}$$

where

$$G(t) = F_i(x) \delta x_i, \qquad J_{ji} = \frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}.$$

The equations of motion for variables x_i, taking into account the principle of stationary action, have the form

$$J_{ji}(x)\, \dot{x}_i = \frac{\partial H}{\partial x_j}. \tag{2.5}$$

If the matrix inverse with respect to matrix J_{ij} does exist, we obtain

$$\dot{x}_i = J_{ij}^{-1}(x) \frac{\partial H}{\partial x_j}. \tag{2.6}$$

Let us define the PB for arbitrary functions $A(x)$ and $B(x)$:

$$\{A, B\} = \frac{\partial A}{\partial x_i} J_{ij}^{-1}(x) \frac{\partial B}{\partial x_j}, \tag{2.7}$$

therefore

$$\dot{x}_i = \{x_i, H\}.$$

It is easy to see, that operation (2.7) satisfies the conditions

$$\{A, B\} = -\{B, A\}, \{AB, C\} = A\{B, C\} + B\{A, C\}$$

and Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \tag{2.8}$$

For deriving (2.8) it is nessesary to take into account formula

$$\frac{\partial J_{ij}}{\partial x_k} + \frac{\partial J_{jk}}{\partial x_i} + \frac{\partial J_{ki}}{\partial x_j} = 0. \tag{2.9}$$

Hamiltonian mechanics written in terms of arbitrary variables was investigated in [2]. There formula, similar to (2.9) was obtained. Now consider finite transformations

$$x_i \longrightarrow x'_i = x'_i(x). \tag{2.10}$$

Transformations (2.10) are called canonical if the difference $F_i(x)dx_i - F_i(x')dx'_i$ is a full differential:

$$\left(F_i(x) - F_j(x')\frac{\partial x'_j}{\partial x_i}\right) dx_i = CQ. \tag{2.11}$$

The equality

$$\frac{\partial F_i(x)}{\partial x_k} - \frac{\partial F_j(x')}{\partial x'_l}\frac{\partial x'_l}{\partial x_k}\frac{\partial x'_j}{\partial x_i} = F_j(x')\frac{\partial^2 x'_j}{\partial x_i \partial x_k} \; (i \leftrightarrow k),$$

or its equivalent form

$$J_{ik}(x) = J_{jl}(x')\frac{\partial x'_j}{\partial x_i}\frac{\partial x'_l}{\partial x_k}. \tag{2.12}$$

is necessary and sufficient condition for (2.11). Let us show, that canonical transformations (2.10), (2.12) keep invariant PB (2.7). For this purpose it is suffisient to show invariance of elementary brackets $\{x_i, x_j\}$. In other words, if

$$\{x_i, x_j\} = J^{-1}_{ij}(x)$$

as it follows from (2.7) then

$$\{x'_i, x'_j\} = J^{-1}_{ij}(x'). \tag{2.13}$$

We note to proof the latter, that

$$\{x'_i, x'_j\} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x'_j}{\partial x_l}J^{-1}_{kl}(x)$$

and at the canonical condition (2.2) indeed the expression (2.13) emerges. Introduce operator PB $\hat{\Lambda}(g)$

$$\hat{\Lambda}(g) = \frac{\partial g}{\partial x_k}J^{-1}_{ki}\frac{\partial}{\partial x_i}, \tag{2.14}$$

where g is an arbitrary function of x. Then, from (2.7) and (2.14) it follows

$$\hat{\Lambda}(g)f = \{g, f\}. \tag{2.15}$$

Using operator PB, the Jacobi identity (2.8) and the general form of canonical transformations (2.9), (2.12) can be written in a compact form. Noting that Jacobi identity (2.8) can be represented in the form

$$\hat{\Lambda}(A)\{B,C\} = \{\hat{\Lambda}(A)B, C\} + \{B, \hat{\Lambda}(A)C\}, \tag{2.16}$$

we obtain

$$[\hat{\Lambda}(A), \hat{\Lambda}(B)]C = \hat{\Lambda}\left(\{A, B\}\right)C,$$

where $[\hat{\Lambda}(A), \hat{\Lambda}(B)]$ is the commutator of $\hat{\Lambda}(A)$ and $\hat{\Lambda}(B)$. Hence since C is an arbitrary quantity, we get

$$[\hat{\Lambda}(A), \hat{\Lambda}(B)] = \hat{\Lambda}\left(\{A, B\}\right). \tag{2.17}$$

Let us consider the transformation

$$x'(x) = \exp(-\hat{\Lambda}(g))x \tag{2.18}$$

and let us show that it is canonical. Due to (16) the formula

$$\exp\left(-\hat{\Lambda}(g)\right)\{f, q\} = \{\exp\left(-\hat{\Lambda}(g)\right)f, \exp\left(-\hat{\Lambda}(g)\right)q\}$$

is valid. Therefore

$$\{x_i', x_j'\} = \exp\left(-\hat{\Lambda}(g)\right)\{x_i, x_j\} = \exp\left(-\hat{\Lambda}(g)\right)J_{ij}^{-1}(x) = J_{ij}^{-1}(x').$$

On the other hand since

$$\{x_i', x_j'\} = \frac{\partial x_i'}{\partial x_k}\frac{\partial x_j'}{\partial x_l}J_{kl}^{-1}(x),$$

then

$$J_{ij}^{-1}(x') = \frac{\partial x_i'}{\partial x_k}\frac{\partial x_j'}{\partial x_l}J_{kl}^{-1}(x),$$

what is equivalent to (2.12). We note that the transformations (2.18) are determined by an arbitrary function $g(x)$ which is called the generating function of canonical transformation. Consider in detail the question of invariance of Hamiltonian equation under the canonical transformations (2.18), the generating function of which obviously depends on time: $g = g(x, t)$. Evidently

$$\dot{x}' = \{x', H\} + \frac{\partial' \exp\left(-\hat{\Lambda}(g)\right)}{\partial t}x, \tag{2.19}$$

(the derivative connected with obvious dependence on t is denoted as $\partial'/\partial t$). Since the following formula (see (2.17)) is valid,

$$\exp\left(\hat{\Lambda}(g)\right)\hat{\Lambda}(f)\exp\left(-\hat{\Lambda}(g)\right) = \hat{\Lambda}\left(\exp(\hat{\Lambda}(g)f)\right),$$

then for partial derivative over time in (2.19) one can obtain an expression

$$\frac{\partial' \exp(-\hat{\Lambda}(g))}{\partial t} x = -\int_0^1 d\lambda \hat{\Lambda} \left(\exp\left(-\hat{\Lambda}(g)\right) \frac{\partial' g}{\partial t}\right) x'.$$

The motion equation (2.19) taking into account (2.15) can be finally written in the form

$$\dot{x}' = \{x', H'\}, H' = H + \int_0^1 d\lambda \exp\left(-\hat{\Lambda}(g)\right) \frac{\partial' g}{\partial t}.$$

Consider the transformations (2.10) under which the kinematic part of Lagrangian (2.2) is invariant. It is easy to see that they satisfy the relationship

$$F_i(x) = F_j(x') \frac{\partial x'_j}{\partial x_i} \tag{2.20}$$

and as it follows from the definition (2.11) are canonical with $Q \equiv$ const. For the infinitesimal transformations $x'_k = x_k + \delta x_k(x)$ the equality (2.20) can be written as follows

$$J_{ik} \delta x_k = \frac{\partial G}{\partial x_i}, \qquad G \equiv F_k \delta x_k,$$

or taking into on account (2.7) it reads

$$\delta x_k = \{x_k, G\}. \tag{2.21}$$

Upon this the quantity G being nonintegral term of the action variation (see (2.4)) which plays role of a generator of considered infinitesimal canonical transformations. This quantity can be obtained from the equation

$$F_i(x) J^{-1}_{ik}(x) \frac{\partial \ln G}{\partial x_k} = 1.$$

If the considered transformations are the symmetry transformations (i.e., the Hamiltoni~n H is invariant under such transformations), then the generator is independent of time. Indeed,

$$G = \{G, H\} = -\{H, G\} = \delta H,$$

where δH is the Hamiltonian variation connected with the transformations (2.21). Hence, if $\delta H = 0$, then and G-0. Note that since the system Lagrangian is defined with accuracy up to the full derivative over time of the arbitrary function $f(x)$

$$L \longrightarrow L' = L + \frac{df(x)}{dt}, \tag{2.22}$$

then the transformation (2.22) leads to the transformation of functions $F_i(x)$

$$F_i(x) \longrightarrow F'_i(x) = F_i(x) + \frac{\partial f(x)}{\partial x_i}.$$

Upon this the tensor J_{ij} remains invariant and a class of variations δx_i leaving invariant the kinematic part of Lagrangian extends and is determined by the relationship

$$J_{ij}\delta x_j = \frac{\partial}{\partial x_i}\left(G + \frac{\partial f}{\partial x_j}\delta x_j\right).$$

For finite transformations (2.10) the invariance condition of the kinematic part of Lagrangian (with taking into account the indefiniteness (2.22)) is written as follows

$$F_i(x) = F_j(x')\frac{\partial x'_j}{\partial x_i} + \frac{\partial f(x')}{\partial x_i} - \frac{\partial f(x)}{\partial x_i}. \tag{2.23}$$

The transformations (2.10),(2.23) are canonical with $Q = f(x') - f(x)$.

Continuum systems. We define the Lagrangian by the expression

$$L = L_k(\varphi, \dot{\varphi}) - H(\varphi) \equiv \int dx F_\alpha(x;\varphi)\,\dot{\varphi}_\alpha(x) - H(\varphi),$$

where $F_\alpha(x;\varphi(x'))$ is an arbitrary functional of dynamical variables $\varphi_\alpha(x)$. Consider the infinitesimal transformations of field $\varphi_\alpha(x)$:

$$\varphi_\alpha(x) \longrightarrow \varphi'_\alpha(x) = \varphi_\alpha(x) + \delta\varphi_\alpha(x), \qquad \delta\varphi_\alpha(x) = \Phi_\alpha\Big(x;\varphi(x',t)\Big). \tag{2.24}$$

Here Φ_α is certain independent explicitly of time functional of the functions $\varphi(x,t)$. Under the transformations (2.24) the Lagrangian variation takes the form

$$\delta L = \frac{dG(\varphi)}{dt} + \int dx'\varphi_\beta(x')\left(\int dx J_{\beta\alpha}(x,x';\varphi)\,\dot{\varphi}_\alpha(x) - \frac{\delta H}{\delta\varphi_\beta(x')}\right),$$

where

$$G(\varphi) = \int dx F_\alpha(x;\varphi)\delta\varphi_\alpha(x), \qquad J_{\alpha\beta}(x,x';\varphi) = \frac{\delta F_\beta(x';\varphi)}{\delta\varphi_\alpha(x)} - \frac{\delta F_\alpha(x';\varphi)}{\delta\varphi_\beta(x)}.$$

From the principle of the stationary action $\delta\int_{t_1}^{t_2} L dt = 0$ it follows that the motion equations for field component $\varphi_\alpha(x)$ has the form:

$$\dot{\varphi}_\alpha(x) = \int dx' J^{-1}_{\alpha\beta}(x,x';\varphi)\frac{\delta H}{\delta\varphi_\beta(x')}. \tag{2.25}$$

Define PB of arbitrary functionals A and B of dynamical variables φ_α by the equality:

$$\{A,B\} = \int dx dx' \frac{\delta A}{\delta\varphi_\alpha(x)} J^{-1}_{\alpha\beta}(x,x';\varphi)\frac{\delta B}{\delta\varphi_\beta(x')}. \tag{2.26}$$

Then the motion equations (2.25) take the Hamiltonian form:

$$\dot{\varphi}_\alpha(x) = \{\varphi_\alpha(x), H\}.$$

Consider the finite transformations

$$\varphi_\alpha(x) \longrightarrow \varphi_\alpha(x') = \varphi'_\alpha\Big(x; \varphi_\alpha(x')\Big). \tag{2.27}$$

We call the transformations (2.27) canonical if the condition

$$\int dx F_\alpha(x;\varphi)\delta\varphi_\alpha(x) - \int dx F_\alpha(x;\varphi')\delta\varphi'_\alpha(x) = \delta Q(\varphi) \tag{2.28}$$

is satisfied. Here $Q(\varphi)$ is the certain functional of φ. It is easy to see that given condition can be written in the following equivalent form

$$J_{\alpha\beta}(x,x';\varphi) = \int dx_1 dx_2 \frac{\delta\varphi'_\lambda(x_1)}{\delta\varphi_\alpha(x)}\frac{\delta\varphi'_\nu(x_2)}{\delta\varphi_\beta(x')} J_{\lambda\nu}(x_1,x_2;\varphi'). \tag{2.29}$$

As in case of systems with finite number of degrees of freedom it isn't hard to verify that PB (2.26) are invariant under canonical transformations (2.27), (2.29). Now consider the transformations (2.27), conserving the kinematic part of the Lagrangian. Such transformations satisfy the relationship

$$F_\alpha(x;\varphi) = \int dx' F_\beta(x';\varphi')\frac{\delta\varphi'_\beta(x';\varphi)}{\delta\varphi_\alpha(x)} \tag{2.30}$$

and with account of (2.28) are canonical with $Q(\varphi) = \text{const}$. For infinitesimal transformations (2.24) the equality (2.30) is written as folows

$$\textstyle\int dx' J_{\alpha\beta}(x,x';\varphi)\delta\varphi_\beta(x') - \frac{\delta G}{\delta\varphi_\alpha(x)},$$

or, taking into account (2.26) does in the form

$$\delta\varphi_\alpha(x) = \{\varphi_\alpha(x), G\}. \tag{2.31}$$

Here G is the generator of the infinitesimal canonical transformations, which should be obtained from the equation

$$\textstyle\int dx dx' F_\alpha(x;\varphi) J^{-1}_{\alpha\beta}(x,x';\varphi)\frac{\delta \ln G}{\delta\varphi_\beta(x')} = 1.$$

Extension of the considered class of variations can be achieved by adding the full derivative over time from the arbitrary functional $\chi(\varphi)$ to Lagrangian; the corresponding equation for extended class of variations takes the form

$$\textstyle\int dx' J_{\alpha\beta}(x,x';\varphi)\delta\varphi_\beta(x') = \frac{\delta}{\delta\varphi_\alpha(x)}\left(G + \int dx' \frac{\delta\chi(\varphi)}{\delta\varphi_\alpha(x')}\delta\varphi_\alpha(x')\right).$$

For finite transformations (2.27), the invariance condition of kinematic part of Lagrangian with account for indefiniteness in the choice of L

$$\textstyle L \longrightarrow L' = L + \int dx \frac{\delta\chi(\varphi)}{\delta\varphi_\alpha(x)}\, \dot{\varphi}_\alpha\,(x)$$

is written in the form

$$F_\alpha(x;\varphi) = \int dx' F_\beta(x';\varphi')\frac{\delta\varphi'_\beta(x';\varphi)}{\delta\varphi_\alpha(x)} + \frac{\delta\chi(\varphi')}{\delta\varphi_\alpha(x)} - \frac{\delta\chi(\varphi)}{\delta\varphi_\alpha(x)}. \tag{2.32}$$

Transformations (2.27), (2.32) are canonical with $Q = \chi(\varphi') - \chi(\varphi)$.

Below we apply the above approach to some physical systems and consider the cases of magnetic system with totally broken symmetry and antiferromagnetic media (see [3,4]).

3 Systems with totally broken symmetry

In this section we apply developed formalism to magnetic systems with totally broken symmetry with respect to spin rotations in spin space. It is well known [5,6], that for adequate description of thermodynamics and kinetics of systems with spontaneously broken symmetry it is necessary to introduce into the theory additional thermodynamic parameters which are not connected with conservation laws but caused by the physical nature of phase state. In the case of magnetic systems with totally broken symmetry under spin rotations such dynamical quantities are the rotation angles φ_α realizing parametrization of three-dimensional rotation group of spin space or connected with them real orthogonal rotation matrix $a(\varphi)$ $(a\tilde{a} = 1)$, which in terms of parameters φ_α can be written in the form

$$a(\varphi) = \exp(-\varepsilon\varphi), \qquad (\varepsilon\varphi)_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\varphi_\gamma.$$

Further it will be covenient for us to consider the matrix a as the matrix of arbitrary affine transformation.

Represent the kinematic part of Lagrangian in the form

$$L_k = \int d^3x \underline{L}_k, \qquad \underline{L}_k = c_{\alpha\beta}(x)\, \dot{a}_{\beta\alpha}(x). \tag{3.1}$$

Here variables $a_{\alpha\beta}$ and $c_{\alpha\beta}$ being the generalized coordinates and momenta of magnetic with spontaneously broken symmetry. Introduce the spin of the system as generator of uniform arbitrary small rotations characterised by angles $\delta\varphi_\alpha$. Since under rotation, described by matrix b, matrices a and c are transformed according to the formulae:

$$a \longrightarrow a' = a\,\tilde{b}, \qquad c \longrightarrow c' = bc, \tag{3.2}$$

we have

$$\delta a_{\alpha\beta} = \varepsilon_{\gamma\beta\nu}.a_{\alpha\gamma}\delta\varphi_\nu, \qquad \delta c_{\alpha\beta} = \varepsilon_{\gamma\alpha\nu}c_{\gamma\beta}\delta\varphi_\nu. \tag{3.3}$$

It is easy to see that transformations (3.3) leave invariant the kinematic part of Lagrangian and the generator

$$G = \delta\varphi_\alpha \int d^3x \varepsilon_{\alpha\beta\gamma}c_{\gamma\nu}(x)a_{\nu\beta}(x) \tag{3.4}$$

corresponds to them. With these into account one obtains:

$$S_\alpha(x) = \varepsilon_{\alpha\beta\gamma}c_{\gamma\nu}(x)a_{\nu\beta}(x). \tag{3.5}$$

In what follows belove it is convenient to introduce tensor $g_{\alpha\beta}$ according to equality

$$g_{\alpha\beta} = c_{\alpha\gamma}a_{\gamma\beta}. \tag{3.6}$$

Then the expression for spin density expressed in terms of antisymmetric part of $g_{\alpha\beta}$ reads:

$$S_\alpha(x) = \varepsilon_{\alpha\mu\nu\gamma}g^a_{\nu\mu}(x), \qquad g^a_{\nu\mu} \equiv \frac{1}{2}(g_{\nu\mu} - g_{\mu\nu}). \tag{3.7}$$

Antisymmetric part of tensor $g_{\alpha\beta}$ as it follows from (3.7) is uniquelly defined by spin density

$$g^{a}_{\alpha\beta} = -\frac{1}{2}\varepsilon_{\alpha\beta\gamma}s_{\gamma}. \tag{3.8}$$

Noting, that

$$g^{s}_{\alpha\beta} \equiv f_{\alpha\beta}(g^{s}_{\alpha\beta}) = \frac{1}{2}(g_{\alpha\beta} + g_{\beta\alpha}), \tag{3.9}$$

one can write

$$g_{\alpha\beta} \equiv f_{\alpha\beta} - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}s_{\gamma}. \tag{3.10}$$

From (3.1), (3.6), (3.10) it follows that density of kinematic part $\underline{L}_k$ of Lagrangian hase the form:

$$\underline{L}_k = \left(f_{\alpha\beta} - \frac{1}{2}\varepsilon_{\alpha\beta\gamma}s_{\gamma}\right)a^{-1}_{\rho\beta}\,\dot{a}_{\beta\alpha}\,. \tag{3.11}$$

Thus, dynamical quantities of magnet with totally broken symmetry are the spin density $s_{\alpha}(x)$, matrix $a_{\alpha\beta}(x)$ of arbitrary affine transformations and symmetrical matrix $f_{\alpha\beta}(x)$. Now let us obtain the PB for dynamical variables $s_{\alpha}, a_{\alpha\beta}, f_{\alpha\beta}$. It is easy to see that the variations

$$\delta a_{\alpha\beta}(x) = \chi_{\alpha\beta}(x), \qquad \delta c_{\alpha\beta}(x) = 0. \tag{3.12}$$

where the functions $\chi_{\alpha\beta}(x)$ are independent of variables $a_{\alpha\beta}$ and $c_{\alpha\beta}$, leave invariant the kinematic part (3.1) and the generator

$$G = \int d^3x c_{\alpha\beta}(x) g_{\beta\alpha}(x) \tag{3.13}$$

corresponds to them. Let us represent variations $\delta a_{\alpha\beta}$ and $\delta c_{\alpha\beta}$ in the form

$$\delta a_{\alpha\beta}(x) = \{a_{\alpha\beta}(x), G\}, \qquad \delta c_{\alpha\beta}(x) = \{c_{\alpha\beta}, G\}. \tag{3.14}$$

Then equalities (3.12), (3.14) yield the following PB:

$$\{a_{\alpha\beta}(x), c_{\mu\nu}(x')\} = \delta_{\alpha\nu}\delta_{\beta\mu}\delta(x - x'), \qquad \{c_{\alpha\beta}(x), c_{\mu\nu}(x')\} = 0. \tag{3.15}$$

We now consider together with density $\underline{L}_k$ the density $\underline{L'}_k$

$$L'_k = -a_{\alpha\beta}(x)\,\dot{c}_{\beta\alpha}\,(x). \tag{3.16}$$

Variations

$$\delta c_{\alpha\beta}(x) = \tilde{\chi}_{\alpha\beta}\,(x), \qquad \delta a_{\alpha\beta}(x) = 0, \tag{3.17}$$

where $\tilde{\chi}$ is independent from $a_{\alpha\beta}, c_{\alpha\beta}$, conserve $\underline{L'}_k$. Representing $\delta a_{\alpha\beta}, \delta c_{\alpha\beta}$ in the form (3.14) with the generator

$$G = -\int d^3x a_{\alpha\beta}(x)\,\tilde{\chi}_{\beta\alpha}\,(x), \tag{3.18}$$

we shall obtain PB, not entering into (3.15):

$$\{a_{\alpha\beta}(x), a_{\mu\nu}(x')\} = 0. \tag{3.19}$$

Taking into account (3.6), (3.15), (3.19) one can obtain the PB for variables $a_{\alpha\beta}(x), g_{\alpha\beta}(x)$:

$$\begin{aligned} \{g_{\alpha\beta}(x), g_{\mu\nu}(x')\} &= (g_{\alpha\nu}(x)\delta_{\beta\mu} - g_{\mu\beta}(x)\delta_{\alpha\nu})\delta(x-x'), \\ \{a_{\alpha\beta}(x), g_{\mu\nu}(x')\} &= a_{\alpha\nu}(x)\delta_{\beta\mu}\delta(x-x'). \end{aligned} \tag{3.20}$$

Variables $s_\alpha(x), f_{\alpha\beta}(x)$ are connected with variables $g_{\alpha\beta}(x)$ by relationships (3.7), (3.9). From here and from (3.20) we can find the PB for dynamic variables $s_\alpha(x), a_{\alpha\beta}(x), f_{\alpha\beta}(x)$:

$$\begin{aligned} \{f_{\alpha\beta}(x), f_{\mu\nu}(x')\} &= \frac{1}{4}(\varepsilon_{\nu\alpha\gamma}\delta_{\beta\mu} + \varepsilon_{\mu\beta\gamma}\delta_{\alpha\nu} + \varepsilon_{\nu\beta\gamma}\delta_{\alpha\mu} + \varepsilon_{\mu\alpha\gamma}\delta_{\beta\nu}) \times \\ &\times\ s_\gamma(x)\delta(x-x'), \\ \{s_\alpha(x), f_{\xi\beta}(x')\} &= \Big(f_{\beta\rho}(x)\varepsilon\xi\rho\alpha + f_{\xi\rho}(x)\varepsilon_{\beta\rho\alpha}\Big)\delta(x-x'), \\ \{s_\alpha(x), s_\beta(x')\} &= \varepsilon_{\alpha\beta\gamma}s_\gamma(x)\delta(x-x'), \\ \{a_{\alpha\beta}(x), s_\mu(x')\} &= \varepsilon_{\mu\rho\beta}a_{\alpha\rho}(x)\delta(x-x'), \\ \{a_{\alpha\beta}(x), f_{\gamma\rho}(x')\} &= \frac{1}{2}\Big(\delta_{\beta\gamma}a_{\alpha\rho}(x) + \delta_{\beta\rho}a_{\alpha\gamma}(x)\Big)\delta(x-x'). \end{aligned} \tag{3.21}$$

Algebra (3.21) contains as its subalgebra the algebra of variables $s_\alpha(x)$ and $a_{\alpha\beta}(x)$:

$$\begin{aligned} \{s_\alpha(x), s_\beta(x')\} &= \varepsilon_{\alpha\beta\gamma}s_\gamma(x)\delta(x-x'), \\ \{a_{\alpha\beta}(x), s_\mu(x')\} &= \varepsilon_{\mu\rho\beta}a_{\alpha\rho}(x)\delta(x-x'), \end{aligned} \tag{3.22}$$

where $a_{\alpha\beta}(x)$ is considered as real orthogonal matrix ($a\,\tilde{a} = 1$) (It is easy to see that equality $\{a_{\alpha\beta}\,\tilde{a}_{\beta\gamma}, s_\nu\} = 0$ is consistent with bracket of variables $s_\alpha(x), a_{\alpha\beta}(x)$). Just this situation was considered in [3]. Note that algebra (3.20), which is necessary for deriving the brackets (3.21), can be obtained directly, if one writes the kinematic part $\underline{L}_k$ of Lagrangian in terms of variables $g_{\alpha\beta}, a_{\alpha\beta}$ and finds transformations, conserving $\underline{L}_k$. As it follows from (3.1)

$$\underline{L}_k = Spg(x)\omega(x), \quad \omega_{\alpha\beta}(x) \equiv a^{-1}_{\alpha\gamma}(x)\dot{a}_{\gamma\beta}(x).$$

It is easy to see that density $\underline{L}_k$ is invariant relatively to arbitrary affine transformations, described by matrix b. Really, since $a \longrightarrow a' = ab^{-1}$, then

$$g \longrightarrow g' = bgb^{-1}, \quad \omega \longrightarrow \omega' = b\omega b^{-1},$$

and

$$\underline{L}_k = \mathrm{Sp}g(x)\omega(x) = \mathrm{Sp}g'(x)\omega'(x).$$

Let us consider arbitrary small transformations

$$b_{\alpha\beta}(x) = \delta_{\alpha\beta} + \varepsilon_{\alpha\beta}(x), \quad |\,\varepsilon_{\alpha\beta}\,| \ll 1.$$

For variations $\delta g(x)$, $\delta a(x)$ we shall find

$$\delta g(x) = [\varepsilon(x), g(x)], \quad \delta a(x) = -a(x)\varepsilon(x). \tag{3.23}$$

Taking into account that according to general formulae generator G equals:

$$G = -\int d^3x \mathrm{Sp} g(x)\varepsilon(x) \tag{3.24}$$

and representing $\delta g(x)$, $\delta a(x)$ in the form

$$\delta g(x) = \{g(x), G\}, \delta a(x) = \{a(x), G\}, \tag{3.25}$$

one can obtain, comparing (3.33) and (3.35) with account of (3.34), the PB (3.20), derived earlier on the base of algebra (3.14), (3.17).

Thus, from (3.21) the motion equations for variables $s_\alpha(x)$, $a_{\alpha\beta}(x)$, $f_{\alpha\beta}(x)$ read:

$$\begin{aligned}
\dot{s}_\alpha(x) &= \Big(f_{\mu\rho}(x)\varepsilon_{\alpha\beta\rho} + f_{\beta\rho}(x)\varepsilon_{\alpha\mu\rho}\Big)\frac{\delta H}{\delta f_{\mu\beta}(x)} + \varepsilon_{\alpha\beta\gamma}\frac{\delta H}{\delta s_\beta(x)}s_\gamma(x) + \\
&+ \varepsilon_{\alpha\beta\rho}a_{\nu\rho}(x)\frac{\delta H}{\delta a_{\nu\beta}(x)}, \\
\dot{f}_{\mu\beta}(x) &= \frac{1}{4}\Big(\varepsilon_{\nu\mu\alpha}\delta_{\gamma\beta} + \varepsilon_{\gamma\beta\alpha}\delta_{\mu\nu} + \varepsilon_{\nu\beta\alpha}\delta_{\mu\gamma} + \varepsilon_{\gamma\mu\alpha}\delta_{\beta\nu}\Big)\frac{\delta H}{\delta f_{\gamma\nu}(x)}s_\alpha(x) + \\
&+ \Big(f_{\mu\rho}(x)\varepsilon_{\beta\gamma\rho} + f_{\beta\rho}(x)\varepsilon_{\mu\gamma\rho}\Big)\frac{\delta H}{\delta s_\gamma(x)} - \frac{1}{2}\Big(\delta_{\lambda\mu}a_{\nu\beta}(x) + \\
&+ \delta_{\lambda\beta}a_{\nu\mu}(x)\Big)\frac{\delta H}{\delta a_{\nu\lambda}(x)}, \\
\dot{a}_{\alpha\beta}(x) &= \varepsilon_{\mu\rho\beta}a_{\alpha\rho}(x)\frac{\delta H}{\delta s_\mu(x)} + \frac{1}{2}\Big(\delta_{\beta\gamma}a_{\alpha\rho}(x) + \delta_{\beta\rho}a_{\alpha\gamma}(x)\Big)\frac{\delta H}{\delta f_{\gamma\rho}(x)}.
\end{aligned} \tag{3.26}$$

Dynamic equations (3.36) and general functional expression for energy density $\varepsilon(x) = \varepsilon\left(x; s(x'), a(x'), f(x')\right)$ describe nonequilibrium properties of magnetic systems with arbitrary character of space nonuniformities and they are very complicated for analysis. Investigation of these equations is considerably simplified in long-wave limit when space nonuniformities of dynamic variables are small. Considering that energy density is a function of quantities $s, a, \nabla a$ (and variable f is cyclic) or is the function of quantities s, a, ω_k $\Big(\omega_{\alpha k} \equiv \frac{1}{2}\varepsilon_{\alpha\beta\gamma}a_{\lambda\gamma}\nabla_k a_{\lambda\beta}\Big)$, where a is the matrix of rotations [3]:

$$\varepsilon\Big(x, s(x'), a(x')\Big) \equiv \varepsilon\Big(s(x), \omega_k(x), a(x)\Big) \tag{3.27}$$

let us derive dynamical equations in local form (on contrary to (3.26) where the non-local form is given). Further we shall assume that energy density satisfies invariance property with respect to uniform spin rotations, described by matrix b:

$$\varepsilon\Big(x; bs(x'), a(x')\,\tilde{b}\Big) = \varepsilon\Big(x, s(x'), a(x')\Big). \tag{3.28}$$

Due to (3.38) we have

$$\varepsilon(s, \omega_k, a) = \varepsilon(as, a\omega_k, 1) \equiv \varepsilon(\underline{s}, \underline{\omega}_k), \tag{3.29}$$

where $\underline{s} = as, \underline{\omega}_k = a\omega_k$. According to (3.22) one should find equalities

$$\{s_\alpha(x), \underline{\omega}_{\beta k}(x')\} = a_{\beta\alpha}(x')\nabla'_k\delta(x-x'), \{s_\alpha(x), \underline{s}_\beta(x')\} = 0$$

leading to invariance of quantities $\underline{s}, \underline{\omega}_k$ with respect to global spin rotations

$$\{S_\alpha, \underline{\omega}_{\beta k}(x)\} = \{S_\alpha, \underline{s}_\beta(x)\} = 0.$$

These expressions correspond to invariance of energy density:

$$\{S_\alpha, \varepsilon(x)\} = 0.$$

Using (3.36) and energy density (3.39) one can obtain dynamical equations for magnets with spontaneously broken symmetry in long-wave limit

$$\dot{s}_\alpha = -\nabla_k \frac{\partial \varepsilon}{\partial \omega_{\alpha k}}, \dot{\varepsilon}_\alpha = -\nabla_k \frac{\partial \varepsilon}{\partial s_\alpha}\frac{\partial \varepsilon}{\partial \omega_{\alpha k}}, \dot{a}_{\alpha\beta} = a_{\alpha\rho}\varepsilon_{\rho\beta\gamma}\frac{\partial \varepsilon}{\partial s_\gamma}. \tag{3.30}$$

Due to the definition of energy density (3.39) it is worthwhile to choose quantities $\underline{s}, \underline{\omega}_k$, as the independent variables. The equations follow:

$$\dot{a}_{\alpha\beta} = a_{\rho\beta}\varepsilon_{\alpha\rho\gamma}\frac{\partial \varepsilon}{\partial \underline{s}_\gamma}, \qquad \dot{\varepsilon}_\alpha = -\nabla_k \frac{\partial \varepsilon}{\partial \underline{s}_\alpha}\frac{\partial \varepsilon}{\partial \underline{\omega}_{\alpha k}},$$

$$\dot{\underline{s}}_\alpha = -\nabla_k \frac{\partial \varepsilon}{\partial \underline{\omega}_{\alpha k}} + \varepsilon_{\alpha\beta\gamma}\left(\underline{s}_\beta \frac{\partial \varepsilon}{\partial \underline{s}_\gamma} + \underline{\omega}_{\beta k}\frac{\partial \varepsilon}{\partial \underline{\omega}_{\alpha k}}\right). \tag{3.31}$$

From equation of motion for matrix a in (3.41) it follows the equation for Cartan form $\underline{\omega}_k$:

$$\underline{\omega}_{\alpha k} = -\nabla_k \frac{\partial \varepsilon}{\partial \underline{s}_\alpha} + \varepsilon_{\alpha\beta\gamma}\underline{\omega}_{\beta k}\frac{\partial \varepsilon}{\partial \underline{s}_\gamma}. \tag{3.32}$$

Complete system of equations (3.31) determines dynamical properties of magnets in the case when the dissipative processes are neglected. Note that we have considered a dynamical variable number being even everywhere in the above formalism. However, there can exist physical systems, dynamics of which can be described by odd number of variables (for example, dynamical variables of purely spin systems are three components of spin density $s_\alpha(x)$). Such cases can also be included to the above formulated scheme if there exists more general PB system containing given algebra of odd number of dynamical variables as its subalgebra exists. So in incited exemple the PB system (3.21) contains the algebra of variables $s_\alpha(x)$ as its subalgebra. Then considering that all rest variables are cyclic one can construct the Hamiltonian dynamics of pure spin systems with obtained in the more general system of PB for the spin densities $s_\alpha(x)$.

4 Antiferromagnet (AFM)

Let us represent the kinematic part of the Lagrangian in the form

$$L_k = \int d^3x L_k, \qquad L_k = \mathbf{m}(x)\,\dot{\mathbf{l}}\,(x), \tag{4.1}$$

where $l_\alpha(x), m_\alpha(x)$ are generalized coordinates and momenta of AFM. A spin density $s_\alpha(x)$ and vector of antiferromagnetism $l_\alpha(x)$ enter into number of dynamical variables describing AFM states. Let us introduce spin density $s_\alpha(x)$ as a function of generalized coordinates $l_\alpha(x)$ and momenta $m_\alpha(x)$. With this purpose we define total spin of AFM $S_\alpha = \int d^3x s_\alpha(x)$ as a generator of infinitesimal homogeneous rotations of magnet. Under infinitesimal rotations $\delta\varphi_\gamma$ the variations $\delta m_\alpha(x), \delta l_\alpha(x)$ read

$$\delta m_\alpha(x) = \varepsilon_{\alpha\beta\gamma}\delta\varphi_\beta m_\gamma(x), \qquad \delta l_\alpha(x) = \varepsilon_{\alpha\beta\gamma}\delta\varphi_\beta l_\gamma(x). \tag{4.2}$$

It is clear that variations (4.2) leave invariant the kinematic part of Lagrangian and the generator

$$G = \delta\varphi_\beta \int d^3x \varepsilon_{\alpha\beta\gamma} m_\alpha(x) l_\gamma(x)$$

corresponds to them. In accordance with above metioned remark

$$S_\alpha = \int d^3x s_\alpha(x), \qquad s_\alpha(x) = \varepsilon_{\alpha\beta\gamma} l_\beta(x) m_\gamma(x). \tag{4.3}$$

According to (4.3) we obtain the generalized momentum m_α

$$\mathbf{m} = \frac{1}{\mathbf{l}^2}(\mathbf{s}\times\mathbf{l} + \gamma\mathbf{l}). \tag{4.4}$$

Second term in (4.4) agrees with a fact that the quantity m_α is determined from the relationship (4.3) only with accuracy up to a collinear vector $\mathbf{l}$. From (4.1), (4.4) it follows, that the kinematic part of Lagrangian of AFM can be represented in the form

$$L_k = \int d^3x L_k, \qquad L_k = \tfrac{1}{\mathbf{l}^2}(\mathbf{s}\times\mathbf{l} + \gamma\mathbf{l})\,\dot{\mathbf{l}}\,.$$

Thus the spin density $s_\alpha(x)$, the vector of antiferromagnetism $l_\alpha(x)$ and the parameter $\gamma(x)$ are the dynamical variables of AFM, and the connection $\mathbf{ls} = 0$ between vectors $\mathbf{l}$ and $\mathbf{s}$ follows from (4.3) (such restriction corresponds to consideration of two-sublattice magnetics with equivalent sublattices). Let us obtain PB of generalized coordinates l_α and momenta m_α. How it is easy to see the variations

$$\delta l_\alpha(x) = g_\alpha(x), \qquad \delta m_\alpha(x) = 0, \tag{4.5}$$

(here functions $g_\alpha(x)$ are independent of variables l_α, m_α) satisfy the equation (3.5), and the generator

$$G = \int d^3x m_\alpha(x) g_\alpha(x) \tag{4.6}$$

corresponds to them. Representing variations $\delta l_\alpha, \delta m_\alpha$ in the form

$$\delta l_\alpha(x) = \{l_\alpha(x), G\}, \qquad \delta m_\alpha(x) = \{m_\alpha(x), G\}, \tag{4.7}$$

and taking into account for (4.5), (4.6), we obtain the follow PB

$$\{l_\alpha(x), m_\beta(x')\} = \delta_{\alpha\beta}\delta(x-x'), \qquad \{m_\alpha(x), m_\beta(x')\} = 0. \tag{4.8}$$

Now if we consider together with density L_k (see(4.1)) the quantity L_k'

$$L_k' = -\mathbf{l}(x)\,\dot{\mathbf{m}}\,(x) \tag{4.9}$$

which differs from (4.1) by the full derivative $\frac{d}{dt}(l_\alpha m_\alpha)$, then as it is easy to see the variations

$$\delta m_\alpha(x) = \tilde{g}_\alpha(x), \qquad \delta l_\alpha(x) = 0, \tag{4.10}$$

where $\tilde{g}_\alpha(x)$ is independent from l_α, m_α, satisfy equation (3.5). The generator of transformations (4.10) takes the form

$$G = -\int d^3x l_\alpha(x)\, \tilde{g}_\alpha(x). \tag{4.11}$$

Again representing variations $\delta m_\alpha, \delta l_\alpha$ in the form (4.7) and taking into account equalities (4.10), (4.11) we get one more PB not involved in (4.8):

$$\{l_\alpha(x), l_\beta(x')\} = 0. \tag{4.12}$$

We express $s_\alpha(x)$ and $\gamma(x)$ in terms of the generalised canonical coordinates and momenta $l_\alpha(x), m_\alpha(x)$ in order to obtain the PB of variables $l_\alpha(x), s_\alpha(x), \gamma(x)$. As it follows from (4.4)

$$\mathbf{s}(x) = \mathbf{l}(x) \times \mathbf{m}(x), \qquad \gamma(x) = \mathbf{l}(x)\mathbf{m}(x).$$

Now using algebra (4.8), (4.12) of variables l_α, m_α we find PB of the variables $l_\alpha(x)$, $s_\alpha(x)$, $\gamma(x)$

$$\begin{aligned}
\{l_\alpha(x), s_\beta(x')\} &= \varepsilon_{\alpha\beta\gamma} l_\gamma(x)\delta(x-x'), \\
\{s_\alpha(x), s_\beta(x')\} &= \varepsilon_{\alpha\beta\gamma} s_\gamma(x)\delta(x-x'), \\
\{l_\alpha(x), \gamma(x')\} &= l_\alpha(x)\delta(x-x'), \\
\{s_\alpha(x), \gamma(x')\} &= 0, \\
\{\gamma(x), \gamma(x')\} &= 0.
\end{aligned} \tag{4.13}$$

Using (4.13), we product the dynamical equations of AFM in the form

$$\dot{l}_\alpha(x) = \varepsilon_{\alpha\beta\gamma}\frac{\delta H}{\delta s_\beta(x)} l_\gamma(x) + \frac{\delta H}{\delta\gamma(x)} l_\alpha(x),$$

$$\dot{s}_\alpha(x) = \varepsilon_{\alpha\beta\gamma}\left(\frac{\delta H}{\delta s_\beta(x)} s_\gamma(x) + \frac{\delta H}{\delta l_\beta(x)} l_\gamma(x)\right), \quad \dot{\gamma}(x) = -\frac{\delta H}{\delta l_\beta(x)} l_\beta(x) \tag{4.14}$$

Investigation of these equations is considerably simplified in long-wave limit where space inhomogeneties of dynamic variables are small. Considering that energy density is a function of quantities $s, l, \nabla l$ (and variable γ is cyclic) or is function of quantities $s, l, v_k (v_{\alpha k} \equiv -\varepsilon_{\alpha\beta\gamma} l_\beta \nabla_k l_\gamma)$ [4]:

$$\varepsilon\Big(x, s, (x'), l(x')\Big) \equiv \varepsilon\Big(s(x), l(x), v_k(x)\Big) \tag{4.15}$$

let us derive the dynamical equations in local form (in (4.14) the nonlocal form is given). Further we shall assume that energy density satisfies the invariance property with respect to uniform spin rotations, described by matrix b:

$$\varepsilon\Big(x; bs(x'), bl(x')\Big) = \varepsilon\Big(x, s(x'), l(x')\Big). \tag{4.16}$$

Using (4.14) and energy density (4.15) one can obtain dynamical equations of antiferromagnet in the form [4]

$$\dot{s}_\alpha = -\nabla_k \frac{\partial \varepsilon}{\partial v_{\alpha k}}, \dot{\varepsilon}_\alpha = -\nabla_k \frac{\partial \varepsilon}{\partial s_\alpha} \frac{\partial \varepsilon}{\partial v_{\alpha k}}, \dot{l}_\alpha = \varepsilon_{\alpha\beta\gamma} \frac{\partial \varepsilon}{\partial s_\beta} l_\gamma. \qquad (4.17)$$

The equation for $v_{\alpha k}$ follows from the equation of motion for the vector of antiferromagnetism:

$$v_{\alpha k} = -\Pi_{\alpha\beta} \nabla_k \frac{\partial \varepsilon}{\partial s_\beta} + \varepsilon_{\alpha\beta\gamma} \frac{\partial \varepsilon}{\partial s_\beta} v_{\gamma k}; \qquad \Pi_{\alpha\beta} = \delta_{\alpha\beta}. - l_\alpha l_\beta. \qquad (4.18)$$

Complete system of equations (4.16) determines dynamical properties of antiferromagnetic in the case when the dissipative processes are neglected.

5 Hydrodynamical asymptotics of the Green functions and spectrum of spin waves.

The retarded two-time Green function specific to quasilocal operators $\hat{a}$ and $\hat{b}$ is

$$G_{ab}(x,t;x',t') = -i\theta(t-t')\mathrm{Sp}w\left[\hat{a}(x,t), \hat{b}(x',t')\right]. \qquad (5.1)$$

where w denotes the equilibrium statistical operator. An external field $\xi(x,t)$ interaction is supposed to be described by the relation

$$V(t) = \int d^3x \xi(x,t) b(x,t).$$

The linear response of the quantity a to an external disturbance is

$$\delta a_\xi(x,t) = \int_{-\infty}^{\infty} dx' \int dx' \xi(x',t') G_{ab}(x-x',t-t').$$

The corresponding Fourier transform takes the form

$$\delta a_\xi(k,\omega) = G_{ab}(k,\omega)\xi(k,\omega). \qquad (5.2)$$

The low-frequency asymptotic expression for the Green function of the system can be derived from (5.2). In order to do this let us note that in presence of the external field the equations of motion specific to quantities s_α and l_α take the forms

$$\begin{aligned} s_\alpha &= -\nabla_k \frac{\partial \varepsilon}{\partial v_{\alpha k}} + \eta_\alpha, \\ \dot{l}_\alpha &= \varepsilon_{\alpha\beta\gamma} \frac{\partial \varepsilon}{\partial s_\beta} l_\gamma + \tilde{\eta}_\alpha, \end{aligned} \qquad (5.3)$$

where sources $\tilde{\eta}_\alpha$ and η_α are defined by the expressions

$$\eta_\alpha = \xi \varepsilon_{\alpha\beta\gamma} \left(\frac{\partial b}{\partial s_\beta} s_\gamma + \frac{\partial b}{\partial l_\beta} l_\gamma + \frac{\partial b}{\partial v_{\beta k}} v_{\gamma k} \right) - \nabla_k \left(\xi \frac{\partial b}{\partial v_{\alpha k}} \right),$$

$$\tilde{\eta}_\alpha = \xi \varepsilon_{\alpha\beta\gamma} \frac{\partial b}{\partial s_\beta} l_\gamma.$$

The equations have been derived on the basis of the equation of motion (4.14), where variable γ is cyclic and

$$\mathcal{H} \equiv \int d^3x\Big(\varepsilon(x) + v(x)\Big), \qquad v(x) = \xi(x)b\Big(s(x), l(x), v_k(x)\Big).$$

Here $v(x)$ denotes the density of the Hamiltonian of the system-external field interaction. Linearizing eqs. (5.3) with respect to the deviations $\delta s(x,t), \delta l(x,t)$ of the quantities $s(x,t)$ and $l(x,t)$ from the equilibrium state $p_k = 0, s_\alpha = 0, v_{\alpha k} = 0$ and using the Fourier representation of the obtained equations one can derive

$$-\imath\omega\delta s_\alpha + k^2\varepsilon_{\gamma\mu\chi}l_\mu\frac{\partial^2\varepsilon}{\partial v_{\alpha l}\partial v_{\gamma l}}\delta l_\chi = \eta_\alpha,$$
$$-\left(\imath\omega\delta_{\alpha\nu} + \varepsilon_{\alpha\beta\gamma}\left[\frac{\partial\varepsilon}{\partial s_\beta}\delta_{\nu\gamma} + \frac{\partial^2\varepsilon}{\partial s_\beta\partial l_\nu}l_\gamma\right]\right)\delta l_\nu - \varepsilon_{\alpha\beta\gamma}\frac{\partial^2\varepsilon}{\partial s_\beta\partial s_\nu}l_\gamma\delta s_\nu = \tilde{\eta}_\alpha, \quad (5.4)$$

where

$$\eta_\alpha = \xi\left(\varepsilon_{\alpha\beta\gamma}\frac{\partial b}{\partial l_\beta}l_\gamma - \imath k_i\frac{\partial b}{\partial v_{\alpha i}}\right),$$
$$\tilde{\eta}_\alpha = \xi\varepsilon_{\alpha\beta\gamma}\frac{\partial b}{\partial s_\beta}l_\gamma.$$

A solution of the system of equations takes the form

$$\begin{aligned}\delta s_\alpha &= \frac{\xi}{\Delta(k,\omega)}\Big\{Ak^2\Pi_{\alpha\nu}\frac{\partial b}{\partial s_\nu} - \omega\Big(\varepsilon_{\alpha\beta}\frac{\partial b}{\partial l_\beta} - \imath k_i\frac{\partial b}{\partial v_{\alpha i}}\Big)\Big\},\\ \delta l_\alpha &= \frac{\xi}{\Delta(k,\omega)}\Big\{\imath\omega\varepsilon_{\alpha\beta}\frac{\partial b}{\partial s_\beta} + B\Big(\frac{\partial b}{\partial l_\alpha} + \imath k_i\varepsilon_{\alpha\beta}\frac{\partial b}{\partial v_{\beta i}}\Big)\Big\}.\end{aligned} \quad (5.5)$$

Here

$$\Delta(k,\omega) = \omega^2 - ABk^2, \qquad A = \frac{1}{6}\frac{\partial^2\varepsilon}{\partial v_k^2}, \qquad B = \frac{1}{2}\Pi_{\beta\gamma}\frac{\partial^2\varepsilon}{\partial s_\beta\partial s_\gamma}$$

and $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}l_\gamma$. In this derivation was takin intu account that due to rotation invariancy of ε and from equalities $p_k = 0, s_\alpha = 0, v_{\alpha k} = 0$ it follows that $l_\beta\frac{\partial\varepsilon}{\partial s_\beta} - \frac{1}{2}\Pi_{\beta\gamma}\frac{\partial^2\varepsilon}{\partial s_\beta\partial l_\gamma} = 0$. In the first order on k and ω (where $kl \ll 1, \omega\tau_r \ll 1$ and l, τ_r are some microscopic quantities, like mean free path or relaxation time), response of quantity a (which is a functional of dynamical variables) to distrubance of an external field can be written in the form

$$\delta a(k,\omega) = \frac{\partial a}{\partial s_\alpha}\delta s_\alpha(k,\omega) + \frac{\partial a}{\partial l_\alpha}\delta l_\alpha(k,\omega) + \frac{\partial a}{\partial v_{\alpha i}}\delta v_{\alpha i}(k,\omega).$$

Comparing it with (5.2), taking into account (5.5) and noting that $\delta v_{\alpha i} = \imath k_i\times$
$\times\varepsilon_{\alpha\beta}\delta l_\beta$, one can derive the low-frequency asymptotical expression for the Green function specific to dynamical variables a and b:

$$G_{ab}(k,\omega) = \frac{B}{\Delta(k,\omega)}\Big\{\frac{\omega}{B}\Pi_{\alpha\rho}\frac{\partial a}{\partial s_\rho} - \imath\varepsilon_{\alpha\rho}\frac{\partial a}{\partial l_\rho} + k_i\frac{\partial a}{\partial v_{\alpha i}}\Big\}\times$$
$$\times\Big\{\frac{\omega}{B}\Pi_{\alpha\gamma}\frac{\partial b}{\partial s_\gamma} + \imath\varepsilon_{\alpha\gamma}\frac{\partial b}{\partial l_\gamma} + k_l\frac{\partial b}{\partial v_{\alpha l}}\Big\} - \frac{1}{B}\frac{\partial a}{\partial s_\alpha}\frac{\partial b}{\partial s_\beta}\Pi_{\alpha\beta}. \quad (5.6)$$

From (5.6), in particular one can obtain

$$\begin{aligned} G_{s_\alpha,s_\beta}(k,\omega) &= \frac{Ak^2}{\Delta(k,\omega)}\Pi_{\alpha\beta}, \\ G_{l_\alpha,s_\beta}(k,\omega) &= \imath\varepsilon_{\alpha\beta}\frac{\omega}{\Delta(k,\omega)}, \\ G_{l_\alpha,l_\beta}(k,\omega) &= \frac{B}{\Delta(k,\omega)}\Pi_{\alpha\beta}. \end{aligned} \tag{5.7}$$

Putting in (5.7) $\omega = 0$, it can be seen that the singularity $\frac{1}{k^2}$ is specific to the Green function G_{l_α,l_β}, in complete correspondence to Bogolubov theorem [7]:

$$G_{l_\alpha,l_\beta}(k,0) = -\frac{\Pi_{\alpha\beta}}{Ak^2}.$$

At $k = 0$ singularities in terms of ω are specific to the Green functions G_{l_α,s_β} and G_{l_α,l_β}

$$G_{l_\alpha,s_\beta}(0,\omega) = \imath\frac{\varepsilon_{\alpha\beta}}{\omega}, \qquad G_{l_\alpha,l_\beta}(0,\omega) = \frac{B}{\omega^2}\Pi_{\alpha\beta}.$$

Note that in the one-particle approximation the Green function specific to Pauly operators of a two-sublattice anysotropic Heisenberg antiferromagnet with spin 1/2 has been considered in [8]. Equating $\Delta(k,\omega)$ from (5.4) to zero, one can obtain spectrum of spin waves of the antiferromagnet systems

$$\omega^2 = c^2k^2, c^2 \equiv AB$$

which coincides with the results of [9]. The temperature dependence of the spin wave velocity has been investigated by means of microscopic approach in [10].

6 Dynamics of manysublattice magnets with exchange interaction

In section 2 we have studied systems with totally broken symmetry with respect to spin rotations, dynamical variables of which, in particular, are the density of total spin $s(x)$ and rotation matrix $a(x)$. In this section we consider manysublattice magnets with exchange interaction. Low-frequency dynamics of such systems just is described by parameters $s(x), a(x)$. It is explained by the fact that at sufficiently large times due to exchange interaction rigid spin complexes are formed, orientations of which is assigned by rotation matrix $a(x)$. It is interesting to obtain dynamical equations coming from some more general equations in the same way as the hydrodynamic equations can be derived from kinetic equation for one-particle distribution function. As such more general equations we shall choose Landau-Lifshits equations and after determining the form of local-equilibrium solution we shall show how the reduction from complete description of magnetic in terms of sublattice spin densities to the reduced one with the help of parameters a, s is carried out. Let the state of manysublattice magnet is described by the densities of sublattice spins $\mathbf{s}_a(x)(a = 1, ..., n; n$ is a number of sublattices). Landau-Lifshits equations, describing the dynamics of spin densities $\mathbf{s}_a(x)$, have the form:

$$\dot{\mathbf{s}}_a = [\mathbf{s}_a, \mathbf{h}_a] + \mathbf{R}_a, \mathbf{h}_a(x) = -\frac{\delta\mathcal{H}\Big(\mathbf{s}_b(x')\Big)}{\delta\mathbf{s}_a(x)}. \tag{6.1}$$

Here $\mathcal{H}$ is the Hamiltonian which is a functional of sublattice spins, $\mathbf{R}_a$ is the dissipative term. In the case of pure exchange interaction (which we shall see only aftewards) total spin is conserved and consequently:

$$\sum_a \int d^3x[\mathbf{s}_a, \mathbf{h}_a] = 0, \qquad \sum_a \int d^3x\mathbf{R}_a = 0. \tag{6.2}$$

Local-equilibrium distribution of spins $\mathbf{s}_a(x)$ must be defined from minimum condition of the energy functional $\mathcal{H}\Big(\mathbf{s}_b(x')\Big)$ under the fixed distribution of spin density $\mathbf{s}(x) = \sum_a \mathbf{s}_a(x)$. From here we come to the next variational problem

$$\delta\Big\{\mathcal{H}\Big(\mathbf{s}_a(x')\Big) + \int d^3x\vec{\lambda}(x)\sum_a \mathbf{s}_a(x)\Big\} = 0 \tag{6.3}$$

where $\vec{\lambda}(x)$ is the corresponding Lagrange factor. Varying (6.3), we get

$$\frac{\delta\mathcal{H}\Big(\mathbf{s}_b(x')\Big)}{\delta\mathbf{s}_a(x)} = -\vec{\lambda}(x) \tag{6.4}$$

and the Lagrange factor is to be found from the relationship

$$\sum_a \mathbf{s}_a(x) = \mathbf{s}(x). \tag{6.5}$$

In accordance with stated above let us choose the relaxation term $\mathbf{R}_a(x)$ in the form

$$\mathbf{R}_a(x) = \sum_b \hat{r}_{ab}\left(\mathbf{h}_b(x) - \vec{\lambda}'(x)\right), \tag{6.6}$$

(compare with the relaxation term in movement equation of magnetic moment in Landau- Lifshits form [11]). Supposing that $\sum_a \mathbf{R}_a = 0$ see (6.2)) we have

$$\vec{\lambda}'(x) = \hat{r}^{-1}\sum_{ab}\hat{r}_{ab}\,\mathbf{h}_b(x), \qquad \hat{r} \equiv \sum_{ab}\hat{r}_{ab}. \tag{6.7}$$

Here $\hat{r}_{ab}$ is the matrix of positively determined quadratic form $(\sum_{ab}\mathcal{Y}_a\,\hat{r}_{ab}\,\mathcal{Y}_b > 0)$. Then according to (6.1), (6.6) we get

$$\dot{\mathcal{H}} = \sum_a \int d^3x \frac{\delta\mathcal{H}}{\delta\mathbf{s}(x)}\,\dot{\mathbf{s}}_a\,(x) =$$
$$= -\sum_{ab}\int d^3x\left(\mathbf{h}_a(x) - \vec{\lambda}'(x)\right)\hat{r}_{ab}\left(\mathbf{h}_b(x) - \vec{\lambda}'(x)\right) < 0,$$

that corresponds to energy dissipation in system. Consider asymptotic ($t \gg \tau_r, \tau_r \sim \frac{1}{r_{ab}}$ is relaxation time) space-uniform solutions of (6.1). If the Hamiltonian is limited from below then solution of (6.1) at $t \gg \tau_r$ satisfies Eqs. (6.4), (6.5) due to monotonic decay (because $\mathcal{H} = 0$ at $t \gg \tau_r$), which in spaceuniform case has the form

$$\frac{\partial \varepsilon(s_b)}{\partial \mathbf{s}_a} = -\vec{\lambda}, \qquad \sum_a \mathbf{s}_a = \mathbf{s}; (\vec{\lambda}' = \vec{\lambda}), \tag{6.8}$$

where $\varepsilon(\mathbf{s}_b) \equiv \varepsilon(x, \mathbf{s}_b)$ is the energy density. Since $\mathbf{R}_a = 0$, then from (6.1) we have

$$\dot{\mathbf{s}}_a = [\mathbf{s}_a, \mathbf{h}_a], \mathbf{h}_a = \vec{\lambda}. \tag{6.9}$$

Due to exchange character of interaction

$$\sum_a \left[\mathbf{s}_a, \frac{\partial \varepsilon}{\partial \mathbf{s}_a}\right] = 0 \tag{6.10}$$

and, consequently

$$\vec{\lambda} = \lambda \mathbf{s}, \qquad \dot{\mathbf{s}} = 0.$$

As density of energy depends only on scalar products $\mathbf{s}_a\mathbf{s}_b, \mathbf{s}^2{}_a$ then most general solution of (6.8) has the form

$$\mathbf{s}_b \equiv \underline{\mathbf{s}}_b(\mathbf{s}) = \tfrac{1}{n}\mathbf{s} + \alpha_b(s)\tfrac{\mathbf{s}}{s} + \beta_b(s)\frac{[\vec{\xi}_0, \mathbf{s}]}{|[\vec{\xi}_0, \mathbf{s}]|} + \gamma_b(s)\frac{\left[[\vec{\xi}_0, \mathbf{s}], \mathbf{s}\right]}{\left|\left[[\vec{\xi}_0, \mathbf{s}], \mathbf{s}\right]\right|},$$
$$\sum_b \underline{\mathbf{s}}_b(\mathbf{s}) = \mathbf{s}. \tag{6.11}$$

where $\vec{\xi}_0$ is the arbitrary unit vector and $\alpha_b(s), \beta_b(s), \gamma_b(s)$ are the functions of $s \equiv |\, \mathbf{s} \,|$, uniquelly defined by exchange Hamiltonian $\mathcal{H}$ and satisfying the relationships:

$$\sum_b \alpha_b(s) = \sum_b \beta_b(s) = \sum_b \gamma_b(s) = 0.$$

Let a be the arbitrary matrix of rotations. Then according to (6.11)

$$\mathbf{s}_b(\mathbf{s}) \equiv \tilde{a}\ \underline{\mathbf{s}}_b(a\mathbf{s}) = \underline{\mathbf{s}}(\mathbf{s})\,|_{\vec{\xi}_0 \to a\vec{\xi}_0}, \qquad \sum_b \mathbf{s}_b(\mathbf{s}) = \mathbf{s}. \tag{6.12}$$

Matrix a can depend on time. This dependence, as follows from (6.9), is defined by equation:

$$\dot{a}_{\alpha\beta} = -a_{\alpha\gamma}\varepsilon_{\gamma\beta\rho}\lambda_\rho. \tag{6.13}$$

Let $\varepsilon(a, \mathbf{s}) \equiv \varepsilon\left(\tilde{a}\ \mathbf{s}_b(a\mathbf{s})\right)$. Then according to (97)

$$\frac{\partial \varepsilon}{\partial \mathbf{s}} = \sum_b \frac{\partial \varepsilon}{\partial s_{b\alpha}} \frac{\partial s_{b\alpha}}{\partial \mathbf{s}} = -\lambda_\alpha \sum_b \frac{\partial s_{b\alpha}}{\partial \mathbf{s}} = -\vec{\lambda}. \tag{6.14}$$

Consequently, equation (6.13) can be rewritten in the form:

$$\dot{a}_{\alpha\beta} = a_{\alpha\gamma}\varepsilon_{\gamma\beta\rho}\frac{\partial\varepsilon}{\partial s_\rho}. \tag{6.15}$$

Let us formulate functional hypothesis. We shall consider that in space-nonuniform case at times $t \gg \tau_r$ spin densities of sublattices depend on time also only via parameters $a(x,t), \mathbf{s}(x,t)$

$$\mathbf{s}_a(x,t)_{t\gg\tau_r} \to \mathbf{s}_a\Big(x; a(x',t), \mathbf{s}(x',t)\Big) \tag{6.16}$$

and

$$\textstyle\sum_a \mathbf{s}_a(x; a, \mathbf{s}) = \mathbf{s}(x).$$

Physically such consideration corresponds to the fact that at suffciently large times due to strong exchange interaction complexes of spins are formed, which practically are not deformed and orientation of which is assigned by the rotation matrix a.

In the main approximation by space gradients from the form of local-equilibrium solution (5.13) for $\mathbf{s}_a(x; a, \mathbf{s})$ we have

$$\mathbf{s}_a\Big(x; a(x'), \mathbf{s}(x')\Big) = \tilde{a}\,(x,t)\underline{\mathbf{s}}_a\Big(a(x,t)\mathbf{s}(x,t)\Big) + \ldots \tag{6.17}$$

There exists close analogy of our consideration with the description of kinetics and hydrodynamics of ordinary liquid. It exact equation of evolution is the Boltsman equation for one-particle distribution function $f(x)$. In our case the analogy of this equation is Landau-Lifshits equation for sublattice spins $\mathbf{s}_a(x)$. Further at times larger than relaxation time, distribution funtion becomes a functional of densities of additive movement integrals (hydrodynamic parameters) and dependence from time of function f is contained only in these densities. For our case for times $t \gg \tau_r$ sublattice spins $\mathbf{s}_a(x)$ become the functionals of parameters $a(x), \mathbf{s}(x)$ and depend on time only via $a(x,t), \mathbf{s}(x,t)$. In what follows below we shall limit ourselvs by consideration of low-frequency phenomenas, ($\omega\tau_r \ll 1$), then reduction occurs in description of manysublattice magnetic state from complete set of variables $\mathbf{s}_a(x)$ to parameters $a(x), \mathbf{s}(x)$. Let us introduce, by definition, reduced spin $\mathbf{s}^r_a(x)$ according to equality

$$\mathbf{s}^r_a(x) = \tilde{a}\,(x)\underline{\mathbf{s}}_a\Big(\underline{\mathbf{s}}(x)\Big), \qquad \underline{\mathbf{s}} \equiv a\mathbf{s}. \tag{6.18}$$

For $\mathbf{s}^r_a(x)$ we have

$$\sum_a \mathbf{s}^r_a(x) = \mathbf{s}(x).$$

If some functional of sublattice spins $\mathbf{s}_a(x)$ is assigned then reduced functional can be received from them by the substitution $\mathbf{s}_a(x) \to \mathbf{s}^r_a(x)$:

$$F\Big(x; \mathbf{s}_a(x')\Big) \to F^r\Big(x; a(x'), \mathbf{s}(x')\Big) \equiv F\Big(x; \mathbf{s}^r_a(x')\Big). \tag{6.19}$$

In particular, if density of energy functional $\varepsilon\,(x; \mathbf{s}_a(x'))$ is assined then reduced density is determined by the formula

$$\varepsilon^r\Big(x; a(x'), \mathbf{s}(x')\Big) = \varepsilon\Big(x; \mathbf{s}^r_a(x')\Big). \tag{6.20}$$

In the small gradient approximation we have

$$a(x)\,\tilde{a}\,(x') = e^{(x'-x)(\nabla+\underline{\omega})} \approx e^{(x'-x)\underline{\omega}(x)}, \qquad \underline{\omega}(x) \equiv a\nabla\,\tilde{a}\,.$$

Therefore we can take the expression

$$\varepsilon^r\left(x; a(x'), \mathbf{s}(x')\right) = \varepsilon\left(x; e^{(x'-x)\underline{\omega}(x)}\underline{s}_c\,(\underline{\mathbf{s}}(x'))\right).$$

as the reduced energy density. Let us obtain the movement equations for short description parameters $a(x)$, $\mathbf{s}(x)$. Poisson brackets for sublattice spins $\mathbf{s}_a(x)$ are of the form

$$\{s_{a\alpha}(x), s_{b\beta}(x')\} = \delta_{ab}\varepsilon_{\alpha\beta\gamma}s_{b\gamma}(x)\delta(x-x'). \tag{6.21}$$

Let us determine PB for dynamic variables $a(x), \mathbf{s}(x)$ by formulae:

$$\begin{aligned} \{s_\alpha(x), s_\beta(x')\}' &= \varepsilon_{\alpha\beta\gamma}s_\gamma(x)\delta(x-x'), \\ \{a_{\alpha\beta}(x), s_\mu(x')\}' &= \varepsilon_{\mu\rho\beta}a_{\alpha\rho}(x)\delta(x-x'), \\ \{a_{\alpha\beta}(x), a_{\mu\nu}(x')\}' &= 0. \end{aligned} \tag{6.22}$$

Derivation of algebra (6.22) was given in section 2. Prime above figure brackets in (6.22) and below means that corresponding PB are calculated in terms of short description parameters $a(x), \mathbf{s}(x)$; if prime above brackets is absent then such brackets are calculated interms of sublattice spins $\mathbf{s}_a(x)$, applying algebra (6.22). Let us introduce reduced PB denoted by $\{...,...\}^r$, which are to be calculated in terms of manysublattice spins $\mathbf{s}_a(x)$ with subsequent substitution of reduced quantities $\mathbf{s}^r_a(x)$ insteed of $\mathbf{s}_a(x)$. The nextrelation holds:

$$\{s_\alpha(x), s^r_{b\beta}(x')\}' = \{s_\alpha(x), s_{b\beta}(x')\}^r$$

It follows now that reduced PB of density of summarised spin $s_\alpha(x)$ with arbitrary functional $F(x;\mathbf{s}_a)$ of sublattice spins equals to PB of spin density $s_\alpha(x)$ with reduced functional $F^r(x;a,\mathbf{s})$, calculated with help of algebra (6.22)

$$\left\{s_\alpha(x), F\left(x', \mathbf{s}_a(x'')\right)\right\}^r = \left\{s_\alpha(x), F^r\left(x'; a(x'')\mathbf{s}(x'')\right)\right\}'. \tag{6.23}$$

Using this statement it is easy to get the movement equation for $\mathbf{s}(x)$. Really,

$$\dot{s}_\alpha\,(x) = \{s_\alpha(x), \mathcal{H}\}.$$

When $t \gg \tau_r$ bracket $\{...,...\}$ in the main approximation (in sence of expansion (6.17)) leads to the bracket $\{...,...\}^r$. Substituting reduced bracket according to (6.23) we get

$$\dot{s}_\alpha\,(x) = \{s_\alpha(x), \mathcal{H}^r\}'; \qquad \mathcal{H}^r = \int d^3x\varepsilon^r\left(x; a(x'), \mathbf{s}(x')\right).$$

or

$$\frac{\partial s_\alpha(x)}{\partial t} = \varepsilon_{\alpha\beta\gamma}\left(\frac{\delta\mathcal{H}^r}{\delta s_\beta(x)}s_\gamma(x) + \frac{\delta\mathcal{H}^r}{\delta a_{\rho\beta}(x)}a_{\rho\gamma}(x)\right). \tag{6.24}$$

Now let us pass to derivation of movement equation for rotation matrix. In the space-uniform case, according to (6.15), we have:

$$\dot{a}_{\alpha\beta}= a_{\alpha\gamma}\varepsilon_{\gamma\beta\rho}\frac{\partial\varepsilon^{r}(a,s)}{\partial s_{\rho}}. \tag{6.25}$$

Using the principle of locality of quasi-equilibrium state this relationship in the space-nonuniform case (in small gradients approximation) can be rewritten in the form:

$$\dot{a}_{\alpha\beta}\,(x) = a_{\alpha\gamma}(x)\varepsilon_{\gamma\beta\rho}\frac{\delta\mathcal{H}^{r}}{\delta s_{\rho}(x)}. \tag{6.26}$$

or in terms of PB (6.22):

$$\tfrac{\partial a(x)}{\partial t} = \{a(x),\mathcal{H}^{r}\}.$$

Thus, closed system of equations for quantities $s(x,t), a(x,t)$ is determined by formulae (6.24), (6.26). These equations coinside with equations (3.40) of section 3 and the Hamiltonian densities of equations (6.24), (6.26), (3.40) which are connected by relationship (6.18).

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