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THE SUPERSYMMETRY REPRESENTATION FOR CORRELATION FUNCTIONS OF DISORDERED SYSTEMS

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The supersymmetry representation for correlation functions of spherical model and non-interacting quantum gas is obtained. Using this representation the configurational averaging can be performed before thermodynamical one and the problem of calculation of configurationally averaged correlation functions of the disordered system is reduced to the calculation of correlation functions for Fermi-Bose interacting regular system.

1. Introduction

It is known that the calculation of configurationally averaged free energy for disordered systems is not an easy problem. In 1975 Edwards and Anderson proposed a new method (the so called replica method) for calculation of free energy of disordered systems [1]. In this method the disordered system is replaced by systems ("replicas") which are identical to the original one. The limit $n \to 0$ gives the initial model. Using replica method the configurational averaging of free energy can be performed before thermodynamical averaging. The partition function of a regular model is obtained after the averaging and standard methods developed for regular systems can be used. However, the replica method has some difficulties. For example, the procedure of analytical continuation of non-zero integer n to $n = 0$ is not simple, the Hamiltonian obtained by the replica method is very complicated and cannot be solved exactly.

Another method, where configurational averaging can be performed at the initial stage of calculations, is the supersymmetry method [2]. This method is applicable to the description of the motion of non-interacting particles in a random potential and is based on the use of functional integrals over both commuting and anticommuting variables. After configurational averaging the problem is reduced to the supersymmetry field theory. The supersymmetry method can be used for an essentially narrower class of problems than the replica one but it is free of the problem of analytical continuation.

Note also that the representation of Green's function in the form of Feynman's path integral [3] is usefull for investigation of electron motion in random potential [4]. Whithin this method the configurational averaging of Green's function can be performed before calculation of the path integrals. In a result the path integrals for a regular system are obtained and the calculation of path integrals is simplified.

In the present paper the correlation functions of disordered systems are considered. The main problem in calculation of the configurationally averaged correlation functions of disordered systems deals with a factor $1/Z$ (Z being the partition funtion) in statistical operator. The idea of the present paper is to rewrite $1/Z$ as a partition function $Z' = Spe^{-\beta H'}$ of some new system with the Hamiltonian H' and then to perform the configurational averaging. In the present paper this idea is realized for spherical model and for non-interacting quantum gas and as a result the supersymmetry representation for correlation functions is obtained. Using this representation the configurational averaging can be performed before the thermodynamic one and the problem of calculation of configurationally averaged correlation functions of disordered systems is reduced to the calculation of correlation functions for interacting Fermi-Bose regular systems.

2. The spherical model

The spherical model is one of the most studied models of statistical mechanics (see, for example, [6] and references therein). The disordered spherical model was investigated by Pastur [7]. It was shown that straightforward generalization of the spherical model to the disordered case for a finite-range interaction has some rather unnatural properties: the phase transition in the model exists even in one dimension, and even in the case of ferromagnetic interaction it does not vanish as a homogeneous external field is switched on and the spontaneous magnetization is zero for $T < T_c$. In ref.[7] for the ferromagnetic interaction, a modification of disordered spherical model was proposed which displays the behaviour expected for the disordered ferromagnets.

In present paper using the supersymmetry method we calculate the correlation functions of the disordered spherical model and modified disordered spherical model. The explicit expresions for critical temperature of both models are obtained. The results are in agreement with [7].

At first we consider spherical model for two-dimensional classical spins. In this case the supersymmetry representation for correlation functions can be performed in the simplest way. The generalization for $2n-$dimensional spins can be easily performed. The supersymmetry representation in the case of $2n+1-$dimensional classical spins faces some difficulties but can be performed as well.

2.1. The supersymmetry representation for correlation functions

The spherical model Hamiltonian [5] of two-dimensional classical spins reads

$$H = -\sum_{i,j} J_{ij}\mathbf{S}_i\mathbf{S}_j + \mu\sum_i \mathbf{S}_i^2, \tag{2.1}$$

where i, j are the site numbers of lattice, $\mathbf{S}_i = (S_i^x, S_j^y)$ is 2-dimensional vector, J_{ij} is the random exchange interaction of i and j spins, μ is determined from spherical condition

$$\frac{1}{N}\sum_i^N \langle \mathbf{S}_i^2 \rangle = 1. \tag{2.2}$$

In present work only finite-range ferromagnetic interaction J_{ij} and the case of bounded fluctuations of exchange interaction are considered.

Let us introduce a complex variables

$$\varphi_j = S_j^x + iS_j^y, \qquad \varphi_j^* = S_j^x - iS_j^y. \tag{2.3}$$

Then the Hamiltonian and the spherical condition read

$$H = -\sum_{i,j} J_{ij}\varphi_i^*\varphi_j + \mu \sum_i \varphi_i^*\varphi_i, \tag{2.4}$$

$$\frac{1}{N}\sum_{i,j}^{N}\langle\varphi_i^*\varphi_i\rangle = 1. \tag{2.5}$$

Now let us consider the spin correlation functions or the mean value of function of spin variables

$$\langle A\rangle = \frac{1}{Z}\int (d\varphi^* d\varphi) A(\varphi^*, \varphi)\ \exp\left(-\sum_{i,j} I_{ij}\ \varphi_i^*\varphi_j\right), \tag{2.6}$$

where

$$(d\varphi^* d\varphi) = \prod_i^N dS_i^x dS_i^y,$$

$$A(\varphi^*, \varphi) = A(\varphi_1^*, \varphi_1, \ldots, \varphi_N^*, \varphi_N) = A(S_1^x, S_1^y, \ldots, S_N^x, S_N^y),$$

$$I_{ij} = (\mu\delta_{ij} - J_{ij})/T.$$

The partition function Z is

$$Z = \int (d\varphi^* d\varphi)\ \exp\left(-\sum_{i,j} I_{ij}\varphi_i^*\varphi_j\right) = \frac{\pi^N}{det\ \|\ I_{ij}\ \|}. \tag{2.7}$$

It should be noted, that we consider only the case of bounded fluctuations. Therefore, there is such $\mu > 0$ that integral (2.7) converges.

We shall be interested in the configurationally averaged mean value $\overline{\langle A\rangle}$, where

$$\overline{(\ldots)} = \int \prod_{i<j} d\Delta J_{ij}\ P(\ldots \Delta J_{ij} \ldots)(\ldots), \tag{2.8}$$

$$\Delta J_{ij} = J_{ij} - \overline{J_{ij}},$$

$P(\ldots \Delta J_{ij} \ldots)$ is the distribution function of exchange integral. We assume that fluctuations of exchange integral are bounded and statistically independent

$$P(\ldots \Delta J_{ij} \ldots) = \prod_{i<j} P_{i,j}(\Delta J_{ij}). \tag{2.9}$$

Note, that the calculation of configurationally averaged quantity $\overline{\langle A\rangle}$ using (2.6) is difficult because of the presence of factor $1/Z$. The main idea is to write $1/Z$ as a partition function of some new system and then to perform configurational averaging.

Using Gaussian Grassman integral we obtain

$$\frac{1}{Z} = \frac{1}{\pi^N}\, det \parallel I_{ij} \parallel = \frac{1}{\pi^N} \int d\eta_1^* d\eta_1 \ldots d\eta_N^* d\eta_N \exp\left(-\sum_{ij} I_{ij}\eta_i^*\eta_j\right) = Z'. \tag{2.10}$$

Note, that Z' can be treated as partition function of spherical model with anticommuting variables.

Then $\langle A \rangle$ reads

$$\langle A \rangle = \frac{1}{\pi^N} \int (d\varphi^* d\varphi)(d\eta^* d\eta) A(\varphi^*, \varphi) \exp\left(-\sum_{ij} I_{ij}\phi_i^+\phi_j\right), \tag{2.11}$$

where we have introduced the supervectors

$$\phi_i = \begin{pmatrix} \varphi_i \\ \eta_i \end{pmatrix}, \qquad \phi_i^+ = (\varphi_i^* \quad \eta_i^*).$$

The functional

$$F = -\sum_{i,j} I_{ij}\, \phi_i^+ \phi_i$$

is invariant under linear transformation

$$\begin{pmatrix} \varphi_i' \\ \eta_i' \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} \begin{pmatrix} \varphi_i \\ \eta_i \end{pmatrix}, \tag{2.12}$$

where ε is Grassman variable.

This transformation can be treated as a rotation in the space of supervectors. That is why the functional has supersymmetry with respect to the group of transformations mixing commuting and anticommuting variables.

Note that important result of supersymmetry is

$$\langle \varphi_i^* \varphi_j \rangle = -\langle \eta_i^* \eta_j \rangle. \tag{2.13}$$

It can be easily proved. Consider (2.11) in the case $A = 1$. Then taking the derivative of (2.11) with respect to I_{ij} we obtain (2.13).

Using (2.11) the configurationally averaged correlation function $\overline{\langle A \rangle}$ can be written in the form

$$\overline{\langle A \rangle} = \frac{1}{\pi^N} \int (d\varphi^* d\varphi)(d\eta^* d\eta) A(\varphi^*\varphi) \times \tag{2.14}$$

$$\times \exp\left(-\sum_{i,j} \overline{I_{ij}}\, \phi_i^+ \phi_j\right) \overline{\exp\left(\frac{1}{T}\sum_{i,j} \Delta J_{ij}\, \phi_i^+ \phi_j\right)}.$$

For distribution function (2.9) one can perform configurational averaging exactly

$$\overline{\exp\left(\frac{1}{T}\sum_{i,j} \Delta J_{ij}\, \phi_i^+ \phi_j\right)} = \exp(U(\varphi, \eta)), \tag{2.15}$$

where

$$U(\varphi,\eta)=\frac{1}{2}\sum_{i,j} ln(U_{ij}((\phi_i^+\phi_j+\phi_j^+\phi_i)/T)),$$

$$U_{ij}(x)=\int d\Delta J_{ij}P_{ij}(\Delta J_{ij})\exp(-\Delta J_{ij}x),$$

and for $\overline{\langle A\rangle}$ we have

$$\overline{\langle A\rangle}=\frac{1}{\pi^N}\int(d\varphi^*d\varphi)(d\eta^*d\eta)A(\varphi^*,\varphi)\ \exp\left(F(\varphi,\eta)\right), \tag{2.16}$$

where

$$F(\varphi,\eta)=F_0(\varphi,\eta)+U(\varphi,\eta),$$
$$F_0(\varphi,\eta)=-\sum_{i,j}\overline{I_{ij}}\ \phi_i^+\phi_j,$$
$$\overline{I_{ij}}=(\mu\delta_{ij}-\overline{J_{ij}})/T=(\mu\delta_{ij}-J(\mathbf{R}_i-\mathbf{R}_j))/T.$$

Because $\phi_i^+\phi_j$ is invariant under linear transformation (2.12) the full functional $F=F_0+U$ is invariant under this transformation too. Thus, the configurational averaging does not break the supersymmetry. Note also that the configurational averaging over the fluctuations of exchange integral leads to Fermi-Bose interaction.

It is convenient to introduce new variables

$$\varphi_{\mathbf{k}}=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\varphi_j e^{-i\mathbf{k}\mathbf{R}_j}, \tag{2.17}$$

$$\eta_{\mathbf{k}}=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\eta_j e^{-i\mathbf{k}\mathbf{R}_j}.$$

Then

$$(d\varphi^*d\varphi)=\prod_{\mathbf{k}}(d\varphi_{\mathbf{k}}^*d\varphi_{\mathbf{k}}),$$
$$(d\eta^*d\eta)=\prod_{\mathbf{k}}(d\eta_{\mathbf{k}}^*d\eta_{\mathbf{k}}),$$

and F_0 is diagonal

$$F_0=-\sum_{\mathbf{k}}I(\mathbf{k})(\phi_{\mathbf{k}}^+\phi_{\mathbf{k}}), \tag{2.18}$$

where

$$I(\mathbf{k})=(\mu-J(\mathbf{k}))/T,$$
$$J(\mathbf{k})=\sum_j J(\mathbf{R}_j)e^{-i\mathbf{k}\mathbf{R}_j}.$$

Because in the limit $N\to\infty$ one has $\frac{1}{N}\sum_i\langle\varphi_i^*\varphi_i\rangle=\overline{\frac{1}{N}\sum_i\langle\varphi_i^*\varphi_i\rangle}$ the spherical condition now can be written as

$$\frac{1}{N}\sum_{\mathbf{k}}\overline{\langle\varphi_{\mathbf{k}}^*\varphi_{\mathbf{k}}\rangle}=1. \tag{2.19}$$

2.2. Quadratic fluctuation approximation

Unfortunately, the calculation of functional integral (2.16) with full functional can not be done exactly. Consider the simplest approximation. We assume that fluctuations of exchange integral are small. During all calculations we shall preserve the terms up to the quadratic fluctuations of exchange integral $\Delta J_{ij}{}^2$. We carry out this approximation for the case $T > T_c$, where T_c is the critical temperature.

In quadratic fluctuation approximation we obtain

$$\exp(U(\varphi,\eta)) = 1 + \frac{1}{4T^2}\sum_{i,j} D_{ij}(\phi_i^+\phi_j + \phi_j^+\phi_i)^2 \tag{2.20}$$

where $D_{i,j} = \overline{\Delta J_{ij}{}^2} = D(\mathbf{R_i} - \mathbf{R_j})$.

In $\mathbf{k}$-space this result reads

$$\exp(U(\varphi,\eta)) = 1 + \frac{1}{2T^2N}\sum_{\mathbf{k_1},\mathbf{k_2}}\sum_{\mathbf{k}'_1,\mathbf{k}'_2}(D(\mathbf{k_1}+\mathbf{k}'_1) + \mathbf{D}(\mathbf{k_1}-\mathbf{k}'_2)) * \tag{2.21}$$
$$\delta(\mathbf{k_1}+\mathbf{k}'_1-\mathbf{k_2}-\mathbf{k}'_2)(\phi^+_{\mathbf{k_1}}\phi_{\mathbf{k_2}})(\phi^+_{\mathbf{k}'_1}\phi_{\mathbf{k}'_2})$$

On the basis of this approximation using Wick's theorem we can easily calculate the pair correlation function. The result is the following

$$\overline{\langle\varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\rangle} = \overline{\langle\varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\rangle_0} + \frac{1}{T}A(\mathbf{k})\overline{\langle\varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\rangle_0}^2, \tag{2.22}$$

where

$$A(\mathbf{k}) = \frac{1}{T}\frac{1}{N}\sum_{\mathbf{q}\neq\mathbf{0}}(D(\mathbf{k}+\mathbf{q}) + D(0))\,\overline{\langle\varphi^*_{\mathbf{q}}\varphi_{\mathbf{q}}\rangle_0},$$

$$\overline{\langle\varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\rangle_0} = \frac{T}{\mu - J(\mathbf{k})}$$

is the correlation function calculated with F_0.

In quadratic fluctuation approximation (linear approximation with respect to D) two term of (2.22) can be involved in the denominator $\overline{\langle\varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\rangle_0}$. Then $\overline{\langle\varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\rangle}$ has the same form as $\overline{\langle\varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\rangle_0}$

$$\overline{\langle\varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\rangle} = \frac{T}{\tilde{\mu} - \tilde{J}(\mathbf{k})} \tag{2.23}$$

where

$$\tilde{J}(\mathbf{k}) = J(\mathbf{k}) + \frac{1}{T}\frac{1}{N}\sum_{\mathbf{q}\neq\mathbf{0}} D(\mathbf{k}+\mathbf{q})\,\overline{\langle\varphi^*_{\mathbf{q}}\varphi_{\mathbf{q}}\rangle_0}, \tag{2.24}$$

$$\tilde{\mu} = \mu - \frac{D(0)}{2T}.$$

Then for $\tilde{\mu}$ we obtain the equation

$$\frac{1}{N}\sum_{\mathbf{k}\neq\mathbf{0}}\frac{T}{\tilde{\mu} - \tilde{J}(\mathbf{k})} = 1. \tag{2.25}$$

The critical temperature can be obtained from the pole of correlation function (2.23) at $\mathbf{k} = 0$. Then

$$\tilde{\mu}_c = \tilde{J}(0). \tag{2.26}$$

Substituting (2.26) into (2.25) for the critical temperature we obtain

$$\frac{1}{T_c} = \frac{1}{N} \sum_{\mathbf{k} \neq 0} \frac{1}{\tilde{J}(0) - \tilde{J}(\mathbf{k})}. \tag{2.27}$$

The asymptotic behaviour of $\tilde{\mu}$ at $T \to \infty$ is $\tilde{\mu} = T$, and for the correlation function in the high temperatures limit we obtain

$$\overline{\langle \varphi^*_{\mathbf{k}} \varphi_{\mathbf{k}} \rangle} \to 1 \quad \text{while} \quad T \to \infty .$$

Then in the site representation

$$\overline{\langle \varphi^*_i \varphi_j \rangle} \to \delta_{ij} \quad \text{while} \quad T \to \infty .$$

Let us consider, for example, the case of nearest-neighbours interaction. In this case the renormalized exchange integral (2.24) can be calculated exactly

$$\tilde{J}(\mathbf{k}) = 2\tilde{J} \sum_{\alpha=1}^{d} \cos k_\alpha a, \tag{2.28}$$

where

$$\tilde{J} = \overline{J} \left(1 + \frac{\overline{\Delta J^2}}{\overline{J}^2} \frac{1}{2d} \left(\frac{\mu}{2\overline{J}} f\left(\frac{\mu}{2\overline{J}}\right) - 1 \right) \right),$$

$$f(x) = \frac{1}{N} \sum_{\mathbf{k} \neq 0} \frac{1}{x - \sum_{\alpha=1}^{d} \cos k_\alpha a} = \frac{1}{(2\pi)^d} \int_0^{2\pi} dy_1 \ldots \int_0^{2\pi} dy_d \frac{1}{x - \sum_{\alpha=1}^{d} \cos y_\alpha},$$

d is lattice dimension, a is lattice constant.

At critical temperature $\mu = \mu_c = 2\overline{J}d$ and then $f(\mu_c/2\overline{J}) = f(d) = 2\overline{J}/T_c^0$, where T_c^0 is the critical temperature of ideal spherical model with exchange integral $\overline{J}$. Using (2.27) and (2.28) for renormalized critical temperature we obtain

$$T_c = T_c^0 \left(1 + \frac{\overline{\Delta J^2}}{\overline{J}^2} \left(\frac{\overline{J}}{T_c^0} - \frac{1}{2d} \right) \right). \tag{2.29}$$

It is interesting to note that $\overline{J}/T_c^0 > 1/2d$ and the fluctuations of exchange integral in the case of nearest-neighbours interaction lead to increase of critical temperature of the spherical model straightforwardly generalized to the the disordered case. The renormalized critical temperature is nonzero even when the critical temperature of the ordered model is equal to zero. Thus the phase transition in disordered model exists even in one and two dimensions. This result is in agreement with the result of paper [7].

2.3. The modified spherical model

The Hamiltonian of modified spherical model proposed in [7] with two-dimensional spins reads

$$H = \frac{1}{2}\sum_{i,j} J_{ij}(\mathbf{S_i} - \mathbf{S_j})^2 + \mu \sum_{\mathbf{i}} \mathbf{S_i^2} = \frac{1}{2}\sum_{i,j} J_{ij}(\varphi_i^* - \varphi_j^*)(\varphi_i - \varphi_j) + \mu \sum_i \varphi_{\mathbf{i}}^* \varphi_{\mathbf{i}}. \tag{2.30}$$

Note, that in ordered case the Hamiltonian of modified spherical model is equivalent to the spherical model. In disordered case the Hamiltonian (2.30) contains the diagonal disorder. In [7] it was shown that such modification of the spherical model leads to the properties expected for disordered ferromagnets and this model can be considered as a good model of disordered ferromagnet.

The supersymmetry method developed for spherical model can be easy used for modified spherical model. Using the same approximation as in the case of spherical model, for correlation function of modified spherical model we obtain

$$\overline{\langle \varphi_{\mathbf{k}}^* \varphi_{\mathbf{k}} \rangle} = \frac{T}{\mu + \tilde{J}(0) - \tilde{J}(\mathbf{k})} \tag{2.31}$$

where

$$\tilde{J}(\mathbf{k}) = J(\mathbf{k}) - \frac{2}{T} D(\mathbf{k}) + \frac{2}{T}\frac{1}{N} \sum_{\mathbf{q}\neq 0} D(\mathbf{k}+\mathbf{q})\, \overline{\langle \varphi_{\mathbf{q}}^* \varphi_{\mathbf{q}} \rangle_0}. \tag{2.32}$$

In the case of nearest-neigbors interaction the result for the critical temperature is the following

$$T_c = T_c^0 \left(1 - \frac{\overline{\Delta J^2}}{\overline{J}^2}\frac{1}{d}\right). \tag{2.33}$$

The fluctuations of exchange integral in the case of modified spherical model in contrast to (2.29) lead to decrease of critical temperature.

2.4. The generalization of supersymmetry representation for arbitrary dimension of spin

The generalization of supersymmetry representation for correlation functions in the case of even dimensions of spins faces no problems. The supersymmetry representation in the case of odd dimensions of spins is more complicated but also can be performed. Let us consider the spherical model with one dimensional spins. In this case $1/Z = \sqrt{det \parallel I_{ij} \parallel}/\pi^N$ and for mean value we have

$$\langle A \rangle = \frac{1}{\pi^{N/2}} \sqrt{det \parallel I_{ij} \parallel} \int (dS^x) \exp\left(-\sum_{i,j} I_{ij} S_i^x S_i^y\right) A(S_1^x, \ldots, S_N^x). \tag{2.34}$$

The main problem is that (2.34) contains $\sqrt{det \parallel I_{ij} \parallel}$. Note, that $\sqrt{det \parallel I_{ij} \parallel}$, where I_{ij} is symmetric matrix, can not be written as Gaussian

integral over Grassman variables. This problem can be solved if we rewrite (2.34) in the following form

$$\langle A\rangle = \tag{2.35}$$

$$= det \parallel I_{ij} \parallel \frac{\pi^{-N/2}}{\sqrt{det \parallel I_{ij} \parallel}} \int (dS^x) \exp\left[-\sum_{i,j} I_{ij} S_i^x S_j^x\right] A(S_1^x, ..., S_N^x) =$$

$$= det \parallel I_{ij} \parallel \int \frac{(dS^x dS^y)}{\pi^N} \exp\left[-\sum_{i,j} I_{ij}(S_i^x S_j^x + S_i^y S_j^y)\right] A(S_1^x, ..., S_N^x).$$

Equation for μ is

$$\frac{1}{N}\sum_{i=1}^{N}\langle (S_i^x)^2\rangle = 1. \tag{2.36}$$

Taking into account that $\langle (S_i^x)^2\rangle = \langle (S_i^y)^2\rangle$ the equation (2.36) can be written as

$$\frac{1}{2N}\sum_{i=1}^{N}\langle \mathbf{S}_i^2\rangle = 1. \tag{2.37}$$

On the basis of (2.35) and (2.37) the calculation of correlation functions of spherical model with one-dimensional spins is reduced to the calculation of correlation functions of spherical model with two-dimensional spins. Therefore the method described in sections 1 and 2 can be used.

3. Non-interacting quantum gas

In [2] the supersymmetry representation for Green's function of electron in a random potential was obtained. In this section we propose the new supersymmetry method for calculation of configurationally averaged correlation functions of non-interacting quantum particles. We develop this method for electrons on a lattice. The proposed supersymmetry method can be used also for description of a non-interacting bosons in random potential.

3.1. The Hamiltonian and the main definition

Consider non-interacting electron gas which is described by the Hamiltonian

$$H_f = \sum_{i\sigma} \epsilon_i a_{i\sigma}^+ a_{i\sigma} + \sum_{\sigma}\sum_{ij} V_{ij} a_{i\sigma}^+ a_{j\sigma}, \tag{3.1}$$

where $a_{i\sigma}^+, a_{i\sigma}$ are creation and annihilation operators of fermions with spin σ on the lattice site i, ϵ_i is the energy of fermion located at site i , V_{ij} is the hopping amplitude between sites i and j.

The electron correlation functions are defined in a standard way

$$\langle A\rangle = \frac{1}{Z_f} Sp_f A e^{-H_f/T+\mu N_f/T}, \tag{3.2}$$

where A denotes the product of creation and annihilation operators,

$$Z_f = Sp_f e^{-H_f/T+\mu N_f/T}$$

is the partition function of fermions,

$$N_f = \sum_{i,\sigma} a^+_{i\sigma} a_{i\sigma}$$

is the operator of fermion occupation number, μ is a chemical potential.

In the case when ϵ_i and V_{ij} take random values we shall be interested in the configurationally averaged correlation functions $\overline{\langle A \rangle}$, where configurational averaging contains the averaging over ϵ_i and V_{ij}

$$\overline{\langle A \rangle} = \int (d\epsilon dV) P(\epsilon, V) < A >, \tag{3.3}$$

$P(\epsilon, V)$ is distribution function of ϵ_i and V_{ij}.

The performing of configurational averaging using (3.2) is difficult because of factor $1/Z$. The aim of next section is to solve this problem.

3.2. The relation between fermion and boson partition functions

In this section we show that partition functions of non-interacting fermions Z_f and non-interacting bosons Z_b are connected by the following formula

$$\frac{1}{Z_f(\mu)} = Z_b(\mu \pm i\pi T). \tag{3.4}$$

For this purpose first of all let us consider the system described by the Hamiltonian of simple fermion harmonic oscillator

$$H_f = \epsilon a^+ a. \tag{3.5}$$

The partition function of this system is

$$Z_f(\mu) = 1 + e^{-(\epsilon-\mu)/T}. \tag{3.6}$$

Now consider the boson harmonic oscillator

$$H_b = \epsilon b^+ b, \tag{3.7}$$

with partition function

$$Z_b(\mu) = \frac{1}{1 - e^{-(\epsilon-\mu)/T}}. \tag{3.8}$$

From (3.8) and (3.6) we see that

$$Z_b(\mu \pm i\pi T) = \frac{1}{1 - e^{\pm i\pi} e^{-(\epsilon-\mu)/T}} = \frac{1}{Z_f(\mu)}$$

and thus (3.4) is true.

It is interesting to note that correlation function of fermion and boson subsystem are connected in the following way

$$\langle a^+ a \rangle_\mu = -\langle b^+ b \rangle_{\mu \pm i\pi T}. \tag{3.9}$$

The formula (3.4) is also true for the case of non-interacting electrons described by the Hamiltonian (3.1) and non-interacting bosons described by the same Hamiltonian, where $a_{i\sigma}, a^+_{i\sigma}$ are replaced by boson operators $b_{i\sigma}, b^+_{i\sigma}$. This can be proved using the procedure of diagonalization. After diagonalization (3.1) takes the form

$$H_f = \sum_{n,\sigma} E_n A^+_{n\sigma} A_{n\sigma}. \tag{3.10}$$

It is obvious that the Hamiltonian of bosons satisfying (3.4) is

$$H_b = \sum_{n,\sigma} E_n B^+_{n\sigma} B_{n\sigma}. \tag{3.11}$$

After returning to initial operators the Hamiltonian of bosons (3.11) takes the form

$$H_b = \sum_{i\sigma} \epsilon_i b^+_{i\sigma} b_{i\sigma} + \sum_{\sigma} \sum_{ij} V_{ij} b^+_{i\sigma} b_{j\sigma}. \tag{3.12}$$

Thus, the partition function of fermions described by the Hamiltonian (3.1) and the partition function of bosons described by (3.12) are connected via formula (3.4).

The correlation functions of fermion and boson subsystems satisfy the following relation

$$\langle a^+_{i\sigma} a_{j\sigma} \rangle_\mu = -\langle b^+_{i\sigma} b_{j\sigma} \rangle_{\mu \pm i\pi T}. \tag{3.13}$$

This also can be proved using the procedure of diagonalization.

3.3. The supersymmetry representation for correlation functions

Using (3.4) for correlation functions of fermions we obtain

$$\begin{aligned} \langle A \rangle &= Z_b(\mu \pm i\pi T) Sp_f A e^{-H_f/T + \mu N_f/T} = \\ &= Sp_b e^{-H_b/T + \mu N_b/T \pm i\pi N_b} Sp_f A e^{-H_f/T + \mu N_f/T} = \\ &= SpA e^{-H/T + \mu N/T \pm i\pi N_b}, \end{aligned} \tag{3.14}$$

where the full Hamiltonian is

$$H = H_f + H_b, \tag{3.15}$$

and the operator of boson occupation number is

$$N_b = \sum_{i,\sigma} b^+_{i,\sigma} b_{i,\sigma},$$

$$N = N_f + N_b.$$

Note, that the full Hamiltonian is supersymmetrical one. It is obvious in the case of a system described by simple Hamiltonian (3.5). Then

$$H = \epsilon(a^+ a + b^+ b). \tag{3.16}$$

In the general case the full Hamiltonian

$$H = \sum_{i,\sigma} \epsilon_i (a^+_{i,\sigma} a_{i,\sigma} + b^+_{i,\sigma} b_{i,\sigma}) + \sum_{i,j} V_{i,j} (a^+_{i,\sigma} a_{j,\sigma} + b^+_{i,\sigma} b_{j,\sigma}) \tag{3.17}$$

is also supersymmetrical one that can be seen after diagonalization.

The Hamiltonian (3.17) can be written in the form of supersymmetric quantum mechanics

$$H = Q^+ Q^- + Q^- Q^+. \tag{3.18}$$

The generators of supersymmetry read

$$Q^+ = \sum_{\sigma} \sum_{i,j} T_{i,j} a^+_{i,\sigma} b_{j,\sigma}, \tag{3.19}$$

$$Q^- = \sum_{\sigma} \sum_{i,j} T^*_{i,j} b^+_{j,\sigma} a_{i,\sigma}, \tag{3.20}$$

where Hermitian matrix T satisfies the equation

$$(T^2)_{ij} = \epsilon_i \delta_{i,j} + V_{ij}. \tag{3.21}$$

Therefore the Hamiltonian (3.17) is supersymmetrical Hamiltonian and that is why the representation for correlation functions (3.14) can be called the supersymmetry representation.

3.4. The configurational averaging

The supersymmetry representation is convenient for performing the configurational averaging. Using (3.14) the configurational averaging can be performed before the thermodynamical averaging

$$\overline{\langle A \rangle} = SpA e^{\pm i\pi N_b + \mu N/T - \tilde{H}/T} \tag{3.22}$$

where the result of configurational averaging can be presented in the following form

$$\overline{e^{-H/T}} = e^{-\tilde{H}/T}. \tag{3.23}$$

The effective Hamiltonian $\tilde{H}$ does not contain random parameters. Therefore $\tilde{H}$ can be considered as the Hamiltonian of regular system. The effective Hamiltonian, generally speaking, includes the interaction between bosons and fermions. Thus the disorder is reduced to Fermi-Bose interaction. In fact, our problem now is to calculate the correlation functions of Fermi-Bose interacting regular systems.

Unfortunately the effective Hamiltonian can not be calculated exactly. Using the well known approximation in the case of high temperatures

$$\overline{e^{-H/T}} \approx e^{-\overline{H}/T + \overline{(\Delta H)^2}/2T^2}$$

we obtain

$$\tilde{H} = \overline{H} - \frac{1}{2T} \overline{(\Delta H)^2}, \tag{3.24}$$

where $\overline{H}$ is the Hamiltonian (3.17) with mean value of random parameters $\overline{\epsilon_i}$ and $\overline{V_{ij}}$. $\overline{(\Delta H)^2}$ contains Fermi-Bose interaction. For example, in the case when only ϵ_i takes random value we have

$$\overline{(\Delta H)^2} = \sum_{\sigma\sigma'} \sum_{i,j} \overline{\Delta\epsilon_i \, \Delta\epsilon_j} \, (a^+_{i\sigma} a_{i\sigma} + b^+_{i\sigma} b_{i\sigma})(a^+_{j\sigma'} a_{j\sigma'} + b^+_{j\sigma'} b_{j\sigma'}) \qquad (3.25)$$

where $\overline{\Delta\epsilon_i \, \Delta\epsilon_j} = D(\mathbf{R}_i - \mathbf{R}_j)$.

Note, that even within the simplest approximation the effective Hamiltonian (3.24) is not simple and can not be solved exactly. But it is the Hamiltonian of regular system and therefore the method developed for regular systems can be used.

3.5. Non-interacting bosons in random potential

It is clear that supersymmetry representation can be done similarly for bosons in random potential. Consider the boson system described by the Hamiltonian

$$H_b = \sum_i \epsilon_i b^+_i b_i + \sum_{ij} V_{ij} b^+_i b_j. \qquad (3.26)$$

In this case the result of supersymmetry representation for correlation functions is the following

$$\langle A \rangle = Sp A e^{-H/T + \mu N/T \pm i\pi N_f}. \qquad (3.27)$$

All notations here are the same as in previous sections. Note, that this representation is similar to (3.14) and only N_b is replaced by N_f.

The supersymmetry representation is convenient for investigation of Bose condensation in random potential and this problem will be the subject of special paper.

Conclusions

Thus, the main idea of present paper is to rewrite $1/Z$, where Z is partition function of system with Hamiltonian H, as a partition function Z' of some new system with the Hamiltonian H'. It is interesting to note that if initial H is the Hamiltonian of Fermi system than H' is the Hamiltonian of Bose system and if H is Bose Hamiltonian than H' is Fermi Hamiltonian. H' describes non-realistic fictitious system. The introducing of this fictitious system is to some extent similar to introducing of Faddeev-Popov ghosts in quantum field theory. Note, that the full Hamiltonian $H + H'$ is supersymmetrical one and it describes non-interacting Fermi and Bose subsystems. Therefore the proposed representation for correlation functions is supersymmetry representation. Using this representation the configurational averaging can be performed before thermodynamic averaging. In result we obtain the regular system with interaction between Fermi and Bose subsystems. Therefore, the methods developed for regular systems can be used.

In this paper we deal only with a non-interacting systems. The interesting question is whether the supersymmetry representation for correlation function can be done in the case of interacting systems. In order to answer this question we must represent $1/Z$ for interacting system as a partition function Z' of some new interacting system. Even when we rewrite $1/Z$ as Z' a question is whether the interaction does not break the supersymmetry. All these problems need a detail investigation.

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СУПЕРСИМЕТРІЙНЕ ПРЕДСТАВЛЕННЯ КОРЕЛЯЦІЙНИХ ФУНКЦІЙ НЕВПОРЯДКОВАНИХ СИСТЕМ

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Отримано суперсиметрійне представлення для кореляційних функцій сферичної моделі та невзаємодіючого квантового газу. Використовуючи це представлення конфігураційне усереднення може бути здійснене перед термодинамічним і проблема розрахунку конфігураційно усереднених кореляційних функцій невпорядкованої системи спрощується до розрахунку кореляційних функцій для фермі-бозе взаємодіючої регулярної системи.