

REFERENCE SYSTEM APPROACH IN THE ELECTRON LIQUID THEORY

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The fundamentals of a reference system approach, which is a renormalized perturbation theory in the terms of n -particle correlation functions for the simplest system (reference system), are expounded in the theory of electron liquid.

Functional representation of the partition function, thermodynamic and correlation functions, and one-particle characteristics for the electron liquid model are represented. The results of energy, structure and dielectric characteristic calculation in paramagnetic and superconductive states are given.

Reference system approach has been used for the description of inhomogeneous electron system. Also it has been generalized on models, which are closely spaced to electron liquid model: two-sort fermion system model, fermion system model with a short-range repulsion potential of interaction between the particles, fermion system model with a manyparticle interaction potential, and so on.

1. Introduction

The problem of taking account of the correlation effects in the solids, plasma, atom-molecule systems and in the collisions of the electrons with the complex structure objects is one of the many vital problems in statistical physics. As it is known, the correlation theory of a weakly non-ideal systems has been created in general in the 60-70s. So far, the strongly non-ideal many-particle interacting system theory has not been built.

One promising line of many-electron system research is the reference system approach, having been worked out in [1-5]. This method is the renormalized statistical perturbation theory, which is formulated in terms of n -particle correlation functions of some simplest model system. The transition from the simplest reference systems to the complex ones permit the most complex many-particle interacting systems to be described. Such functional hierarchy of models can be realized in the theory of many-particle interacting systems: the non-interacting electron model - the homogeneous interacting electron model - the inhomogeneous interacting electron model.

Every previous model can be used as a reference system in this scheme. Generally speaking, the most difficult scheme can be used (see chapter 9).

This paper is dedicated to a reference system approach in the electron liquid theory, which is one of the fundamental models in the statistical physics and the degenerate Fermi system models which is closely spaced to it.

2. Non-interacting electron model as a reference system in the collective variable method

Naturally, the non-interacting electron model can be used as a reference system under the electron liquid characteristic computation by collective variable method. We consider a system of N electrons in the volume V in the thermodynamical limit $V, N \rightarrow \infty$, $N/V = \text{const}$. The Hamiltonian, being written by the second quantization representation, has the form

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{V}, \\ \hat{H}_0 &= \sum_{\mathbf{k},s} \epsilon_{\mathbf{k}} a_{\mathbf{k},s}^+ a_{\mathbf{k},s}, \\ \hat{V} &= (2V)^{-1} \sum_{\mathbf{q} \neq 0} V_{\mathbf{q}} \sum_{\mathbf{k}_1, \mathbf{k}_2} \sum_{s_1, s_2} a_{\mathbf{k}_1 - \mathbf{q}, s_1}^+ a_{\mathbf{k}_2 + \mathbf{q}, s_2}^+ a_{\mathbf{k}_2, s_2} a_{\mathbf{k}_1, s_1}.\end{aligned}\quad (2.1)$$

Here $\epsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m$, $V_{\mathbf{q}} = 4\pi e^2 / q^2$, $a_{\mathbf{k},s}^+$, $a_{\mathbf{k},s}$ are the Fermi operators, which represent the state with wave vector \mathbf{k} and spin $s = \pm 1/2$.

Let us consider the partition function for our model (2.1) in grand canonical ensemble

$$Z_{\mu} = \text{Tr}\{\exp[-\beta(\hat{H} - \mu\hat{N})]\}. \quad (2.2)$$

Here β is the reciprocal temperature, μ is the chemical potential variable, \hat{N} is the operator of particle number.

As a matter of convenience, we use the interaction representation for the depicting of the statistical operator

$$\begin{aligned}\exp[-\beta(\hat{H} - \mu\hat{N})] &= \exp(-\beta\hat{H}_{\mu}) \cdot \hat{S}, \\ \hat{S} &= \tilde{T} \exp \left\{ \int_0^{\beta} d\beta' \int_0^{\beta} \dots \int d\beta_1 \dots d\beta_4 \prod_{i=1}^4 \delta(\beta' - \beta_i) \times \right. \\ &\left. (2V)^{-1} \sum_{\mathbf{q} \neq 0} V_{\mathbf{q}} \sum_{\mathbf{k}_1, \mathbf{k}_2, s_1, s_2} a_{\mathbf{k}_1 - \mathbf{q}, s_1}^+(\beta_1) a_{\mathbf{k}_1, s_1}(\beta_2) a_{\mathbf{k}_2 + \mathbf{q}, s_2}^+(\beta_3) a_{\mathbf{k}_2, s_2}(\beta_4) \right\}.\end{aligned}\quad (2.3)$$

Here $\hat{H}_{\mu} \equiv \hat{H}_0 - \mu\hat{N}$, \tilde{T} is the ordering symbol of operators $a_{\mathbf{k},s}(\beta')$ and $a_{\mathbf{k},s}^+(\beta')$. Apart from the usual properties (see [2,4]) it possesses the property of reducing the Fermi operator products to a normal form, when the arguments $\beta_1 = \beta_2$ coincide

$$\tilde{T}\{a_{\mathbf{k}_1, s_1}(\beta_1) a_{\mathbf{k}_2, s_2}^+(\beta_2)\} = -a_{\mathbf{k}_2, s_2}^+(\beta_2) a_{\mathbf{k}_1, s_1}(\beta_1). \quad (2.4)$$

Let us introduce the system of functions of variable β' on segment $0 \leq \beta' \leq \beta$

$$\Psi_{\nu}(\beta') = \beta^{-1/2} \exp(-i\nu\beta') \quad (2.5)$$

which satisfies the periodic condition $\Psi_{\nu}(\beta') = \pm \Psi_{\nu}(\beta' + \beta)$.

These conditions let quantize the frequency

$$1). \quad \nu_n = 2\pi n\beta^{-1}. \quad 2). \quad \nu_n = (2n + 1)\pi\beta^{-1}. \quad n = 0; \pm 1; \pm 2; \dots \quad (2.6)$$

Every frequency set originates system of functions, which satisfies the condition of orthogonality and completeness

$$\int_0^\beta d\beta' \Psi_{\nu_1}^*(\beta') \Psi_{\nu_2}(\beta') = \delta_{\nu_1, \nu_2}, \quad (2.7)$$

$$\sum_\nu \Psi_\nu^*(\beta_1) \Psi_\nu(\beta_2) = \delta(\beta_1 - \beta_2).$$

Using the second condition for representation of δ -function in formula for $a_{\mathbf{k},s}$ and integrating over β' variable, one can obtain following form of S -operator

$$\hat{S} = \tilde{T} \exp \left\{ -(2\beta V)^{-1} \cdot \sum_{\mathbf{q} \neq 0; \nu} V_{\mathbf{q}} \hat{\rho}_x \hat{\rho}_{-x} \right\}, \quad (2.8)$$

$$\hat{\rho}_x \equiv \hat{\rho}_{\mathbf{q},\nu} = \sum_{\mathbf{k},s;\nu^*} a_{\mathbf{k}+\mathbf{q},s}^+(\nu^* + \nu) a_{\mathbf{k},s}(\nu^*). \quad (2.9)$$

Here $\hat{\rho}_x$ is the spectral representation of electron density operator and

$$a_{\mathbf{k},s}(\nu^*) = \int_0^\beta a_{\mathbf{k},s}(\beta_i) \Psi_{\nu_i}^*(\beta_i) d\beta_i. \quad (2.10)$$

Here ν is the difference of two initial frequencies and thus it is pair ($\nu_n = 2n\pi\beta^{-1}$).

Let us consider one-particle Green's function of a reference system, which is built on the $a_{\mathbf{k},s}^+(\nu^*)$ and $a_{\mathbf{k},s}(\nu^*)$ operators

$$- \left\langle \tilde{T} \{ a_{\mathbf{k}_1, s_1}(\beta_1) a_{\mathbf{k}_2, s_2}^+(\beta_2) \} \right\rangle_{\mathcal{H}_\mu} = G_{\mathbf{k}_1, s_1}^0(\beta_1 - \beta_2) \delta_{\mathbf{k}_1, \mathbf{k}_2} \delta_{s_1, s_2}. \quad (2.11)$$

Taking into account the bilinear expansion

$$G_{\mathbf{k},s}^0(\beta_1 - \beta_2) = \sum_\nu G_{\mathbf{k},s}^0(\nu|\mu) \Psi_\nu(\beta_1) \Psi_\nu^*(\beta_2) \quad (2.12)$$

and transformation (2.9), one can prove that

$$- \left\langle \tilde{T} \{ a_{\mathbf{k}_1, s_1}(\nu_1^*) a_{\mathbf{k}_2, s_2}^+(\nu_2^*) \} \right\rangle_{\mathcal{H}_\mu} = G_{\mathbf{k}_1, s_1}^0(\nu_1^*|\mu) \delta_{\mathbf{k}_1, \mathbf{k}_2} \delta_{s_1, s_2} \delta_{\nu_1^*, \nu_2^*}, \quad (2.13)$$

where

$$G_{\mathbf{k},s}^0(\nu^*|\mu) = e^{i\nu^*\delta} \{ i\nu^* - \epsilon_{\mathbf{k}} + \mu \}^{-1} \quad (2.14)$$

and $\delta \rightarrow +0$ (see [6]). It is obvious that, the transformation (2.9) has been built on the function set $\Psi_\nu^*(\beta_i)$ with odd frequencies.

The representation (2.7) transforms the partition function to the form

$$Z(\mu) = Z_0(\mu) \left\langle \tilde{T} \exp \left\{ -(2\beta V)^{-1} \sum_{\mathbf{q}; \nu} V_{\mathbf{q}} \hat{\rho}_x \hat{\rho}_{-x} \right\} \right\rangle_{\mathcal{H}_\mu}, \quad (2.15)$$

where

$$Z_0(\mu) = \text{Tr}\{\exp[-\beta\hat{\mathcal{H}}_\mu]\} = \exp(-\beta\Omega_0(\mu)) \quad (2.16)$$

is the partition function of noninteracting electron system, $\Omega_0(\mu)$ is its thermodynamical potential.

The formula (2.15) is convenient for writing of $Z(\mu)$ functional representation. For that we go over from the space of the second quantization operators to that of the dynamic collective variables

$$\rho_x \equiv \rho_{\mathbf{q},\nu} = \rho_{\mathbf{q},\nu}^c + i\rho_{\mathbf{q},\nu}^s \quad (2.17)$$

by the help of transition operator [1,3,4]

$$\hat{J}(\rho - \hat{\rho}) = \prod_{C(\mathbf{q},\nu)} \delta(\rho_{\mathbf{q},\nu} - \hat{\rho}_{\mathbf{q},\nu}). \quad (2.18)$$

The domain of variables $C(\mathbf{q},\nu)$ includes half of the possible values of \mathbf{q}, ν . The operator $\hat{J}(\rho - \hat{\rho})$ is the analogy of the transition function for the classical statistic systems (see [7]). Any operator $\hat{F}(\hat{\rho})$, being the integer function of operator $\hat{\rho}_x$, has such a form in the collective variable representation

$$\hat{F}(\hat{\rho}) = \int F(\rho) \hat{J}(\rho - \hat{\rho})(d\rho), \quad (d\rho) = \prod_{C(\mathbf{q},\nu)} d\rho_x^c d\rho_x^s. \quad (2.19)$$

After all following representation for the partition function can be obtained

$$Z(\mu) = Z_0(\mu) \int (d\rho) J_0(\rho) \exp\left\{-(2\beta V)^{-1} \sum_x V_{\mathbf{q}} \rho_x \rho_{-x}\right\}, \quad (2.20)$$

where

$$J_0(\rho) = \langle \tilde{T} \hat{J}(\rho - \hat{\rho}) \rangle_{\mathcal{H}_\mu} \quad (2.21)$$

is the Jacobian of the transition to the collective variables. Let us use the integral representation of δ -function, for calculating of $J_0(\rho)$. We obtain

$$J_0(\rho) = \int (d\omega) J(\omega|\mathbf{k}) \exp\left\{i\pi \sum_x \omega_x \rho_x\right\}, \quad (2.22)$$

here $\omega_x \equiv \omega_x^c - i\omega_x^s$ is a variable conjugated to the collective variable ρ_x ,

$$J(\omega|\mu) = \langle \tilde{T} e^{-i\pi \sum_x \omega_x \hat{\rho}_x} \rangle_{\mathcal{H}_\mu} = \exp \sum_{n \geq 1} D_n(\omega|\mu), \quad (2.23)$$

$$D_n(\omega|\mu) = \beta(n!)^{-1} (-\pi i)^n \sum_{x_1, \dots, x_n} \omega_{x_1} \dots \omega_{x_n} \tilde{\mu}_n^0(x_1, \dots, x_n|\mu).$$

where

$$\tilde{\mu}_n^0(x_1, \dots, x_n|\mu) = \beta^{-1} \langle \tilde{T} \{\hat{\rho}_{x_1} \hat{\rho}_{x_2} \dots \hat{\rho}_{x_n}\} \rangle_{\mathcal{H}_\mu}^c, \quad (2.24)$$

are the semi-invariant correlation functions of a reference system. The symbol $\langle \dots \rangle_{\mathcal{H}_\mu}^c$ means statistical averaging over the states of the reference system with Hamiltonian \mathcal{H}_μ .

As one can see(2.15, 2.21-2.23), the non-interacting electron system is the reference system. The Jacobian of the transition $J(\rho)$ and the n -particle correlation functions $\tilde{\mu}_n^0(x_1, \dots, x_n|\mu)$ are the universal characteristics of a reference system.

Integrating over ω_x variable in the formulae (2.21), one obtains

$$J_0(\rho) = \exp \left\{ b_0^0(\mu) + \sum_{n \geq 2} \frac{1}{n!} \sum_{x_1, \dots, x_n} \tilde{b}_n^0(x_1, \dots, x_n|\mu) \rho_{x_1} \dots \rho_{x_n} \right\}, \quad (2.25)$$

where

$$\begin{aligned} b_0^0(\mu) &= \ln \left\{ \int (d\omega) J(\omega|\mu) \right\}, \\ \tilde{b}_2^0(x, -x|\mu) &\equiv b_2^0(x, -x|\mu), \\ \tilde{b}_3^0(x_1, x_2, x_3|\mu) &\equiv b_3^0(x_1, x_2, x_3|\mu), \\ \tilde{b}_4^0(x_1, x_2, x_3, x_4|\mu) &\equiv b_4^0(x_1, x_2, x_3, x_4|\mu) - \\ &\quad 3b_2^0(x_1, -x_1|\mu)b_2^0(x_2, -x_2|\mu)\delta_{x_1, x_3}\delta_{x_2, x_4}, \\ &\dots \\ \tilde{b}_n^0(x_1, \dots, x_n|\mu) &\equiv \\ &\quad \left\{ \int (d\omega) J(\omega|\mu) \right\}^{-1} \int (d\omega) J(\omega|\mu) (i\pi)^n \omega_{x_1} \dots \omega_{x_n} = \\ &\quad (-1)^n \beta^{-1} \frac{\delta}{\delta \tilde{\mu}_n^0(x_1, \dots, x_n|\mu)} \exp\{b_0^0(\mu)\}. \end{aligned} \quad (2.26)$$

The functional representation (2.20-2.24) of the partition function is valid for any Fermi systems with any interaction potentials. Changing the order of integrating over ω_x and ρ_x variables, the following form of $Z(\mu)$ in the case of interacting electron liquid $V_q > 0$ is obtained

$$\begin{aligned} Z(\mu) &= Z_0(\mu) \left\{ \int (d\omega) f(\omega) \right\}^{-1} \int (d\omega) f(\omega) J(\omega|\mu), \\ f(\omega) &= \exp \left\{ -\frac{\pi^2 \beta V}{2} \sum_x V_q^{-1} \omega_x \omega_{-x} \right\}. \end{aligned} \quad (2.27)$$

The whole structure of representation (2.20), (2.26) corresponds to the structure of configuration integral in the collective variable method for classical systems of interaction particles under the basic consideration of short-range interaction [7]. The correlation functions $\tilde{\mu}_n^0(x_1, \dots, x_n|\mu)$ are the result of quantum effects in reference system and are not subjected to interaction and particle structure. This functions depend not only on impulse, but also on frequency ν . It is a characteristic peculiarity of quantum systems that results from non-commutativity of kinetic energy and interaction operators.

The straight calculation of $\tilde{\mu}_n^0(x_1, \dots, x_n|\mu)$ is the central point of reference system approach. By the way, the investigation has its own value expanding physical characteristics of ideal systems.

According to (2.23) and (2.12) - (2.13) we will represent $\tilde{\mu}_n^0(x_1, \dots, x_n|\mu)$ through Green functions in symmetrical form

$$\tilde{\mu}_n^0(x_1, \dots, x_n|\mu) = (-1)^{n+1}(\beta n)^{-1} \delta_{\nu_1+\dots+\nu_n,0} \delta_{\mathbf{q}_1+\dots+\mathbf{q}_n,0} \times \quad (2.28)$$

$$\times \sum_{\mathbf{k},s,\nu^*} \sum_{i_1 \neq i_2 \neq \dots \neq i_n=1}^n \prod_{j=1}^n G_{\mathbf{k}-\kappa_j,s}^0(\nu^* - \omega_j|\mu),$$

that represents their symmetry concerning their arguments. Here $\kappa_j = \mathbf{q}_{i_1} + \mathbf{q}_{i_2} + \dots + \mathbf{q}_{i_j}$, $\omega_j = \nu_{i_1} + \nu_{i_2} + \dots + \nu_{i_j}$. Functions $\tilde{\mu}_n^0(x_1, \dots, x_n|\mu)$ are subject to the thermodynamical parameters β, μ, V . When μ is equal to chemical potential of reference system $\mu_0 \equiv \mu_0(\beta, \frac{N}{V})$, we obtain correlation functions in terms of canonical ensemble.

In the (2.27) we always have that $\kappa_n = 0, \omega_n = 0$. In the higher order semi-invariant ($n \geq 4$) degeneration may occur, when $\kappa_j = 0, \omega_j = 0$ takes place at $j \leq n-2$. The simplest semi-invariant $\tilde{\mu}_2^0(x, -x|\mu_0)$ is the polarization operator in the random phase approximation. It is introduced in (\mathbf{q}, ν) representation in the paper [8]. In the works [9,10,11] parametric representation of static semi-invariant $\tilde{\mu}_n^0(x_1, \dots, x_n|\mu_0)$ for the case of $T = 0K$ has been introduced. $\tilde{\mu}_3^0(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, 0, 0, 0|\mu_0)$ and $\tilde{\mu}_4^0(\mathbf{q}_1, -\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_2, 0, 0, 0, 0|\mu_0)$ are represented in elementary functions. Simultaneously in the paper [11] static semi-invariant of the third order is obtained. Dynamic semi-invariants $\tilde{\mu}_n^0(x_1, \dots, x_n|\mu_0)$ at $n \geq 3$ have been investigated in the works [1,2,4,12] in the case of $T = 0K$. The functions $\tilde{\mu}_3^0(x_1, x_2, x_3|\mu_0)$ and $\tilde{\mu}_4^0(x_1, -x_1, x_2, -x_2|\mu_0)$ are represented in elementary functions. In the case of non zero temperature it is impossible to obtain functions $\tilde{\mu}_n^0(x_1, \dots, x_3|\mu_0)$ at $n \geq 3$ in terms of elementary functions. But the lower order of semi-invariant may be represented in terms of Fermi integral of elementary functions. For example, [13]

$$\tilde{\mu}_2^0(x, -x|\mu_0) = \quad (2.29)$$

$$-2\Re e \sum_{\mathbf{k},s} n_{\mathbf{k},s}^0(\mu_0) (i\nu + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}})^{-1} = \frac{3N}{2\epsilon_F} I_{2,0}(q, u|\mu_0, \beta),$$

$$I_{2,0}(q, u|\mu_0, \beta) = \int_0^\infty dk k n_{\mathbf{k}}^0(\mu_0, \beta) \Gamma_2(k|q, u),$$

$$\tilde{\mu}_3^0(x_1, x_2, x_3|\mu_0) = \delta_{\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3,0} \delta_{\nu_1+\nu_2+\nu_3,0} \times$$

$$\left\{ \gamma_3(x_1, -x_2) + \gamma_3(x_2, -x_3) + \gamma_3(x_3, -x_1) \right\},$$

$$\gamma_3(x_1, x_2) =$$

$$-2\Re e \sum_{\mathbf{k},s} n_{\mathbf{k},s}^0(\mu_0) (i\nu_1 + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}_1})^{-1} (i\nu_2 + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}_2})^{-1} =$$

$$\frac{3N}{(2\epsilon_F)^2} \int_0^\infty dk k n_{\mathbf{k}}^0(\mu_0, \beta) \Gamma_3(k|q_1, q_2, u_1, u_2, t), \dots$$

The dimensionless variables are used here: $k \equiv |\mathbf{k}|k_F^{-1}$, $q_i \equiv |\mathbf{q}_i|k_F^{-1}$, $u_i \equiv \nu_i(2\epsilon_F q_i)^{-1}$, $\epsilon_F \equiv \hbar^2 k_F^2/2m$; t is the cosine of the angle between the vectors

\mathbf{q}_1 and \mathbf{q}_2 ,

$$n_{\mathbf{k},s}^0(\mu_0) \equiv n_{\mathbf{k},s}^0(\mu_0, \beta) = \left\{ 1 + \exp[\beta^*(k^2 - \mu_0^*)] \right\}^{-1}, \quad (2.30)$$

assuming $\beta^* \equiv \beta\epsilon_F$, $\mu_0^* \equiv \mu_0\epsilon_F^{-1}$. Functions $\Gamma_n(\dots)$ for $n = 2, 3$ are represented by the following expressions

$$\begin{aligned} \Gamma_2(k|q; u) &= (2q)^{-1} \ln \frac{u^2 + (k + q/2)^2}{u^2 + (k - q/2)^2}, \\ \Gamma_3(k|q_1, q_2; u_1, u_2; t) &= (q_1 q_2)^{-1} \Re e \left\{ \Delta(k)^{-1} \ln \frac{\delta(k) + k\Delta(k)}{\delta(k) - k\Delta(k)} \right\}, \\ \delta(k) &= k^2 t - \xi_1 \xi_2, \quad \Delta(k) = \left\{ k^2(t^2 - 1) + \xi_1^2 + \xi_2^2 - 2t\xi_1\xi_2 \right\}^{1/2}, \\ \xi_j &= \frac{1}{2} q_j - i u_j; \quad j = 1, 2. \end{aligned} \quad (2.31)$$

The temperature dependence of the semi-invariants are illustrated by Fig. 1a, 1b where $\tilde{\mu}_2^0(x, -x|\mu_0)$ and $\tilde{\mu}_3^0(x, -x, 0|\mu_0)$ are represented in the case of very low frequency ($\tilde{\nu} \equiv \nu(2\epsilon_F)^{-1} = 0.01$ using different $\beta^* \in [0, 100]$). The chemical potential of the reference system μ_0 at given temperature β is obtained from

$$\int_0^\infty dk k^2 n_k^0(\mu_0, \beta) = \frac{1}{3}. \quad (2.32)$$

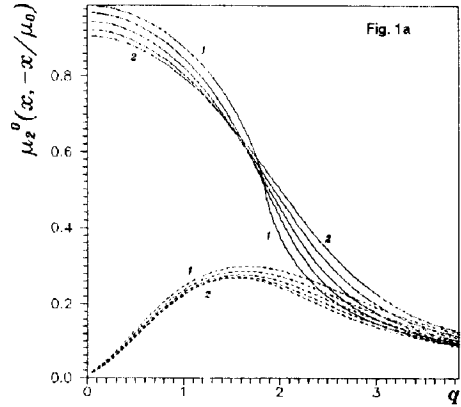
As we can see from Fig. 1a function $\tilde{\mu}_2^0(x, -x|\mu_0)$ is monotonously dependent on variable β^* . Function $\tilde{\mu}_3^0(x, -x, 0|\mu_0)$ reveals the same dependence only in the region of big wave numbers and non-monotonous dependence in the region of small wave numbers (see Fig.1b). We can easily obtain the limit of the static semi-invariants using Sommerfeld method [14] assuming $\mathbf{q}_i \rightarrow 0$:

$$\begin{aligned} \tilde{\mu}_n^0(0, \dots, 0|\mu_0) &= N(2\epsilon_F)^{1-n} \times \\ &\left\{ R_n(\mu_0^*)^{\frac{5}{2}-n} + C_n \frac{\pi^2}{8} (\mu_0^*)^{\frac{1}{2}-n} (\beta^*)^{-2} + D_n \frac{7\pi^4}{1920} (\mu_0^*)^{-\frac{3}{2}-n} (\beta^*)^{-4} + \dots \right\}, \\ R_1 &= 1, \quad R_2 = R_3 = 3, \quad R_n = (-1)^{n-1} 3(2n-7)!! \quad \text{at } n \geq 4, \\ C_1 &= 1, \quad C_n = (-1)^{n-1} (2n-3)!! \quad \text{at } n \geq 2, \\ D_n &= (-1)^{n-1} (2n+1)!! \quad \text{at } n \geq 1. \end{aligned} \quad (2.33)$$

In the case of degeneration in functions $\mu_n^0(x_1, \dots, x_n|\mu)$ along with $\gamma_n(x_1, \dots, x_{n-1})$ there are derivatives of μ from $\gamma_m(x_1, \dots, x_{m-1})$ of lower order, that forms "anomalous" parts of these functions. So, in the common case

$$\tilde{\mu}_n^0(x_1, \dots, x_n|\mu) = \mu_{n,1}^0(x_1, \dots, x_n|\mu) + \mu_{n,2}^0(x_1, \dots, x_n|\mu), \quad (2.34)$$

where $\mu_{n,1}^0(\dots)$ is a "normal" and $\mu_{n,2}^0(\dots)$ is "anomalous" parts.



3. The functional representation of other characteristics

The functional representation of the structure (2.16) are valid also for integral, local and one-particle characteristics of the model. For example let us do the representation for Fourier images of n-particle correlation functions

$$\mu_n(\mathbf{q}_1, \dots, \mathbf{q}_n) = Z^{-1}(\mu^*) \text{Tr} \left\{ \hat{I}_n(\mathbf{q}_1, \dots, \mathbf{q}_n) \exp \left[-\beta(\hat{H} - \mu^* \hat{N}) \right] \right\}, \quad (3.1)$$

$$\hat{I}_n(\mathbf{q}_1, \dots, \mathbf{q}_n) = \sum_{\substack{\mathbf{k}_1, \dots, \mathbf{k}_n \\ s_1, \dots, s_n}} \hat{a}_{\mathbf{k}_1 + \mathbf{q}_1, s_1}^+ \hat{a}_{\mathbf{k}_2 + \mathbf{q}_2, s_2}^+ \dots \hat{a}_{\mathbf{k}_n + \mathbf{q}_n, s_n}^+ \hat{a}_{\mathbf{k}_n, s_n} \dots \hat{a}_{\mathbf{k}_1, s_1},$$

where $n \geq 1$, and $\mu^* \equiv \mu^*(\frac{N}{V}, \beta)$ is a chemical potential, that is certain for temperature and particle density. The transformation in collective variables leads to such representation [2]

$$\mu_n(\mathbf{q}_1, \dots, \mathbf{q}_n) = \beta^{1-n} \sum_{\nu_1, \dots, \nu_n} \delta_{\nu_1 + \dots + \nu_n, 0} \mu_n(x_1, \dots, x_n | \mu^*), \quad (3.2)$$

$$\mu_n(x_1, \dots, x_n | \mu^*) = \beta^{-1} \langle \hat{T} \hat{S} \rangle_{\mathcal{H}_\mu}^{-1} \left\langle \hat{T} \left\{ \hat{\rho}_{x_1} \hat{\rho}_{x_2} \dots \hat{\rho}_{x_n} \hat{S} \right\} \right\rangle_{\mathcal{H}_\mu} =$$

$$(-1)^n (\pi i)^{-n} \beta^{-1} \left\{ \int (d\omega) f(\omega) J(\omega | \mu^*) \right\}^{-1} \times$$

$$\int (d\omega) f(\omega) \frac{\delta^n}{\delta \omega_{x_1} \dots \delta \omega_{x_n}} J(\omega | \mu^*).$$

Calculating a variation derivative of $J(\omega | \mu^*)$ in the straight form we shall find the equations for correlation functions of interaction system

$$\mu_1(x_1) = \quad (3.3)$$

$$- \sum_{n \geq 2} (-1)^n \left[(n-1)! \right]^{-1} \sum_{x_2, \dots, x_n} \tilde{\mu}_n^0(x_1, \dots, x_n | \mu^*) b_{n-1}(x_2, \dots, x_n),$$

$$\mu_2(x_1, x_2) = \tilde{\mu}_2^0(x_1, -x_1 | \mu^*) \delta_{x_1 + x_2, 0} +$$

$$\sum_{n \geq 3} (-1)^n \left[(n-2)! \right]^{-1} \sum_{x_3, \dots, x_n} \tilde{\mu}_n^0(x_1, x_2, \dots, x_n | \mu^*) b_{n-2}(x_3, \dots, x_n) +$$

$$\beta \sum_{n, m \geq 2} (-1)^{n+m} \left[(n-1)! (m-1)! \right]^{-1} \sum_{z_2, \dots, z_n} \tilde{\mu}_n^0(x_1, z_2, \dots, z_n | \mu^*) \times$$

$$\sum_{y_2, \dots, y_m} \tilde{\mu}_m^0(x_2, y_2, \dots, y_m | \mu^*) b_{n+m-2}(z_2, \dots, z_n, y_2, \dots, y_m), \dots$$

Here we use the analogues of functions $b_n^0(x_1, \dots, x_n)$ that are represented by the equation

$$b_n(x_1, \dots, x_n) = (\pi i)^n \left\{ \int (d\omega) f(\omega) J(\omega | \mu^*) \right\}^{-1} \times \quad (3.4)$$

$$\int (d\omega) f(\omega) J(\omega | \mu^*) \omega_{x_1} \dots \omega_{x_n}.$$

and are the many-particle effective interaction potentials. They may be obtained in the form of series for reference system functions $\tilde{\mu}_n^0(x_1, \dots, x_n | \mu^*)$. The other way is to represent $b_n(x_1, \dots, x_n)$ through functions $\mu_n(x_1, \dots, x_n | \mu^*)$ and to obtain the system of the integral equations. In the case of non-degeneration of the set x_1, \dots, x_n we find that

$$b_n(x_1, \dots, x_n) = \beta^{1-n} V^{-n} V_{q_1} \dots V_{q_n} \mu_n(x_1 \dots x_n | \mu^*). \tag{3.5}$$

In the case of degeneration the following equations are valid

$$b_2(x, -x) = -(\beta V)^{-1} V_q \left\{ 1 - \frac{V_q}{V} \mu_2(x_1, -x_1) \right\}, \tag{3.6}$$

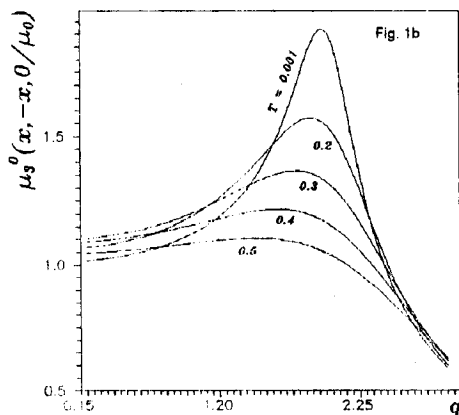
$$b_4(x_1, -x_1, x_2, -x_2) = (\beta V)^{-2} V_{q_1} V_{q_2} \left\{ 1 - \frac{1}{V} \sum_{i=1,2} V_{q_i} \mu_2(x_i, -x_i) + V^{-2} V_{q_1} V_{q_2} \mu_4(x_1, -x_1, x_2, -x_2) \right\}, \dots$$

Along with functions (3.2) it is useful to consider semi-invariant correlation functions

$$\tilde{\mu}_n(x_1, \dots, x_n) = \beta^{-1} \langle \tilde{T} \hat{S} \rangle_{H_{\mu^*}} \langle \tilde{T} \{ \hat{\rho}_{x_1} \dots \hat{\rho}_{x_n} \hat{S} \} \rangle_{H_{\mu^*}}^c, \tag{3.7}$$

where we mean connection of the operators $\hat{\rho}_x$. (3.7) is the analogue of basic functions $\tilde{\mu}_n^0(x_1, \dots, x_n | \mu)$.

One-particle characteristics in the reference system approach need a special consideration. To such characteristics belong the distribution of the particles through impulses, one-particle Green's function, the spectrum of single particle excitations, the state density, etc.



To calculate the mean

$$n_{\mathbf{k},s} = \langle a_{\mathbf{k},s}^+ a_{\mathbf{k},s} \rangle_{H - \mu^* N} = Z^{-1}(\mu^*) \text{Tr} \left\{ a_{\mathbf{k},s}^+ a_{\mathbf{k},s} \exp \left[-\beta (\hat{H} - \mu^* \hat{N}) \right] \right\} \tag{3.8}$$

we should substitute the operator of the number of particles $a_{\mathbf{k},s}^+, a_{\mathbf{k},s}$ by the variation derivative in function $\varepsilon_{\mathbf{k}}$. So

$$n_{\mathbf{k},s} = -(2\beta)^{-1} \frac{\delta}{\delta \varepsilon_{\mathbf{k}}} \ln Z(\mu^*) |_{\mu^* = \text{const.}} \tag{3.9}$$

According to (2.26) we obtain

$$n_{\mathbf{k},s} = \tag{3.10}$$

$$n_{\mathbf{k},s}^0(\mu^*) - (2\beta)^{-1} \left\{ \int (d\omega) f(\omega) J(\omega | \mu^*) \right\}^{-1} \int (d\omega) f(\omega) \frac{\delta}{\delta \varepsilon_{\mathbf{k}}} J(\omega | \mu^*) =$$

$$n_{\mathbf{k},s}^0(\mu^*) - \frac{1}{2} \sum_{n \geq 1} (-1)^n (n!)^{-1} \sum_{x_1, \dots, x_n} b_n(x_1, \dots, x_n) \frac{\delta}{\delta \varepsilon_{\mathbf{k}}} \tilde{\mu}_n^0(x_1, \dots, x_n | \mu^*).$$

Let us consider Green's function as another example

$$G_{\mathbf{k},s}(\beta_1 - \beta_2) = -Z^{-1}(\mu^*) \text{Tr} \left\{ \tilde{T} \left[a_{\mathbf{k},s}(\beta_1) a_{\mathbf{k},s}^\dagger(\beta_2) e^{-\beta(\hat{H} - \mu^* \hat{N})} \right] \right\}. \quad (3.11)$$

Representing operators $a_{\mathbf{k},s}(\beta')$ through $a_{\mathbf{k},s}(\nu^*)$ and changing S -matrix in terms of collective variables we obtain the representation

$$G_{\mathbf{k},s}(\beta_1 - \beta_2) = \sum_{\nu^*} G_{\mathbf{k},s}(\nu^* | \mu^*) \Psi_{\nu^*}(\beta_1) \Psi_{\nu^*}^\dagger(\beta_2), \quad (3.12)$$

where

$$G_{\mathbf{k},s}(\nu^* | \mu^*) = \left\{ \int (d\omega) f(\omega) J(\omega | \mu^*) \right\}^{-1} \times \int (d\omega) f(\omega) \left\langle \tilde{T} \left\{ \hat{n}_{\mathbf{k},s}(\nu^*) \exp \left[-i\pi \sum_x \omega_x \hat{\rho}_x \right] \right\} \right\rangle_{H_\mu^*}. \quad (3.13)$$

There $\hat{n}_{\mathbf{k},s}(\nu^*) = a_{\mathbf{k},s}^\dagger(\nu^*) a_{\mathbf{k},s}(\nu^*)$. Separating the connected parts of the means and considering the (2.23) we shall get

$$\begin{aligned} \langle \hat{n}_{\mathbf{k},s}(\nu^*) \exp \left[-i\pi \sum_x \omega_x \hat{\rho}_x \right] \rangle_{H_\mu^*} &= \\ \langle \hat{n}_{\mathbf{k},s}(\nu^*) \exp \left[-i\pi \sum_x \omega_x \hat{\rho}_x \right] \rangle_{H_\mu^*}^c J(\omega | \mu^*), & \\ \langle \hat{n}_{\mathbf{k},s}(\nu^*) \exp \left[-i\pi \sum_x \omega_x \hat{\rho}_x \right] \rangle_{H_\mu^*}^c &= \langle \hat{n}_{\mathbf{k},s}(\nu^*) \rangle_{H_\mu^*} + \\ \sum_{n \geq 1} (-1)^n (n!)^{-1} \sum_{x_1, \dots, x_n} \omega_{x_1} \dots \omega_{x_n} \tilde{\mu}_{n+1}^0(x_1, \dots, x_n, 0 | \mu^* | \mathbf{k}, s, \nu^*). & \end{aligned} \quad (3.14)$$

By the definition

$$\tilde{\mu}_{n+1}^0(x_1, \dots, x_n, 0 | \mu^* | \mathbf{k}, s, \nu^*) = \beta^{-1} \left\langle \tilde{T} \left\{ \hat{n}_{\mathbf{k},s}(\nu^*) \hat{\rho}_{x_1} \dots \hat{\rho}_{x_n} \right\} \right\rangle_{H_\mu^*}^c \quad (3.15)$$

is a (\mathbf{k}, s, ν^*) -component of $(n+1)$ -particle correlation functions of the reference system under the condition $x_{n+1} = 0$:

$$\tilde{\mu}_n^0(x_1, \dots, x_n | \mu^*) = \sum_{\mathbf{k}, s, \nu^*} \tilde{\mu}_n^0(x_1, \dots, x_n | \mu^* | \mathbf{k}, s, \nu^*). \quad (3.16)$$

Substituting (3.14) in (3.13) we will obtain the analogy of Dayson equation

$$\begin{aligned} G_{\mathbf{k},s}(\nu^* | \mu^*) &= G_{\mathbf{k},s}^{(0)}(\nu^* | \mu^*) + \\ \beta \sum_{n \geq 2} (-1)^n (n!)^{-1} \sum_{x_1, \dots, x_n} b_n(x_1, \dots, x_n) \tilde{\mu}_{n+1}^0(x_1, \dots, x_n, 0 | \mu^* | \mathbf{k}, s, \nu^*). & \end{aligned} \quad (3.17)$$

The precise integrating of expressions (2.26), (3.2), (3.4) is impossible because of complex dependence of functions $J(\omega | \mu)$ on their arguments. Different variants of approximate calculations correspond to different levels of summing up of ordinary perturbation theory diagrams based on Coulomb's

potential V_q . For the simplest calculation it is necessary to extract from the product $f(\omega)J(\omega|\mu)$ such a factor $P(\omega)$, which may be integrated precisely. Then the factor $f(\omega)J(\omega|\mu^*)P^{-1}(\omega)$ will be considered by moment method over $P(\omega)$ distribution.

The choice

$$P(\omega) = f(\omega) \exp D_2(\omega|\mu) = \exp \left\{ \frac{-\pi^2 \beta V}{2} \sum_x \omega_x \omega_{-x} V_0^{-1}(x) \right\},$$

$$V_0(x) = V_{RPA}(x) = V_q \varepsilon_0^{-1}(x),$$

$$\varepsilon_0(x) = 1 + V^{-1} V_q \tilde{\mu}_2^0(x, -x|\mu) \quad (3.18)$$

corresponds to the random phase approximation (RPA). The consideration of factor $\exp \sum_{n \geq 3} D_n(\omega|\mu)$ by the moment method results in quantum group series by many-particle functions $\tilde{\mu}_n^0(x_1, \dots, x_n|\mu)$ ($n \geq 3$). More difficult self-consistent and variation methods of calculation of such functional are proposed in works [3,15].

One of comfortable form often used in different characteristic calculations is expression of free energy F with the help of pair correlation function

$$F = F_0 + (2\beta V)^{-1} \sum_{\nu, q \neq 0} V_q \int_0^1 d\lambda \mu_2(x, -x|\lambda), \quad (3.19)$$

where F_0 is free energy of basis system, $\mu_2(x, -x|\lambda)$ a pair correlation function of the model with the potential λV_q .

4. Post-random phase approximation

Using (3.2) or system (3.3) and summing up series by functions $\mu_n^0(x_1, \dots, x_n|\mu_0)$ we obtain

$$\mu_2(x, -x) = M_2(x, -x) \left\{ 1 + V^{-1} V_q M_2(x, -x) \right\}^{-1}, \quad (4.1)$$

where pair polarization operator has form

$$M_2(x, -x) = \tilde{\mu}_2^0(x, -x|\mu_0) + \sum_{n \geq 1} \mathcal{M}_n(x, -x), \quad (4.2)$$

and $\mathcal{M}_n(x, -x)$ are based on n-particle correlation functions of reference system and screened interaction potentials [3,4].

After substitution $\mathcal{M}_n(x, -x) = 0$ for $n \geq 1$ we obtain RPA for even correlation function, that defines all characteristics in this approximation.

The simplest development of RPA is calculation of $\mathcal{M}_1(x, -x)$ and $\mathcal{M}_2(x, -x)$ which are defined by reference system correlation function of third and fourth order (Post RPA) [5].

$$M_2^{RPA}(x, -x) = \tilde{\mu}_2^0(x, -x|\mu_0) + \mathcal{M}_1(x, -x) + \mathcal{M}_2(x, -x), \quad (4.3)$$

$$\mathcal{M}_1(x, -x) = -(2\beta V)^{-1} \sum_{x_1} V_0(x_1) \mu_4^0(x, -x, x_1, -x_1|\mu_0),$$

$$\mathcal{M}_2(x, -x) = (2\beta V^2)^{-1} \sum_{x_1} V_0(x_1) V_0(x + x_1) \tilde{\mu}_3^0(x, x_1, -x - x_1|\mu_0) \times$$

$$\tilde{\mu}_3^0(-x, -x_1, x + x_1|\mu_0).$$

These formulae directly define polarization operator in terms of RPA. Polarization operator in this approximation when $T = 0K$ at the first time was researched in works [5,16,17] in the wide region of non-ideality parameter $r_s = a_0^{-1}(3V/4\pi N)^{1/3}$, where a_0 is Bohr's radius.

By this basic correlation energy $\epsilon_{cor}(r_s)$, chemical potential μ^* , compressibility, pressure, binary distribution function and so on were calculated in the region $0 \leq r_s \leq 10$. The precision of calculation may be controlled by comparison of obtained $\epsilon_{cor}(r_s)$ with the method. The deviation is equal to 0,25% at $r_s = 1$, 1,0% at $r_s = 2$, 3,7% at $r_s = 5$. These results are illustrated also by the Fig. 2, where $\epsilon_{cor}(r_s)$ is pointed in RPA (curve 1), at taking into account $\mathcal{M}_1(x, -x)$ (curve 2) and in approximation (4.3) (curve 3). The dashed curve is the result from work [18]. In the Tab.1 the results of calculation $\epsilon_{cor}(r_s)$ from more authentic works of other authors are represented [18-24].

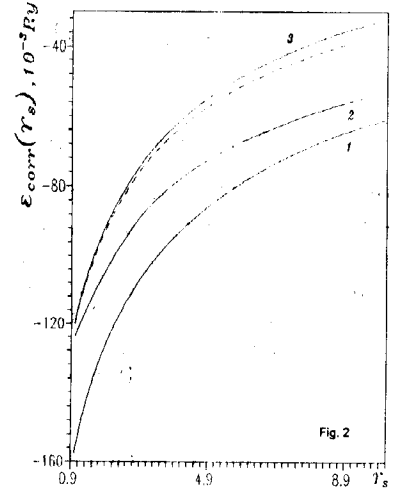


Fig. 2

As one can see from calculation, approximation (4.3) is very well in the region of weakly non-ideality.

5. Local-field correction function

The next approximation is the diagram summation for the polarization function and reducing of the obtained result to the form of a local-field conception

$$M_2(x, -x) = \tilde{\mu}_2^0(x, -x|\mu_0) \left\{ 1 - V^{-1} V_q \tilde{\mu}_2^0(x, -x|\mu_0) G(x) \right\}^{-1}, \quad (5.1)$$

where $G(x)$ is the dynamic local-field correction function.

The expression for the two-particle correlation function has the form

$$\mu_2(x, -x) = \tilde{\mu}_2^0(x, -x|\mu_0) \left\{ 1 + V^{-1} V_q (1 - G(x)) \tilde{\mu}_2^0(x, -x|\mu_0) \right\}^{-1}. \quad (5.2)$$

The local-field correction function is the universal characteristic of the interacting many-particle systems. This function permits the integral and local characteristics of a model to compute. But one-particle characteristic computing demands the variation derivative of $G(x)$. For example, let us write the particle momentum distribution for the electron liquid model. As one can see (f.(3.9)) the expression for the particle momentum distribution has the form

$$n_{\mathbf{k},s} = \frac{1}{2} \frac{\delta}{\delta \epsilon_{\mathbf{k}}} \Omega(\mu^*)|_{\mu^*} = \frac{1}{2} \left\{ \frac{\delta \Omega(\mu^*)}{\delta \epsilon_{\mathbf{k}}} - \frac{\delta \mu^*}{\delta \epsilon_{\mathbf{k}}} \times \frac{d\Omega(\mu^*)}{d\mu^*} \right\} = \quad (5.3)$$

$$\frac{1}{2} \frac{\delta}{\delta \epsilon_{\mathbf{k}}} \left\{ \Omega(\mu^*) + N\mu^* \right\} = \frac{1}{2} \frac{\delta F}{\delta \epsilon_{\mathbf{k}}} =$$

$$n_{\mathbf{k},s}^0 + (4\beta V)^{-1} \sum_{\nu} \sum_{\mathbf{q} \neq 0} V_{\mathbf{q}} \int_0^1 d\lambda \left\{ 1 + \lambda L(x) [1 - G(x)] \right\}^{-2} \times \\ \left\{ \frac{\delta}{\delta \varepsilon_{\mathbf{k}}} \tilde{\mu}_2^0(x, -x | \mu_0) + \lambda(x) \tilde{\mu}_2^0(x, -x | \mu_0) \frac{\delta}{\delta \varepsilon_{\mathbf{k}}} G(x | \lambda) \right\}.$$

Here $L(x) \equiv V^{-1} V_{\mathbf{q}} \tilde{\mu}_2^0(x, -x | \mu_0)$ and $G(x | \lambda)$ is the local-field correction function with the interaction potential $\lambda V_{\mathbf{q}}$ and $G(x | 1) \equiv G(x)$. In such a manner, for some model characteristic computing, the universal function basis must be built. This basis consists of $G(x)$ - the local-field correction function and its variation derivatives $\frac{\delta}{\delta \varepsilon_{\mathbf{k}}} G(x)$, $\frac{\delta}{\delta V_{\mathbf{q}}} G(x)$, $\frac{\delta}{\delta n_{\mathbf{k},s}^0} G(x)$. Unfortunately, the modern local-field conception is limited to the only function $G(x)$, besides, as a rule in its static variant $G(\mathbf{q}, 0)$ [20,25-30].

For the first time the integral equation set for $G(x) \equiv G_1(x) + G_2(x)$ has been obtained in [5,15]

$$G_i(x) + \sum_{x_1} \Phi_i(x, x_1) G_2(x) = G_i^{RPA}(x), \quad i = 1, 2; \quad (5.4)$$

$$\Phi_1(x, x_1) = -(2\beta V_{\mathbf{q}})^{-1} \left[\tilde{\mu}_2^0(x, -x | \mu_0) \right]^{-2} V_{\mathbf{q}_1} \mu_{4,1}^0(x, -x, x_1, -x_1 | \mu_0);$$

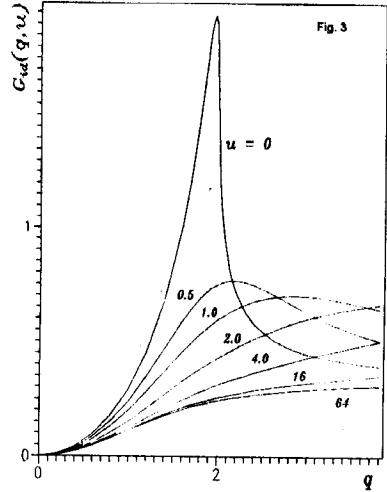
$$\Phi_2(x, x_1) = -(2\beta V V_{\mathbf{q}})^{-1} \left[\tilde{\mu}_2^0(x, -x | \mu_0) \right]^{-2} V_{\mathbf{q}_1} V_0(x + x_1) \times \\ \left[\tilde{\mu}_3^0(x, x_1, -x - x_1 | \mu_0) \right]^2;$$

$$G_i(x)^{RPA} = \sum_{x_1} \Phi_i(x, x_1) \varepsilon_0^{-1}(x_1).$$

The (\mathbf{q}, ν) representation of the local-field correction function was studied in [5,31,32]. As one can see, $G_1^{RPA}(x)$ corresponds to a weakly non-ideal system limit (Geldart-Taylor's limit). This is the universal characteristic which does not depend on the coupling parameter

$$G_{id}(x) \equiv G_1^{RPA}(x)|_{r_s=0} = \frac{1}{4} q^2 \times I_{2,0}^{-2}(q, u) \times \\ \left\{ -\frac{1}{2\pi} \int_0^{\infty} dq_1 q_1 \int_0^{\infty} du_1 \int_{-1}^{+1} dt I_{4,0}(q, q_1; u, u_1; t) \right\}. \quad (5.5)$$

The dimensionless variables and functions are defined by the following relations $q \equiv |\mathbf{q}| k_F^{-1}$, $u = (2\varepsilon_F q)^{-1} \nu I_{2,0}(q, u) = (3N)^{-1} 2\varepsilon_F \times \tilde{\mu}_2^0(x, -x | \mu_0)$, $I_{4,0} = (3N)^{-1} (2\varepsilon_F)^3 \tilde{\mu}_4^0(x, -x, x_1, -x_1 | \mu_0)$. The graph set of $G_{id}(x)$ as a function of the variables q, u is represented in Fig. 3.



In view of the fact that $G_1(x)$ expresses from $G_2(x)$, let us consider the question of existence and unique solution to the equation (5.4). This equation belongs to Fredholm integral equations of the second kind [33].

Let us use the substitution

$$y(x) = G_2(x)V_q\tilde{\mu}_2^0(x, -x|\mu_0), \quad f(x) = G_2^{RPA}(x)V_q\tilde{\mu}_2^0(x, -x|\mu_0). \quad (5.6)$$

The function $G_2(x)$ satisfies the equation with the positive symmetrical Fredholm kernel

$$y(x) + \sum_{x_1} K(x, x_1)y(x_1) = f(x), \quad (5.7)$$

$$K(x, x_1) = (2\beta V)^{-1}V_0(x + x_1)\left[\tilde{\mu}_2^0(x, -x|\mu_0)\tilde{\mu}_2^0(x_1, -x_1|\mu_0)\right]^{-1} \times \\ \tilde{\mu}_3^0(x, x_1, -x - x_1|\mu_0)\tilde{\mu}_3^0(-x, -x_1, x + x_1|\mu_0) \geq 0.$$

Taking into account the definition and asymptotic of the function $\tilde{\mu}_3^0(x, x_1, -x - x_1|\mu_0)$, one can see the similarity of the function $K(x, x_1)$ to the function

$$(2\beta V)^{-1}(3N)^{-2}V_0(x + x_1)\left\{\tilde{\mu}_2^0(x + x_1, -x - x_1|\mu_0)\right\}^2, \quad (5.8)$$

and the equation (5.7) to the convolution-type integral equation. In this way, the corresponding homogeneous equation $f(x) = 0$ has only the trivial solution when the kernel is positive.

From the Fredholm theorem [34] the unique non-trivial solution for the equation (5.7) exists. In such a manner, the equation system (5.4) has the unique non-trivial solution.

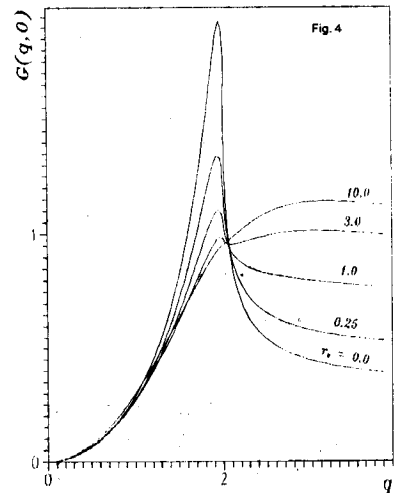
The local-field correction function was studied by the numerical method for the case of weakly non-ideal systems and the case of strong non-ideal ones (see [35]).

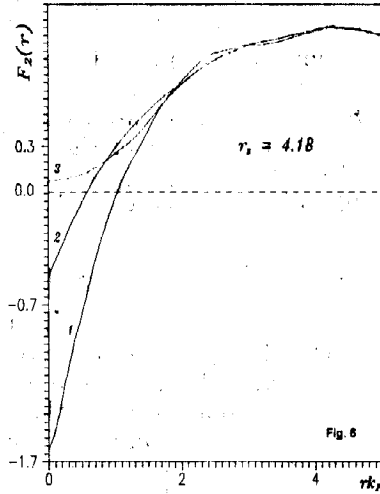
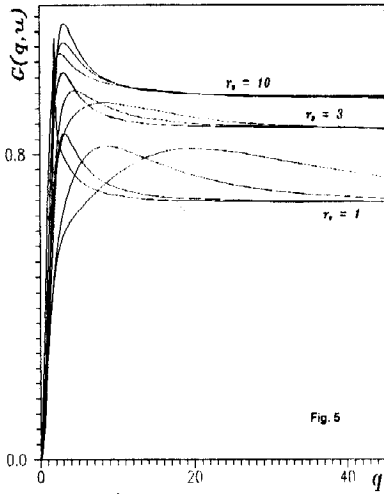
As one can see (Fig. 4) the strong dependence of $G(x)$ on the coupling parameter occurs. The form of the function is obviously changed, especially near point $q = 2k_F$.

Taking into account the maximum value of the functions $\tilde{\mu}_3^0(x, x_1, -x - x_1|\mu_0)$ and $\tilde{\mu}_4^0(x, -x, x_1, -x_1|\mu_0)$ in the vicinity of the supersurface $x_1 + x_2 = 0$, the approximate solutions of the system (5.4) have been obtained in quadratures (see [32])

$$G(x) \cong \left\{G_1^{RPA}(x) + G_2^{RPA}(x)\right\} \left\{1 + \phi(x)G_2^{RPA}(x)\right\}^{-1}, \quad (5.9)$$

where $\phi(x) \approx 1$. The dynamic local-field correction function in this approximation is presented in Fig. 5. An influence of the local-field correction function on the binary distribution function computation under $r_s = 4, 18$ is





shown in Fig. 6. Curve 1 corresponds to the random phase approximation. Curve 2 corresponds to the $G_{id}(x)$ approximation. Curve 3 is calculated on basis of the formula (5.9). The result of some model characteristic computations (correlation energy, compressibility, and binary distribution function $F_2(0)$ under $T = 0K$, and $0 \leq r_s \leq 10$) is represented in tab. 2.

The relative deviation of the correlation energy obtained in [18] $\delta_1(r_s)$ and $\delta_2(r_s)$ in [36] from one obtained in the present paper is found

$$\delta_1(r_s) = \epsilon_c^{-1} \{ \epsilon_c^{MC} - \epsilon_c \}, \tag{5.10}$$

$$\delta_2(r_s) = \epsilon_c^{-1} \{ \epsilon_c^{WVN} - \epsilon_c \}. \tag{5.11}$$

The small values of $\delta_1(r_s)$ and $\delta_2(r_s)$ prove that $\epsilon_c(r_s)$, $\epsilon_c^{CA}(r_s)$, $\epsilon_c^{WVN}(r_s)$ are calculated with high, almost identical precision. The interpolation of Monte-Carlo results [18], presented in [36] is the most precise analytic form for the electron liquid correlation energy. The curve set of the binary distribution function

$$F_2(\mathbf{r}) = 1 + [N(N-1)\beta]^{-1} \sum_x \mu_2(x, -x) \exp(i\mathbf{q}\mathbf{r}), \tag{5.12}$$

having been calculated on basis of the formula (5.9), is shown in Fig. 7.

The behavior of $F_2(\mathbf{r})$ on the small distance is very similar to that computed in [37].

The variation derivative $\frac{\delta G(x)}{\delta \epsilon_{\mathbf{k}}}$ is introduced in [33]. One obtains the following form of $\frac{\delta G(x)}{\delta \epsilon_{\mathbf{k}}}$ on the basis of the equation system (5.4)

$$g(x|\mathbf{k}) = \frac{\delta}{\delta \epsilon_{\mathbf{k}}} G(x) = g_1(x) + g_2(x), \tag{5.13}$$

$$g_i(x|\mathbf{k}) + \sum_{x_1} \Phi_i(x, x_1) g_2(x_1|\mathbf{k}) = g_i^{RPA}(x|\mathbf{k}) - \sum_{x_1} G_2(x_1) \frac{\delta}{\delta \epsilon_{\mathbf{k}}} \Phi_i(x, x_1).$$

Here $g_i^{RPA}(x|\mathbf{k}) \equiv \frac{\delta}{\delta \epsilon_{\mathbf{k}}} G_i^{RPA}(x)$. An existence of unique solution to the (5.13) system can be proved like in the case of (5.4). The $g(x|\mathbf{k})$ was studied by the different approximations. This is a function of the four variables ($|\mathbf{q}|, |\mathbf{k}|, t = (\mathbf{q}, \mathbf{k})[qk]^{-1}$, and ν). The function of three variables

$$\Gamma(q, u|k) = \varepsilon_F \frac{V}{8\pi^3} k_F^3 \int d\Omega_{\mathbf{q}} g(x|\mathbf{k}) \quad (5.14)$$

is needed for the $n_{\mathbf{k},s}$ computation. It was investigated both by the numerical and the analytical methods under $T = 0\text{K}$. The $\Gamma_{id}(q, u|k)$ function singularities are represented in Fig. 8a, 8b, were $\Gamma_{id}(q, u|k) \equiv \Gamma(q, u|k)$ at $r_s \rightarrow 0$.

This function is an universal characteristic of the reference system. In [33] using $G(x)$ and $g(q, u|k)$ the residual Fermi surface variation of the interacting electron liquid model under $T=0\text{K}$ was studied in the wide region of the coupling parameter.

$$\Delta(r_s) = n_{\mathbf{k}_F - \delta, s} - n_{\mathbf{k}_F + \delta, s} \quad \text{at } \delta \rightarrow +0. \quad (5.15)$$

As it is known, the condition $\Delta(r_s) > 0$ violates from the random phase approximation under $r_s > 8,7$ (curve 1 in Fig. 9). Curve 2 has been calculated on the basis of $G(\mathbf{q}, 0)$, which is computed in [21]. As it is shown, in the weak and medium non-ideality region taking into account $G(\mathbf{q}, 0)$ improves the RPA. But at big values of r_s in this approximation $\Delta(r_s)$ becomes negative too. Curve 3 is obtained with $G_{id}(x)$ and its derivative $\Gamma_{id}(q, u|k)$. In this approximation $\Delta(r_s) < \Delta_{RPA}(r_s)$. The dependence $\Delta(r_s)$ on the coupling parameter at the exact account at $G_1^{RPA}(q, u)$ and its derivative $g_1^{RPA}(x|\mathbf{k})$ is shown by curve 4. Asymptotically at small values of r_s this curve coincides with curve 3. But it is of the different character from curve 1 and 2. It leads to the constant value at $r_s \rightarrow \infty$. Here $\Delta(r_s) \simeq -0.3 \dots$ Curve 5 is built in the following approximation: $\Delta(r_s)$ is calculated in (2.12) approximation with the help of $G_1^{RPA}(q, u)$ and $\frac{\delta}{\delta \epsilon_{\mathbf{k}}} G_1^{RPA}(q, u)$. Curve 6 corresponds to (5.9) approximation for the local-field correction and (5.13) for $\Gamma(q, u|k)$. In this approximation $\Delta(r_s)$ has not become negative in the strong non-ideality region.

6. Model characteristics at non-zero temperatures

As far as we know, the computation of the temperature dependence of the model characteristics is a more intricate problem in comparison with the case $T = 0\text{K}$. Up to the present, the temperature dependence of the important characteristics has not been investigated very thoroughly. There are few works in this field (see [38,39]). In this chapter we shall show the reference system approach using for the temperature dependence characteristics of the homogeneous interacting electron liquid model computation. Primarily, we modify the partition function functional representation (2.27). Let us

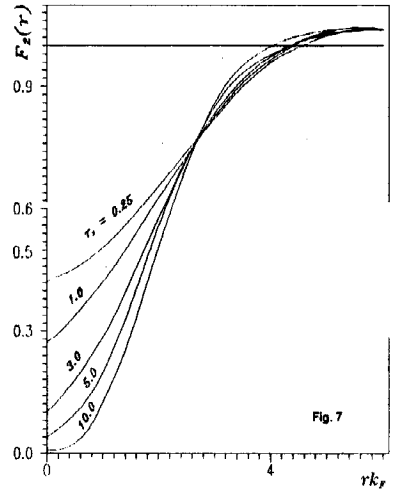
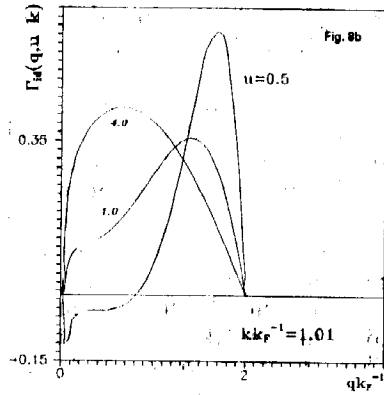
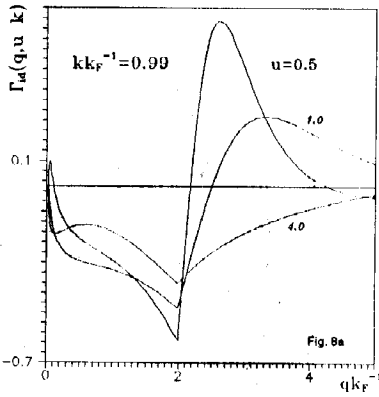


Fig. 7



consider the interaction representation of the statistical operator on the basis of the operator $\hat{H}_0 - \mu_0 \hat{N}$, where μ_0 is the chemical potential of a reference system under $T = 0K$.

$Z(\mu)$ has the form

$$Z(\mu) = \frac{Z_0(\mu_0)}{\{(d\omega)f(\omega)\}} \int (d\omega) f(\omega) J(\omega|\mu_0) \exp \left\{ \sum_{n \geq 1} \frac{(\mu - \mu_0)^n}{n!} d_n(\omega|\mu_0) \right\},$$

$$d_n(\omega|\mu_0) = \beta \tilde{\mu}_n(0, \dots, 0|\mu_0) + \frac{\partial^n}{\partial \mu_0^n} \sum_{m \geq 2} D_m(\omega|\mu_0). \tag{6.1}$$

Taking advantage of equation

$$N = \left[\beta Z(\mu^*) \right]^{-1} \frac{\partial}{\partial \mu^*} Z(\mu^*) \tag{6.2}$$

we can compute the chemical potential and write the statistical operator by the canonical ensemble variables. Let us write the chemical potential in the form

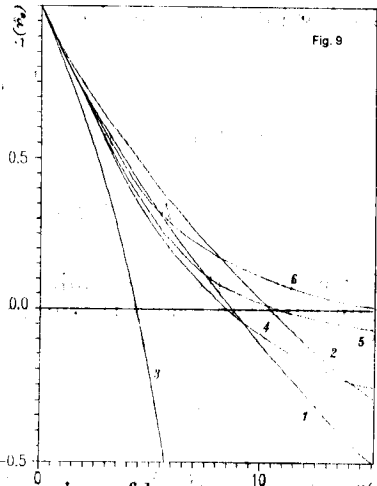
$$\mu^* = \mu_0 + \Delta\mu + \delta\mu, \tag{6.3}$$

where $\Delta\mu \equiv \Delta\mu(r_s)$ is the chemical potential shift of the model. $\Delta\mu(r_s)$ describes the interaction in the chemical potential behavior. The following form for $\delta\mu$ has been obtained in the region of low temperature (it is less than the degenerate one $T_F = k_B^{-1} \epsilon_F$) and in the linear approach over $\delta\mu$

$$\delta\mu = -\frac{\kappa_0}{\kappa} \left[\tilde{\mu}_2^0(0, 0|\mu_0) \right]^{-1} \sum_{n \geq 1} [\Delta\mu]^{n-1} \beta^{-1} [(n-1)!]^{-1} \langle \Delta d_n(\omega|\mu_0) \rangle_0, \tag{6.4}$$

where κ is the electron liquid model compressibility, κ_0 is the reference system one under $T = 0K$,

$$\Delta d_n(\omega|\mu_0) \equiv d_n(\omega|\mu_0) - d_n^0(\omega|\mu_0). \tag{6.5}$$



Here $d_n^0(\omega|\mu_0)$ is the $d_n(\omega|\mu_0)$ function limit under $T = 0K$. The symbol $\langle \dots \rangle_0$ means statistical averaging over ω variable with the weight function $f(\omega)J_1^0(\omega|\mu_0)$, which does not depend on the temperature. This function has been obtained from $f(\omega)J^0(\omega|\mu_0)$ replacing $\mu_n^0(x_1, \dots, x_n|\mu_0)$ on its "normal" component under $T = 0K$.

Let us confine the first term of a series (6.4) in a weak non-ideality region. This situation corresponds only to the low correlation functions consideration. $\delta\mu$ as a function of the variables r , (coupling parameter) and T/T_F (dimensionless temperature) is shown in Fig. 10.

Let us expand the free energy of the model

$$F = \Omega(\mu^*) + N\mu^* \quad (6.6)$$

as a function of the variable μ^* by Taylor's theorem near the point $\mu_0^* = \mu_0 + \Delta\mu$:

$$F = \left\{ \Omega(\mu_0^*) + \mu_0^* N \right\} + \delta\mu \left\{ N + \Omega'(\mu_0^*) \right\} + \frac{1}{2}(\delta\mu)^2 \Omega''(\mu_0^*) + \dots \quad (6.7)$$

Taking into account the equation (6.2), we obtain the equality

$$N + \Omega'(\mu_0^*) = -\delta\mu \Omega''(\mu_0^*) + \dots \quad (6.8)$$

In this way, the linear part over $\delta\mu$ is absent in the following expansion

$$F = \left\{ \Omega(\mu_0^*) + \mu_0^* N \right\} - \frac{1}{2}(\delta\mu)^2 \Omega''(\mu_0^*) + \dots \quad (6.9)$$

Let us use only the chemical potential, evaluated at $T = 0K$, for the transition with the grand thermodynamical potential to the free energy, because $\delta\mu \sim T^2$ is in the low temperature region.

$$\exp(-\beta F) \cong Z(\mu_0 + \Delta\mu) \exp \left\{ -\beta N[\mu_0 + \Delta\mu] \right\} = \quad (6.10)$$

$$\exp \left[-\beta \left(\Omega_0(\mu_0) + \mu_0 N \right) \right] \left\{ \int (d\omega) f(\omega) \right\}^{-1} \times \\ \int (d\omega) f(\omega) J_1^0(\omega|\mu_0) \exp \left\{ \sum_{n \geq 2} \Delta D_n(\omega|\mu_0) + \sum_{n \geq 1} \frac{(\Delta\mu)^n}{n!} \Delta d_n(\omega|\mu_0) \right\}.$$

The mutual compensation of the "anomalous" part of the correlation functions and the chemical potential displacement $\Delta\mu$ occurred under $T = 0K$.

The notation

$$\Delta D_n(\omega|\mu_0) = D_n(\omega|\mu_0) - D_n^0(\omega|\mu_0), \quad (6.11)$$

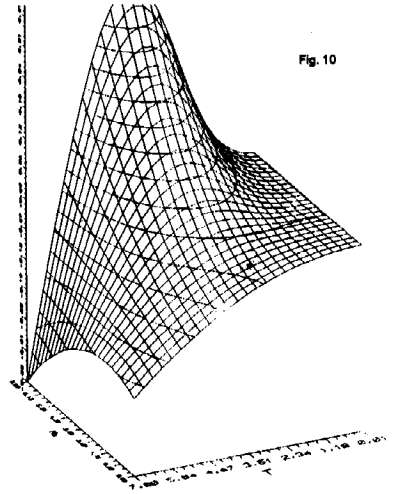


Fig. 10

has been used. Here $D_n^0(\omega|\mu_0) \equiv \lim_{\beta \rightarrow \infty} D_n(\omega|\mu_0)$. Let us expand the last multiplier in formula (6.10) by Taylor's theorem, because $\Delta d_n(\dots)$ and $\Delta D_n(\dots)T^2$ are in the low temperature region. As a result, we obtain the free energy in the form

$$F \cong E_0 + \Delta F, \quad (6.12)$$

where

$$E_0 = \left\{ \Omega_0^0(\mu_0) + \mu_0 N \right\} - \frac{1}{\beta} \ln \left\{ \int (d\omega) f(\omega) \right\}^{-1} \int (d\omega) f(\omega) J_1^0(\omega|\mu_0) \quad (6.13)$$

is energy of the model under $T = 0K$.

$$\begin{aligned} \Delta F = & \Omega_0(\mu_0) - \Omega_0^0(\mu_0) - \\ & \frac{1}{\beta} \left\{ \sum_{n \geq 2} \langle \Delta D_n(\omega|\mu_0) \rangle_0 + \sum_{n \geq 1} \frac{(\Delta\mu)^n}{n!} \langle \Delta d_n(\omega|\mu_0) \rangle_0 \right\} + \dots \end{aligned} \quad (6.14)$$

is the temperature correction, which is caused by the temperature dependence of the functions $\tilde{\mu}_n^0(x_1, \dots, x_n|\mu_0)$ where $\Omega_0^0(\mu_0) \equiv \lim_{\beta \rightarrow \infty} \Omega_0(\mu_0)$.

7. Superconductive phase of electron liquid model

It is well known, that in the region of the strong non-ideality the electron liquid is unsteady because its compressibility is negative and static dielectric function has negative branch in the region of low wave vectors.

In works [40,41] it was noted, that it may be reasoned by the transition in superconductive state. In the work [42] many-electron method was proposed for the description of superconductive phase of systems, which include strong non-ideal degenerated electron subsystem. The model of electron liquid serves as a statistic basis in the description of systems of that kind. On the other hand, the problem of superconductive phase electron liquid (SPEL) is important itself, because it is a fundamental model of the statistic physics.

The basic method for description of SPEL is based on the assumption about an active role of subsystem of electrons their wave vectors being within a small region $\delta(\mathbf{k})$ near the Fermi surface.

The subsystem of electrons with wave vectors $\mathbf{k} \notin \delta(\mathbf{k})$ is considered as normal one (paramagnetic). It plays a part of a medium which forms the state of the active subsystem.

To describe an active subsystem of electrons Bogolubov's (u, v) - transformation [43] is used

$$a_{2s\mathbf{k},s} = u_{\mathbf{k}} \alpha_{\mathbf{k},\frac{1}{2}-s} + 2sv_{\mathbf{k}} \alpha_{\mathbf{k},\frac{1}{2}+s}^{\dagger}, \quad (7.1)$$

were $u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$. To organize a reference system the Hamiltonian of quasi-particles is introduced

$$\hat{h} = \sum_{\mathbf{k}; \sigma=0,1} E_{\mathbf{k}} \alpha_{\mathbf{k},\sigma}^{\dagger} \alpha_{\mathbf{k},\sigma}, \quad \mathbf{k} \in \delta(\mathbf{k}), \quad (7.2)$$

where $E_{\mathbf{k}}$ is an unknown spectrum. The subsystem of quasi-particles together with paramagnetic subsystem of electrons form reference system with the Hamiltonian

$$\hat{H}_R = \hat{H}_0 + \hat{V}_P, \quad \hat{H}_0 = \sum_{s, \mathbf{k} \notin \delta(\mathbf{k})} (\varepsilon_{\mathbf{k}} - \mu) a_{\mathbf{k},s}^+ a_{\mathbf{k},s} + \hat{h}, \quad (7.3)$$

$$\hat{V}_P = (2V)^{-1} \sum_{\mathbf{q} \neq 0} V_{\mathbf{q}} \sum_{\mathbf{k}_1, \mathbf{k}_2, s_1, s_2} a_{\mathbf{k}_1 - \mathbf{q}, s_1}^+ a_{\mathbf{k}_2 + \mathbf{q}, s_2}^+ a_{\mathbf{k}_2, s_2} a_{\mathbf{k}_1, s_1}.$$

and $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_1 - \mathbf{q}, \mathbf{k}_2 + \mathbf{q} \notin \delta(\mathbf{k})$. In the calculation of a partition function the operator $\hat{H} - \mu \hat{N} - \hat{H}_R$ is taken into account by the moments method relatively reference system. Its thermodynamical and correlative functions may be calculated by methods which were elaborated for paramagnetic phase of the model. Thermodynamical potential $\Omega(\mu)$ is a functional of $u_{\mathbf{k}}, v_{\mathbf{k}}$ and $E_{\mathbf{k}}$. In the ideal case, when all diagrams of perturbation theory are taken into account, $\Omega(\mu)$ is invariant under their choice, that is

$$\frac{\delta \Omega(\mu)}{\delta v_{\mathbf{k}}} = 0, \quad \frac{\delta \Omega(\mu)}{\delta E_{\mathbf{k}}} = 0. \quad (7.4)$$

In practice $\Omega(\mu)$ may be calculated only approximately and (7.4) are conditions of the extreme, which define unknown functions.

In the approximation of first correlations of reference system and temperatures, which are low as compared with superconductive transition temperature T_c , conditions (7.4) may be reduced to the following system of integral equations

$$u_{\mathbf{k}} v_{\mathbf{k}} \xi_{\mathbf{k}} = (2V)^{-1} (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2) \sum_{\mathbf{k}_1} u_{\mathbf{k}_1} v_{\mathbf{k}_1} (1 - 2n_{\mathbf{k}_1}) Q(\mathbf{k}, \mathbf{k}_1), \quad (7.5)$$

$$E_{\mathbf{k}} = (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2) \xi_{\mathbf{k}} + 2u_{\mathbf{k}} v_{\mathbf{k}} v^{-1} \sum_{\mathbf{k}_1} u_{\mathbf{k}_1} v_{\mathbf{k}_1} (1 - 2n_{\mathbf{k}_1}) Q(\mathbf{k}, \mathbf{k}_1),$$

Here $n_{\mathbf{k}} = [1 + \exp(\beta E_{\mathbf{k}})]^{-1}$ is the momentum distribution of quasi-particles.

$$\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu - (2V)^{-1} \sum_{\mathbf{q}, s} n_{\mathbf{k} + \mathbf{q}, s}^P Q(\mathbf{k}, \mathbf{k} + \mathbf{q}) + \quad (7.6)$$

$$V^{-1} \sum_{\mathbf{k}_1} Q(\mathbf{k}, \mathbf{k}_1) [u_{\mathbf{k}_1} \tilde{u}_{\mathbf{k}_1} + v_{\mathbf{k}_1} \tilde{v}_{\mathbf{k}_1}] -$$

$$(2V^2)^{-1} \sum_{\mathbf{q}} v_{\mathbf{q}}^2 \mu_P(\mathbf{q} | E_{\mathbf{k}} + E_{\mathbf{k} + \mathbf{q}}) + \dots$$

Kernel $Q(\mathbf{k}, \mathbf{k}_1)$ is taken with the opposite sign screened potential of interaction

$$Q(\mathbf{k}_1, \mathbf{k}_2) = -V_{\mathbf{k}_1 - \mathbf{k}_2} \left\{ 1 - \frac{1}{V} V_{\mathbf{k}_1 - \mathbf{k}_2} \mu_P(\mathbf{k}_1 - \mathbf{k}_2 | E_{\mathbf{k}_1} + E_{\mathbf{k}_2}) \right\}. \quad (7.7)$$

Here $\mu_P(\mathbf{q} | E)$ is a retarded correlation function of paramagnetic subsystem

$$\mu_P(\mathbf{q} | E) \equiv \mu_2^P(x, -x) \quad \text{at} \quad i\nu = E. \quad (7.8)$$

If \mathbf{k} and \mathbf{k}_1 are close to \mathbf{k}_F , then $E_{\mathbf{k}}$ and $E_{\mathbf{k}_1}$ reach minimum and in such case $\mu_P(\mathbf{k}_1 - \mathbf{k}_2 | E_{\mathbf{k}_1} + E_{\mathbf{k}_2})$ is close to the model of electron liquid in the paramagnetic phase ($\mu_2(x, -x)$ when $\nu = 0$). Then under enough great significance of coupling parameter $Q(\mathbf{k}, \mathbf{k}_1)$ is positive. When \mathbf{k} and \mathbf{k}_1 are essentially different from \mathbf{k}_F then $Q(\mathbf{k}, \mathbf{k}_1)$ is negative even when r_s is great enough. Thus, retardation effects are essential and the criterion of rise of superconductive state has the form

$$\frac{1}{V} V_{\mathbf{k}_1 - \mathbf{k}_2} \mu_P(\mathbf{k}_1 - \mathbf{k}_2 | E_{\mathbf{k}_1} + E_{\mathbf{k}_2}) \geq 1 \quad \text{at } k_1, k_2 \approx k_F. \quad (7.9)$$

The research of possibility of existence of system (7.5) solutions is fulfilled with the help of a simple model, where region $\delta(\mathbf{k})$ is a spherical layer

$$k_F - \delta_1 \leq k \leq k_F + \delta_2. \quad (7.10)$$

By means of standard substitution

$$v_{\mathbf{k}}^2 = \frac{1}{2} \left\{ 1 - \xi_{\mathbf{k}} \left[\xi_{\mathbf{k}}^2 + c_{\mathbf{k}}^2 \right]^{-1/2} \right\}, \quad E_{\mathbf{k}} = \left[\xi_{\mathbf{k}}^2 + c_{\mathbf{k}}^2 \right]^{1/2} \quad (7.11)$$

we come to new convenient functions $C_{\mathbf{k}}$, $\xi_{\mathbf{k}}$, which satisfies within the model (7.10) such a system of one-dimensional nonlinear equations

$$\begin{aligned} C_{\mathbf{k}} - \int_{k_F - \delta_1}^{k_F + \delta_2} k_1^2 q(k, k_1 | [C_{\mathbf{k}}, \xi_{\mathbf{k}}]) C_{\mathbf{k}_1}, \Phi(C_{\mathbf{k}_1}, \xi_{\mathbf{k}_1}) dk_1 &= 0, \\ \xi_{\mathbf{k}} + \int_{k_F - \delta_1}^{k_F + \delta_2} k_1^2 q(k, k_1 | [C_{\mathbf{k}}, \xi_{\mathbf{k}}]) \xi_{\mathbf{k}_1}, \Phi(C_{\mathbf{k}_1}, \xi_{\mathbf{k}_1}) dk_1 &= \tilde{E}_{\mathbf{k}}. \end{aligned} \quad (7.12)$$

Here the following notation were used

$$\begin{aligned} \Phi(C_{\mathbf{k}_1}, \xi_{\mathbf{k}_1}) &= \left[\xi_{\mathbf{k}_1}^2 + C_{\mathbf{k}_1}^2 \right]^{-1/2} \text{th} \left\{ \frac{\beta}{2} \left[\xi_{\mathbf{k}_1}^2 + C_{\mathbf{k}_1}^2 \right]^{1/2} \right\}, \\ q(k, k_1 | [C_{\mathbf{k}}, \xi_{\mathbf{k}}]) &= (8\pi^2)^{-1} \int_{-1}^{+1} Q(\mathbf{k}, \mathbf{k}_1) dt, \\ \tilde{E}_{\mathbf{k}} &\equiv \varepsilon_{\mathbf{k}} - \mu + (2V)^{-1} \sum_{\mathbf{k}_1} Q(\mathbf{k}, \mathbf{k}_1) \left[1 + \sum_s n_{\mathbf{k}_1, s}^P \right] - \\ &(2V^2)^{-1} \sum_{\mathbf{q}} \mu_p(\mathbf{q} | E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{q}}) + \dots, \end{aligned} \quad (7.13)$$

and t is cosine of angle between vectors \mathbf{k} and \mathbf{k}_1 .

Kernel $q(k, k_1 | [C_{\mathbf{k}}, \xi_{\mathbf{k}}])$ is a functional of unknown $C_{\mathbf{k}}$ and $\xi_{\mathbf{k}}$ but its general form may be illustrated by the kernel of null approximation $q_0(k, k_1) \equiv q(k, k_1 | [C_{\mathbf{k}}, \xi_{\mathbf{k}}])$ when $C_{\mathbf{k}} = 0$, $\xi_{\mathbf{k}} = \xi_{\mathbf{k}}^P$ where $\xi_{\mathbf{k}}^P$ corresponds to the function $\xi_{\mathbf{k}}$ in the normal state.

In the Fig. 11 family of curves $q_0(k, k_1) \equiv 0$ was represented for some significance of r_s . Kernel is positive inside every "butterfly" and negative - outside it. The first of equations (7.12) belongs to the type of Hammerstein equations [34], and an existence and unique of its solutions is conditioned by the positive form of the kernel and by the positively and limitation, of the functions $C_k \Phi(C_k, \xi_k)$.

As one can see in the figure, under great enough r_s , the region δ_k may be chosen so as to guarantee the existence of non-trivial C_k solutions.

In the work [44] it was shown, that where F_s is the free energy of superconductive phase, F_p the normal phase, and functions C_k and ξ_k are solutions of equations (7.12) when $\mu = \mu_p$, where μ_p - the chemical potential of paramagnetic phase of the model.

$$F_s - F_p \leq -\frac{1}{2} \sum_{\delta(\mathbf{k})} (1 - 2n_{\mathbf{k},s}) [\xi_k^2 + C_k^2]^{-1/2} \{ [\xi_k^2 + C_k^2]^{1/2} - |\xi_k| \}^2. \quad (7.14)$$

The inequality (7.14) is true when the region δ_k is small as compared with the Fermi sphere. In Fig. 12 solutions of equation for C_k in the case of absolutely zero temperature which were found by the general method are shown. Here region δ_k was defined by the condition of the minimum of value $\Delta F = F_s - F_p$.

The temperature of superconductive transition T_c is found on the condition of disappearance of non-trivial solutions of equation for C_k (look Tab.3).

The peculiarity of superconductive phase of the electron liquid model is the strong dependence of its characteristics on the value of coupling parameter. Therefore the general physical picture, that is obtained by means of basic method looks like the model BCS [45].

Under very great values of the coupling parameter the separation of the model in two subsystems loses the sense and the basic method stops to be effective. Under such conditions one-component Bogolyubov's model must be used [46].

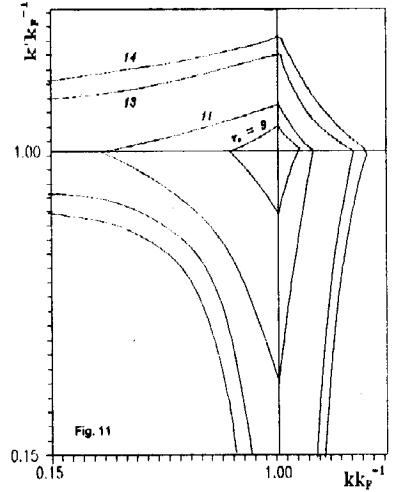


Fig.11

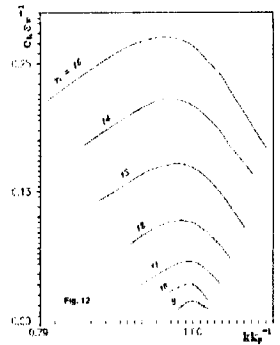


Fig.12

8. Reference system approach in the inhomogeneous electron system theory

Spatially inhomogeneous electron liquid model is considered as the basic in modern microscopic metal theory.

The method of the density functional is one of the many approximate methods for the description of inhomogeneous electron systems [47]. The construction of the density functional with many-particle correlation accounting is the vital problem in the theory of inhomogeneous electron systems. To solve this problem the reference system approach has been considered in [48]. Having suggested the correct taking account of the inhomogeneity, let us consider the simple model, in which the interaction between the electrons and the positive charge is described by local potential. The model Hamiltonian has the form

$$\tilde{H} = \hat{H} + \hat{V}_1 + \hat{V}_2 + C. \quad (8.1)$$

in the second quantization representation. Here \hat{H} is the Hamiltonian of the homogeneous electron liquid model. Operator

$$\hat{V}_1 = V^{-1} \sum_{\mathbf{q}} V_1(\mathbf{q}) S_{-\mathbf{q}} \sum_{\mathbf{k},s} a_{\mathbf{k},s}^\dagger a_{\mathbf{k},s} \quad (8.2)$$

describes the interaction between the electrons and the inhomogeneous component of the density of the positive charge

$$S_{\mathbf{q}} = \int d\mathbf{r} \left[\rho_+(\mathbf{r}) - \frac{N}{V} \right] \exp(i\mathbf{q}\mathbf{r}) = \int d\mathbf{r} \Delta\rho_+(\mathbf{r}) \exp(i\mathbf{q}\mathbf{r}), \quad (8.3)$$

and $S_{\mathbf{q}=0} = 0$. \hat{V}_2 is the operator of the electrostatic interaction, which is caused by inhomogeneity of positive charge

$$\hat{V}_2 = (2V)^{-1} \sum_{\mathbf{q} \neq 0} V_2(\mathbf{q}) \{ S_{\mathbf{q}} S_{-\mathbf{q}} - N \}. \quad (8.4)$$

The constant C is determined by deviation of interaction potentials with Coulomb interactions in a homogeneous system

$$C = (2V)^{-1} N^2 \{ V_{\mathbf{q}} + 2V_1(\mathbf{q}) + V_2(\mathbf{q}) \} |_{\mathbf{q}=0}. \quad (8.5)$$

The frequency representation will be used for the model (8.1) partition function computing relatively the electron variables (see section 2)

$$\tilde{Z}(\mu) = \text{Tr} \left\{ \exp \left[-\beta(\hat{H}_0 - \mu\hat{N}) \right] \tilde{T} \left[\hat{S}(\nu) \hat{S}_1(\nu) \right] \right\} \exp \left[-\beta(C + V_2) \right]. \quad (8.6)$$

Here $\hat{S}(\nu)$ is defined by formulae (2.15) and

$$\hat{S}_1(\nu) = \exp \left\{ -V^{-1} \sum_{\mathbf{q}} V_1(\mathbf{q}) S_{-\mathbf{q}} \hat{\rho}_{\mathbf{q},0} \right\}. \quad (8.7)$$

Using the homogeneous electron liquid model as a reference system (all thermodynamical characteristics and the correlation functions of this model are known) let us represent the partition function $\tilde{Z}(\mu)$ in the form

$$\tilde{Z}(\mu) = Z(\mu) \langle \hat{S}_1(\nu) \rangle_h \exp \left[-\beta(C + V_2) \right]. \quad (8.8)$$

Here $Z(\mu)$ is the partition function of the reference system, which is defined on the basis of formula (2.15). The symbol $\langle \dots \rangle_h$ means statistical averaging over the states of the reference system.

Expanding the operator $\hat{S}_1(\nu)$ in a power series, averaging each of this expansion and representing the result in the form of the exponential function, we obtain the thermodynamical potential of the model (8.1) in the form of expansion in the static n -particle correlation functions of the reference system

$$\tilde{\Omega}(\mu) = -\frac{1}{\beta} \ln \hat{Z}(\mu) = \Omega(\mu) + C + V_2 + \Omega_1(\mu), \quad (8.9)$$

$$\Omega_1(\mu) = \sum_{n \geq 2} (-1)^{n-1} \frac{1}{n!} \frac{1}{V^n} \sum_{\mathbf{q}_1, \dots, \mathbf{q}_n} \tilde{\mu}_n(\mathbf{q}_1, \dots, \mathbf{q}_n | \mu) \prod_{i=1}^n S_1(\mathbf{q}_i),$$

where $S_1(\mathbf{q}) \equiv V_1(\mathbf{q})S\mathbf{q}$. $\Omega(\mu)$ is the thermodynamical potential of the reference system

$$\tilde{\mu}(\mathbf{q}_1, \dots, \mathbf{q}_n | \mu) \equiv \frac{1}{\beta} \langle \tilde{T} \hat{S}(\nu) \rangle_{\mathcal{H}_\mu}^{-1} \left\langle \tilde{T} \left\{ \hat{S}(\nu) \prod_{i=1}^n \hat{\rho}_{\mathbf{q}_i, 0} \right\} \right\rangle_{\mathcal{H}_\mu}^C. \quad (8.10)$$

Let us calculate the free energy $\tilde{F} = \tilde{\Omega}(\mu^*) + \mu^* N$ of the model in variables β, V, N , where μ^* is the root of equation

$$-N = \frac{d}{d\mu^*} \tilde{\Omega}(\mu^*). \quad (8.11)$$

Let us write the chemical potential in the form $\mu^* = \mu + \Delta\mu$, (where μ is the chemical potential of a reference system) and expand $\tilde{\Omega}(\mu^*)$ by Taylor's theorem near the point μ_0 . This allows us equation (8.11) into algebraic form to transform

$$\sum_{n \geq 2} [(n-1)!]^{-1} (\Delta\mu)^{n-2} \left\{ \Delta\mu M_n(0) - (n-1) \frac{d^{n-1}}{d\mu^{n-1}} \Omega_1(\mu) \right\} = 0. \quad (8.12)$$

Here

$$M_n(0) = -\frac{d^n}{d\mu^n} \Omega(\mu) \quad (8.13)$$

is the static limit at $\mathbf{q}_1, \dots, \mathbf{q}_n \rightarrow 0$ of the polarization function. Solving the equation (8.12), we obtain the result in the first iteration

$$\Delta\mu = M_2^{-1}(0) \sum_{n \geq 2} (-1)^{n-1} V^{-n} \sum_{\mathbf{q}_1, \dots, \mathbf{q}_n} \tilde{\mu}_{n+1}(\mathbf{q}_1, \dots, \mathbf{q}_n, 0 | \mu) \prod_{i=1}^n S_1(-\mathbf{q}_i). \quad (8.14)$$

Now let us separate the contribution of inhomogeneity F_1 to the free energy, and write

$$\begin{aligned} \tilde{F} &= F + F_1 + C + V_2, & F &\equiv F(n_0) = \Omega(\mu) + \mu N, & (8.15) \\ F_1 &= F_1[S_q] = \Omega_1(\mu) + \sum_{n \geq 2} \frac{(\Delta\mu)^{n-1}}{(n!)} \left\{ n \frac{d^{n-1}}{d\mu^{n-1}} \Omega_1(\mu) - \Delta\mu M_n(0) \right\}. \end{aligned}$$

Here $F(n_0)$ is the free energy of the homogeneous electron liquid with the density $n_0 = N/V$. The model characteristic computation based on representation (8.15). For example, let us consider the unary distribution function

$$f_1(\mathbf{r}) = 1 + f_1(\mathbf{r}), \quad f_1(\mathbf{r}) = \sum_{\mathbf{q} \neq 0} f_1(\mathbf{q}) \exp(i\mathbf{q}\mathbf{r}), \quad (8.16)$$

Representing the Fourier transform of $f_1(\mathbf{q})$ as a variation derivative, we obtain

$$\begin{aligned} f_1(\mathbf{q}) &= \frac{1}{N} \langle \sum_{\mathbf{k}, s} a_{\mathbf{k}+\mathbf{q}, s}^+ a_{\mathbf{k}, s} \rangle_{\bar{H}} = VN^{-1} \left\{ \frac{\delta}{\delta S_1(\mathbf{q})} \bar{\Omega}(\mu^*) \right\}_{|\mu^*} = \quad (8.17) \\ &\frac{1}{N} \sum_{n \geq 2} \frac{(-1)^{n-1}}{[(n-1)!]} V^{1-n} \sum_{\mathbf{q}_2, \dots, \mathbf{q}_n} \tilde{\mu}_n(\mathbf{q}, \mathbf{q}_2, \dots, \mathbf{q}_n | \mu) \prod_{i=2}^n S_1(-\mathbf{q}_i) + \\ &\frac{1}{NM_2(0)} \sum_{n \geq 2} \frac{(-1)^{n-1}}{[(n-1)!]} V^{1-n} \sum_{\mathbf{q}_2, \dots, \mathbf{q}_n} \tilde{\mu}_{n+1}(\mathbf{q}, \mathbf{q}_2, \dots, \mathbf{q}_n, 0 | \mu) \times \\ &\prod_{i=2}^n S_1(-\mathbf{q}_i) \sum_{m \geq 2} \frac{(-1)^{m-1}}{V^m} \sum_{\mathbf{q}'_2, \dots, \mathbf{q}'_m} \tilde{\mu}_{m+1}(\mathbf{q}', \mathbf{q}'_2, \dots, \mathbf{q}'_m | \mu) \times \\ &\prod_{j=1}^m S_1(-\mathbf{q}'_j) + \dots \end{aligned}$$

The component $F_1[S_{\mathbf{q}}]$ is the functional over $S_{\mathbf{q}}$, which describes the inhomogeneous. We limit our treatment only in the weak non-ideality region

$$f_1(\mathbf{q}) \approx -(NV)^{-1} V_1(\mathbf{q}) S_{\mathbf{q}} \tilde{\mu}_2(\mathbf{q}, -\mathbf{q} | \mu) + \dots \quad (8.18)$$

Using the iterating method and (8.18) as a zeroth iteration let us solve the equation (8.17). We obtain

$$\begin{aligned} V^{-1} S_{\mathbf{q}} V_1(\mathbf{q}) &= -f_{\mathbf{q}} + \tilde{\mu}_2^{-1}(\mathbf{q}, -\mathbf{q} | \mu) \times \quad (8.19) \\ &\sum_{n \geq 3} [(n-1)!]^{-1} \sum_{\mathbf{q}_2, \dots, \mathbf{q}_n} \tilde{\mu}_n(\mathbf{q}, \mathbf{q}_2, \dots, \mathbf{q}_n | \mu) \prod_{i=2}^n \left\{ f_{\mathbf{q}_i} - \tilde{\mu}_2^{-1}(\mathbf{q}_i, -\mathbf{q}_i | \mu) \times \right. \\ &\left. \sum_{m \geq 3} [(m-1)!]^{-1} \sum_{\mathbf{q}'_2, \dots, \mathbf{q}'_m} \tilde{\mu}_m(\mathbf{q}_i, \mathbf{q}'_2, \dots, \mathbf{q}'_m | \mu) \times \prod_{j=2}^m f_{\mathbf{q}'_j} + \dots \right\} - \\ &M_2^{-1}(0) \tilde{\mu}_2^{-1}(\mathbf{q}, -\mathbf{q} | \mu) \sum_{n, m \geq 2} [n!(m-1)!]^{-1} \times \\ &\sum_{\mathbf{q}_1, \dots, \mathbf{q}_n} \sum_{\mathbf{q}'_2, \dots, \mathbf{q}'_m} \tilde{\mu}_{n+1}(\mathbf{q}_1, \dots, \mathbf{q}_n, 0 | \mu) \times \\ &\tilde{\mu}_{m+1}(\mathbf{q}, \mathbf{q}'_2, \dots, \mathbf{q}'_m, 0 | \mu) \prod_{i=1}^n f_{\mathbf{q}_i} \prod_{j=2}^m f_{\mathbf{q}'_j} + \dots \end{aligned}$$

Here $f_{\mathbf{q}} \equiv f_1(\mathbf{q}) N \tilde{\mu}_2^{-1}(\mathbf{q}, -\mathbf{q} | \mu)$. Substituting the value of $S_{\mathbf{q}}$ in formula (8.15), we obtain the expression for F_1 as the functional of $f_1(\mathbf{q})$

$$\tilde{F} = F + F_{cl} + \sum_{j \geq 2} F^{(j)}[f_1]. \quad (8.20)$$

The electrostatic interaction energy is extracted

$$\begin{aligned}
 F_{cl} = C + & \tag{8.21} \\
 (2V)^{-1} \sum_{\mathbf{q}} \left\{ V_2(\mathbf{q}) S_{\mathbf{q}} S_{-\mathbf{q}} + 2V_1(\mathbf{q}) N S_{-\mathbf{q}} f_1(\mathbf{q}) + N^2 V_{\mathbf{q}} f_1(\mathbf{q}) f_1(-\mathbf{q}) \right\} = \\
 \frac{1}{2} \int \int d\mathbf{r}_1 d\mathbf{r}_2 \left\{ V_2(\mathbf{r}_1 - \mathbf{r}_2) \rho_+(\mathbf{r}_1) \rho_+(\mathbf{r}_2) + 2V_1(\mathbf{r}_1 - \mathbf{r}_2) \rho_+(\mathbf{r}_1) n(\mathbf{r}_2) + \right. \\
 \left. V(\mathbf{r}_1 - \mathbf{r}_2) n(\mathbf{r}_1) n(\mathbf{r}_2) \right\}.
 \end{aligned}$$

The components $F^{(j)}[f_1]$ are represented in the form of an expansion in the reference system correlation functions.

$$\begin{aligned}
 F^{(2)}[f_1] = \frac{N^2}{2} \sum_{\mathbf{q}} f_1(\mathbf{q}) f_1(-\mathbf{q}) M_2^{-1}(\mathbf{q}, -\mathbf{q}) - & \tag{8.22} \\
 \sum_{n \geq 3} (n!)^{-1} \sum_{\mathbf{q}_1, \dots, \mathbf{q}_n} \tilde{\mu}_n(\mathbf{q}_1, \dots, \mathbf{q}_n | \mu) \prod_{i=1}^n f_{\mathbf{q}_i} + \\
 \frac{1}{2} \sum_{n, m \geq 3} \left[(n-1)! (m-1)! \right]^{-1} \sum_{\mathbf{q}_1, \dots, \mathbf{q}_n} \sum_{\mathbf{q}'_1, \dots, \mathbf{q}'_m} \tilde{\mu}_n(\mathbf{q}_1, \dots, \mathbf{q}_n | \mu) f_{\mathbf{q}_1} f_{\mathbf{q}_2} \times \dots \\
 f_{\mathbf{q}_{n-1}} f_{\mathbf{q}'_2}, \dots, f_{\mathbf{q}'_m} \tilde{\mu}_2^{-1}(\mathbf{q}_n, -\mathbf{q}_n | \mu) \tilde{\mu}_m(-\mathbf{q}_n, \mathbf{q}'_2, \dots, \mathbf{q}'_m | \mu) + \dots
 \end{aligned}$$

Representing every term of the series (8.22) in the form of a sum of the n elements, let us make the transformation of the effective n -particle potentials for summing of these series:

$$\begin{aligned}
 M_2^{-1}(\mathbf{q}, -\mathbf{q}) \equiv M_2^{-1}(0) + [M_2^{-1}(\mathbf{q}, -\mathbf{q}) - M_2^{-1}(0)], & \tag{8.23} \\
 \tilde{\mu}_3(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 | \mu) \left\{ \prod_{i=1}^3 \tilde{\mu}_2(\mathbf{q}_i, -\mathbf{q}_i | \mu) \right\}^{-1} = \delta_{\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3, 0} \left\{ M_3(0) M_2^{-3}(0) + \right. \\
 \left[\sum_{i=1}^3 M_3(\mathbf{q}_i, -\mathbf{q}_i, 0) M_2^{-1}(0) M_2^{-2}(\mathbf{q}_i, -\mathbf{q}_i) - 3M_3(0) M_2^{-3}(0) \right] + \\
 \left[M_3(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) \prod_{i=1}^3 M_2^{-1}(\mathbf{q}_i, -\mathbf{q}_i) - \right. \\
 \left. \left. \sum_{i=1}^3 M_3(\mathbf{q}_i, -\mathbf{q}_i, 0) M_2^{-1}(0) M_2^{-2}(\mathbf{q}_i, -\mathbf{q}_i) + 2M_3(0) M_2^{-3}(0) \right] \right\}, \dots
 \end{aligned}$$

As a result the first element does not depend on the vector \mathbf{q} , the second one depends only on the vector \mathbf{q}_i , the third one depends on the two vectors \mathbf{q}_i and \mathbf{q}_j , etc. Let us represent $F^{(2)}[f_1]$ in the form of series

$$F^{(2)}[f_1] = \sum_{j=1}^{\infty} F_j^{(2)}, \tag{8.24}$$

where $F_1^{(2)}$ is the series sum and is built on components (8.23), which does not depend on \mathbf{q} . $F^{(2)}_2$ is the series sum, built on components, which

depend on \mathbf{q}_i and \mathbf{q}_j , etc. Traditionally the theory of inhomogeneous models has been developed mostly in R -representation

$$F + F_1^{(2)} = \int d\mathbf{r} g_0(\mathbf{r}), \quad g_0(\mathbf{r}) = n(\mathbf{r}) f_e(n_{\mathbf{r}}), \quad (8.25)$$

where $f_e(n_0) = N^{-1} F[n_0]$ is the free energy per one particle of a reference system with density n_0 . In this way, the element $F_1^{(2)}$ brings us to the local density approximation [47]. Other terms of the series (8.24) can be represented in the form

$$F_j^{(2)} = \frac{1}{j!} \int \dots \int d\mathbf{r}_1 \dots d\mathbf{r}_j \Delta n(\mathbf{r}_1) \dots \Delta n(\mathbf{r}_j) \times \quad (8.26)$$

$$\sum_{\mathbf{q}_1, \dots, \mathbf{q}_{j-1}} \tilde{K}_j(\mathbf{q}_1, \dots, \mathbf{q}_{j-1} | n(\mathbf{r}_j)) \exp \left\{ i \sum_{m=1}^{j-1} (\mathbf{q}_m, \mathbf{r}_m - \mathbf{r}_j) \right\}.$$

The kernels of (8.24) are determined by many-particle polarization functions of the reference system in the local approximation. Let us represent the expressions of the lower order kernels $\tilde{K}_2(\mathbf{q}, -\mathbf{q} | n)$ and $\tilde{K}_3(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1 - \mathbf{q}_2 | n)$

$$\tilde{K}_2(\mathbf{q}, -\mathbf{q} | n) = K_2(\mathbf{q}, -\mathbf{q} | n) - K_2(0, 0 | n), \quad (8.27)$$

$$K_2(\mathbf{q}, -\mathbf{q} | n) = M_2^{-1}(\mathbf{q}, -\mathbf{q} | n) = [\tilde{\mu}_2^0(\mathbf{q}, -\mathbf{q} | n)]^{-1} - V_{\mathbf{q}} V^{-1} G(\mathbf{q} | n),$$

$$\tilde{K}_3(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1 - \mathbf{q}_2 | n) = K_3(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1 - \mathbf{q}_2 | n) + 2K_3(0, 0, 0 | n) - K_3(\mathbf{q}_1, -\mathbf{q}_1, 0 | n) - K_3(\mathbf{q}_2, -\mathbf{q}_2, 0 | n) - K_3(\mathbf{q}_1 + \mathbf{q}_2, -\mathbf{q}_1 - \mathbf{q}_2, 0 | n),$$

$$K_3(\mathbf{q}_1, \mathbf{q}_2, -\mathbf{q}_1 - \mathbf{q}_2 | n) = -V^2 \tilde{\mu}_3^0(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 | n) \times \delta_{\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3, 0} \times$$

$$\prod_{i=1}^3 \tilde{\mu}_2^{-1}(\mathbf{q}_i, -\mathbf{q}_i | n).$$

In such a manner, the free energy of the inhomogeneous model can be represented in the form

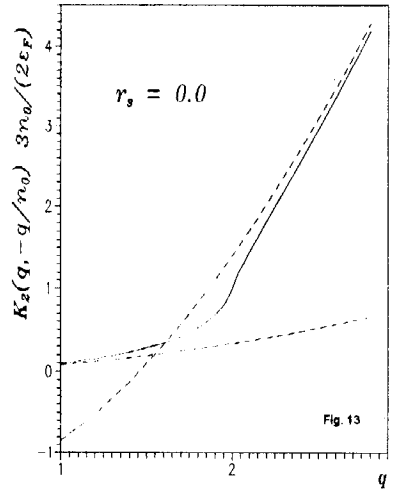
$$\tilde{F} = F_{cl} + g[n], \quad (8.28)$$

where $g[n]$ is the universal density functional of a reference system, which does not depend on the potential $V_1(\mathbf{r})$:

$$g[n] = \int d\mathbf{r} g_0(\mathbf{r}) + \sum_{j \geq 2} F_j^{(2)}[n] + \dots \quad (8.29)$$

The kernels $\tilde{K}_j(\mathbf{q}_1, \dots, \mathbf{q}_j | n(\mathbf{r}))$ have been studied in [50]. The expansion $\tilde{K}_j(\mathbf{q}_1, \dots, \mathbf{q}_j | n(\mathbf{r}))$ in the powers of variables \mathbf{q}_n occurred on the small and medium value region wave vector. This expansion originates the gradient expansion [49]. But these expansion takes place only in the segment $|\mathbf{q}_n| \leq 2k_F$. The kernel $\tilde{K}_2(\mathbf{q}, -\mathbf{q} | n_0)$ and its approximation are shown for the case $r_s = 0$ in Fig. 13. The extraction of the kernel part method has been proposed in works [49].

Using the asymptotical expansion of functions $\tilde{\mu}_2^0(\mathbf{q}, -\mathbf{q} | \mu)$ and local-field correction function, let us show it by the example of the kernel $\tilde{K}_2(\mathbf{q}, -\mathbf{q} | n_0)$



$$G(\mathbf{q} | n_0) \rightarrow \begin{cases} \gamma(n_0)q^2 + \alpha(n_0)q^4 + \beta(n_0)q^6 + \delta(n_0)q^8 + \dots, & q \ll 1, \\ G_{(\infty | n_0)} + \frac{G_2(n_0)}{q^2} + \dots, & q \gg 1, \end{cases} \quad (8.30)$$

where $q \equiv |\mathbf{q}|k_F^{-1}$. The asymptotic

$$\tilde{K}_2(\mathbf{q}, -\mathbf{q} | n_0) \xrightarrow{q \rightarrow \infty} R_2(\mathbf{q}, -\mathbf{q} | n_0) = \frac{\epsilon_F}{2n_0} \left[q^2 - \frac{32}{15} \right] + \frac{4\pi e^2}{k_F^2} \gamma(n_0) \quad (8.31)$$

enables to use the numerical method for its Fourier transform computing. Let us extract the singularity in the $\tilde{K}_2(\mathbf{q}, -\mathbf{q} | n_0)$ and represent it as a sum of two elements

$$\begin{aligned} \tilde{K}_2(\mathbf{q}, -\mathbf{q} | n_0) &= R_2(\mathbf{q}, -\mathbf{q} | n_0) + \Phi_2(\mathbf{q}, -\mathbf{q} | n_0), \\ \Phi_2(\mathbf{q}, -\mathbf{q} | n_0) &\equiv K_2(\mathbf{q}, -\mathbf{q} | n_0) - K_2(0, 0 | n_0) - R_2(\mathbf{q}, -\mathbf{q} | n_0). \end{aligned} \quad (8.32)$$

The residual kernel $\Phi_2(\mathbf{q}, -\mathbf{q} | n_0)$ is shown for the $r_s = 0, 4, 10$ in Fig. 14. It has the regular asymptotic

$$\frac{3n_0}{2\epsilon_F} \Phi_2(\mathbf{q}, -\mathbf{q} | n_0) \rightarrow \begin{cases} 4 \left[\frac{2}{5} - r_s \frac{\gamma(n_0)}{\pi \eta} \right] - q^2 \left[\frac{2}{3} + \frac{4r_s}{\pi \eta} \beta(n_0) + \dots \right] & q \ll 1, \\ -\frac{1}{q^2} \left[\frac{96}{105} + \frac{4r_s}{\pi \eta} G_{(\infty | n_0)} \right] + \dots & q \gg 1 \end{cases} \quad (8.33)$$

and corresponds the negative screen potential in the region of large \mathbf{q} . The sign of the expression can exchange at the region $q \leq 1$ under the different value of coupling parameter r_s . The detailed characteristic of this kernel is the deep minimum near the point $q = 2k_F$. The Fourier transform can be calculated by numerical method for the $\Phi_2(\mathbf{r}|n_0)$ in the form (8.32)

$$\Phi_2(\mathbf{r}|n_0) = \frac{1}{V} \sum_{\mathbf{q}} \Phi_2(\mathbf{q}, -\mathbf{q}|n_0) \exp(i\mathbf{q}\mathbf{r}). \tag{8.34}$$

As one can see (Fig.14) $\Phi_2(\mathbf{r}|n_0)$ has the character of an attractive screen potential with Coulomb asymptotic at small distance

$$\Phi_2(\mathbf{r}|n_0) \simeq -\frac{\varepsilon_F}{\mathbf{r}k_F} \frac{\pi}{2} \left\{ \frac{96}{105} + \frac{4r_s G'_\infty(n_0)}{\pi\eta} \right\} + \dots \tag{8.35}$$

The oscillations are caused by the divergence of $\Phi_2(\mathbf{r}|n_0)$ near the point $q = 2k_F$. The divergence component $R_2(\mathbf{r}|n_0)$ has the following analytical form

$$R_2(\mathbf{r}|n_0) = \left\{ \frac{\hbar^2}{4mn_0} \left[(\nabla_1, \nabla_2) - \frac{32}{15} k_F^2 \right] + \frac{4\pi e^2}{k_F^2} \gamma(n_0) \right\} \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{8.36}$$

In this way, $F_2^{(2)}[n]$ consists of the two components: the correction to the local density approximation and two-particle component

$$F_2^{(2)} = \int d\mathbf{r} \left\{ \frac{\hbar^2}{8m} n_0 \left| \frac{\nabla n}{n} \right|^2 - \frac{\hbar^2}{m} \frac{4}{15} (3\pi^2)^{2/3} \frac{(\Delta n)^2}{n^{1/3}} + (\Delta n)^2 \frac{2\pi e^2}{3\pi^2)^{2/3}} \frac{\gamma(n)}{n^{2/3}} \right\} + \frac{1}{2!} \int \int d\mathbf{r}_1 d\mathbf{r}_2 \Delta n(\mathbf{r}_1) \Delta n(\mathbf{r}_2) R_2(\mathbf{r}_1 - \mathbf{r}_2 | n(\mathbf{r}_2)). \tag{8.37}$$

The three-particle kernel $\tilde{K}_3(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3|n)$ has been studied both by the gradient expansion method and by the divergence component extracting method in [49].

9. Generalization to related models

Reference system approach, which was developed for electron-liquid theory, can be simply generalized for the case of a Fermi-system with a local interaction between particles [51-53]

The model with many-particle local interaction was investigated in paper [2]. Any system of one-particle functions $\{\varphi_\sigma(\mathbf{r})\}$ is used. On this basis the Hamiltonian has the form

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \sum_{\sigma,s} \varepsilon_\sigma a_{\sigma,s}^\dagger a_{\sigma,s}, \tag{9.1}$$

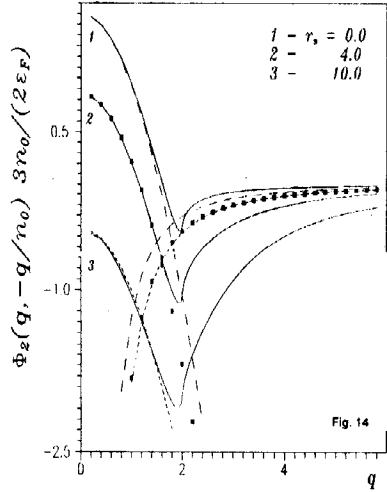


Fig. 14

$$\hat{V} = \sum_{n \geq 1} (n!)^{-1} \times \sum_{\sigma_1, \dots, \sigma_n, s_1, \dots, s_n} V_n(\sigma_1, \dots, \sigma_n) a_{\sigma_1, s_1}^+ a_{\sigma_2, s_2}^+ \dots a_{\sigma_n, s_n}^+ a_{\sigma_{n+1}, s_{n+1}} \dots a_{\sigma_{2n}, s_{2n}}.$$

Fermi-operators $a_{\sigma, s}$ correspond to functions $\varphi_{\sigma}(\mathbf{r})$ and spin- projection $s = \pm \frac{1}{2}$ and $V_n(\sigma_1, \dots, \sigma_n)$ are the matrix elements of n -particle interaction potentials on this basis. For representation S -matrix in the collective variables we shall use Fourier series for interaction potentials. As a result, we obtain the spectral representation of particle density operator

$$\hat{\rho}_x = \sum_{s, \nu^*} \sum_{\sigma_1, \sigma_2} R_{\sigma_1, \sigma_2}(\mathbf{q}) a_{\sigma_1, s}^+(\nu^* + \nu) a_{\sigma_2, s}(\nu^*), \quad (9.2)$$

$$R_{\sigma_1, \sigma_2}(\mathbf{q}) = \int \varphi_{\sigma_1}(\mathbf{r}) e^{i\mathbf{q}\mathbf{r}} \varphi_{\sigma_2}(\mathbf{r}) d\mathbf{r}.$$

The partition function in terms of the collective variables can be represented in the form

$$Z(\mu) = Z_0(\mu) \int (d\rho) J(\rho) \exp[-V(\rho)], \quad (9.3)$$

$$V(\rho) = \sum_{n \geq 1} (n!)^{-1} (\beta V)^{1-n} \sum_{x_1, \dots, x_n} \delta_{\nu_1 + \dots + \nu_n, 0} V_n(\mathbf{q}_1, \dots, \mathbf{q}_n) \rho_{x_1}, \dots, \rho_{x_n}.$$

Here $Z_0(\mu)$ is the partition function of the reference system with Hamiltonian \hat{H}_0 . The transition Jacobian is determined by the relation (2.22) and function $J(\rho)$ is formed on the operators (9.2).

In the work [53] the dependence of a local field correction function on type of the interaction potential between particles has been investigated on the example of the fermion model with a short range interaction potential. It was modelled by the Yukawa potential $V(\mathbf{r}) = A e^2 r^{-1} \exp(-\xi r/a_0)$, where parameters A and ξ have the quantum- mechanic sense, but not statistical. The electron-liquid model is the partial case of such model at $A = 1$, $\xi = 0$. The Fourier-component of the potential in this case is equal zero at $q = 0$. Being the ratio of an average potential energy of a particle to its kinetic one, the coupling parameter in that model has the form

$$\left(\frac{3}{5} \epsilon_F\right)^{-1} \frac{N}{V} \int d\mathbf{r}' V(\mathbf{r} - \mathbf{r}') |F_2^{id}(\mathbf{r}) - 1| = \quad (9.4)$$

$$\frac{2\pi A^2 e^2}{3\epsilon_F} \frac{N}{V} \int dr r \exp\{-\xi r/a_0\} = \frac{10A^2 r_s m^*}{\xi^2} f(\xi^*),$$

$$f(\xi^*) = (\xi^*)^{-2} \{1 - [1 + \xi^*] \exp(-\xi^*)\}.$$

Here $r_s = r_0/a_0$ is the coupling parameter for the non-ideal electron liquid, $\xi^* \equiv \frac{\xi r_s}{\eta}$, $\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$, $r_0 = (3V/4\pi N)^{1/3}$, $m^* = m/m_0$ (m_0 - is a mass of the electron), $F_2^{id}(\mathbf{r})$ is the binary distribution function of ideal Fermi system (without interaction), $R_F = k_F^{-1}$ is the correlation function radius of the degenerate ideal fermion system.

In order to obtain the equation for local field correction function we must replace in (5.4) Fourier-image of the Coulomb potential with

$$\tilde{V}_{\mathbf{q}} = 4\pi e^2 A \left\{ \mathbf{q}^2 + (\xi/a_0)^2 \right\}^{-1}. \quad (9.5)$$

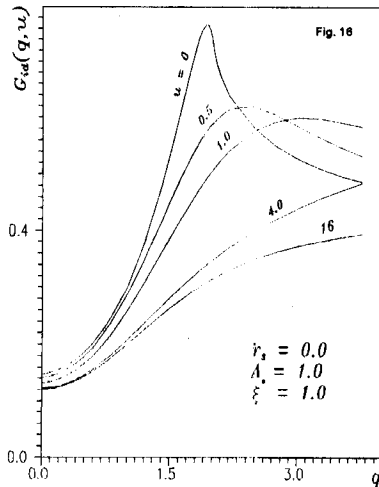
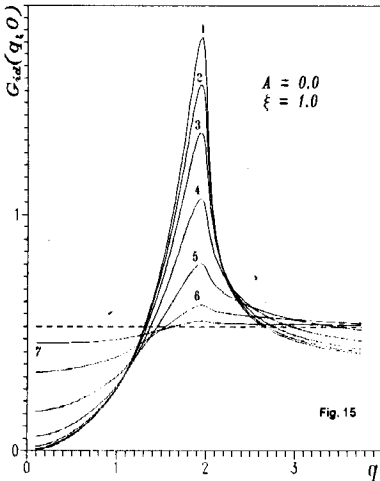
For the case of weakly non-ideal system we have the following local field correction function representation

$$\begin{aligned} \tilde{G}'(x) = & -\frac{1}{4\pi} \left\{ \mathbf{q}^2 + \left(\frac{\xi r_s}{\eta}\right)^2 \right\} I_{2,0}^{-2}(q, u) \times \\ & \int_0^\infty dq_1 q_1^3 \int_0^\infty du \int_{-1}^{+1} dt I_{4,1}(q, q_1; u, u_1; t) \times \\ & \left\{ q_1^2 + \left(\frac{\xi r_s}{\eta}\right)^2 + 4\pi A \alpha_s I_{2,0}(q_1, u_1) \right\}^{-1}, \\ & \alpha_s = r_s (\pi^2 \eta)^{-1}, \quad \eta = (9\pi/4)^{1/3}. \end{aligned} \tag{9.6}$$

The coupling parameter r_s in the electron liquid model is slightly formal. It has only statistical sense and is defined at $T = 0$ by the particle density. The model with a short-range potential has both r_s parameter and two other parameter A and ξ . We can realize the limit of weakly non-ideality through this parameters, assuming $A \rightarrow 0$. In this way for the model with Yukawa interaction potential we may realize three weakly non-ideal limit by three different pathways

- 1). $r_s \rightarrow 0$ at $A, \xi^* = \text{const}$,
- 2). $A \rightarrow 0$ at $r_s, \xi^* = \text{const}$,
- 3). $\xi^* \rightarrow \infty$ at $A, r_s = \text{const} \dots$

In the first case $\tilde{G}(x)$ is equal to the local field correction function $G'_{id}(x)$



for the electron liquid. In the second case the dependence on the parameter ξ remains and only by the following transition $r_s \rightarrow 0$ function $\tilde{G}(x)$ set to $G'_{id}(x)$. In the third case $\tilde{G}(x) = 1/2$ is independent on potential parameters, but it is completely different from $\tilde{G}(x)$ in the region of the small momenta. This peculiarity of function $\tilde{G}(x)$ is illustrated by Fig. 15, 16. The results of numerical calculation $\tilde{G}(x)$ in the approximation (5.9)

are demonstrated in Fig. 17, 18, 19. As is known, local correction function for electron liquid has the following asymptotic [25,26,27]:

$$G(q, u) \rightarrow \begin{cases} \gamma(u, r_s)(q/k_F)^2 & \text{at } q \ll k_F, \\ G_\infty(r_s) & \text{at } q \gg k_F, \end{cases} \quad (9.8)$$

Local correction function for model with sort range potential has slightly other asymptotic

$$\tilde{G}(q, u) \rightarrow \begin{cases} a(u, r_s, \xi^*)[(q/k_F)^2 + (\xi^*)^2] & \text{at } q \ll k_F, \\ G_\infty(A, r_s, \xi^*) & \text{at } q \gg k_F. \end{cases} \quad (9.9)$$

The potential character causes a distinctive behavior of $\tilde{G}(x)$ in the small wave vectors region: $\tilde{G}(x)$ tends to non-zero value around the point $q \rightarrow 0$. In the region of medium and great wave vectors $\tilde{G}(x)$ is similar to the local correction function for an electron liquid. The short-wave asymptotic $\tilde{G}(x)$ increases with the rise of r_s and A parameters and is irrespective of ξ parameter.

In the paper [53] the dependence of ground state energy and binary distribution function on model parameters was investigated on the basis of $\tilde{G}(x)$.

The generalization of the reference system approach to the case of many-electron Fermi-system models is obvious. Let us consider, model with the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \sum_{\mathbf{k}, s} \sum_{1 \leq \alpha \leq M} \epsilon_{\mathbf{k}}^\alpha C_{\mathbf{k}, s}^{+\alpha} C_{\mathbf{k}, s}^\alpha, \quad (9.10)$$

$$\hat{V} = (2V)^{-1} \sum_{\alpha_1, \alpha_2} \sum_{\mathbf{k}_1, \mathbf{k}_2, s_1, s_2} \sum_{\mathbf{q}} V_{\alpha_1 \alpha_2}(\mathbf{q}) C_{\mathbf{k}_1 - \mathbf{q}, s_1}^{+\alpha_1} C_{\mathbf{k}_2 - \mathbf{q}, s_2}^{+\alpha_2} C_{\mathbf{k}_2, s_2}^{\alpha_2} C_{\mathbf{k}_1, s_1}^{\alpha_1},$$

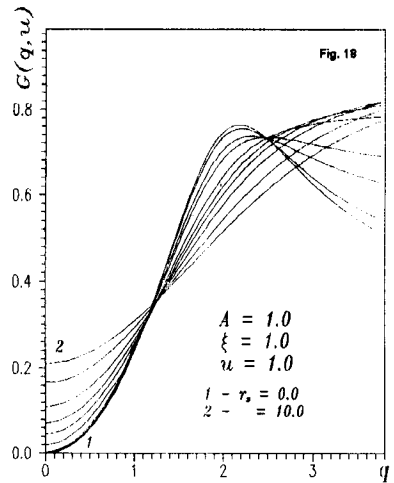
where $\epsilon_n^\alpha = \hbar k^2/2m_\alpha$, and $C_{\mathbf{k}, s}^\alpha$ are the second quantization operators in the plane waves basis for α -species Fermions. The transition to the frequency representation is the same, as in the case of one-species system, but the particle density operators for each sort we are to be taken into account

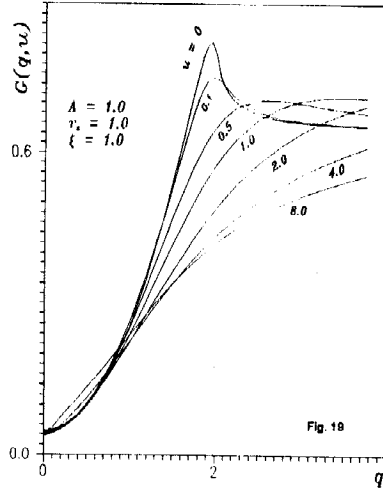
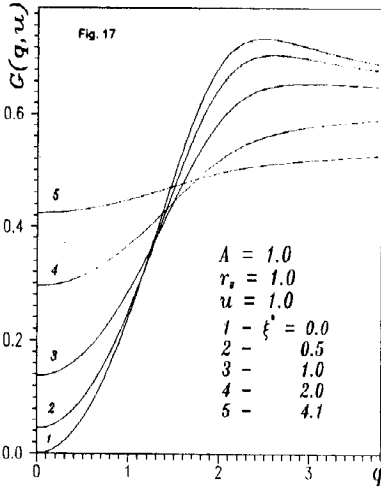
$$\hat{\rho}_x^\alpha = \sum_{\mathbf{k}, s; \nu^*} C_{\mathbf{k} + \mathbf{q}, s}^{+\alpha} (\nu^* + \nu) C_{\mathbf{k}, s}^\alpha (\nu^*), \quad (9.11)$$

$$C_{\mathbf{k}, s}^\alpha (\nu^*) = \int_0^\beta \Psi_{\nu^*}^\alpha(\beta_1) C_{\mathbf{k}, s}^\alpha(\beta_1) d\beta_1.$$

As a result, we have such representation of the partition function in the collective variables

$$Z(\mu) = \prod_{1 \leq \alpha \leq M} Z_0^\alpha(\mu) \int (d\rho) \prod_\alpha J^\alpha(\rho) \times \quad (9.12)$$





$$\exp \left\{ -(2\beta V)^{-1} \sum_x \sum_{\alpha_1, \alpha_2} V_{\alpha_1, \alpha_2}(\mathbf{q}) \rho_x^{\alpha_1} \rho_x^{\alpha_2} \right\}.$$

Here $Z_0^\alpha(\mu)$ is the partition function of a reference α -subsystem, and $J^\alpha(\rho)$ is the partial transition Jacobian

$$J^\alpha(\rho) = \int (d\omega) J^\alpha(\omega | \mu_\alpha) \exp \left\{ i\pi \sum_x \omega_x^\alpha \rho_x^\alpha \right\}, \quad (9.13)$$

$$J^\alpha(\omega | \mu_\alpha) = \exp \left\{ \sum_{n \geq 2} D_n^\alpha(\omega | \mu_\alpha) \right\},$$

$$D_n^\alpha(\omega | \mu_\alpha) = \beta(n!)^{-1} \sum_{x_1, \dots, x_n} \omega_{x_1}^\alpha \dots \omega_{x_n}^\alpha \tilde{\mu}_n^\alpha(x_1, \dots, x_n | \mu_\alpha),$$

where $\tilde{\mu}_n^\alpha(x_1, \dots, x_n | \mu_\alpha)$ are the correlation functions of the reference α -subsystem

In the work [51] the functional representation for partition and distribution function of two-sort Fermi system are received. The general relation are illustrated on the example of the simple electron-hole liquid model in semiconductors. It consists of N electrons and N holes with static screen Coulomb interaction $V_{aa}(\mathbf{q}) = V_{bb}(\mathbf{q}) = V_q \epsilon_0^{-1}$, $V_{ab}(\mathbf{q}) = -V_q \epsilon_0^{-1}$ ($V_q = \frac{4\pi e^2}{q^2}$).

The typical scales for this models are exciton's Bohr-radius $a_{ex} = \epsilon_0 \hbar^2 \times (\mu e^2)^{-1}$ and exciton's Rydberg $E_{ex} = e^2 (2\epsilon_0 a_{ex})^{-1} = \mu e^4 (2\epsilon_0^2 \hbar^2)^{-1}$ (here $\mu^{-1} = m_e^{-1} + m_p^{-1}$) [54]. In the region of low temperature both subsystems are degenerated and their Fermi-impulse are the same. In contrast to electron liquid two-component system has three dimensionless parameters $m_c^* = m_c/m_0$ ($c = e, p$) and coupling parameter

$$r_s^* = \left(\frac{3V}{4\pi N} \right)^{1/3} a_{ex}^{-1} = r_s \mu (\epsilon_0 m_0)^{-1}, \quad (9.14)$$

where m_0 is a mass of the free electron, r_s is the coupling parameter for the electron liquid at the same particle density. As it is known, the concentration of the electron-hole pairs in such semiconductor as Ge or Si corresponds to $r_s^* = 1 \div 2$ [54]. In view point of the electron liquid theory such model is weakly non ideal. However, in (5.16) the fact, that the system has two-component is not taken into account. As it is shown in work [51], the real coupling parameter is

$$r_s^{**} = r_s^*(m_e + m_p)^2(m_e m_p)^{-1}. \quad (9.15)$$

In this way, $r_s^{**} \geq 4r_s^*$, therefore the electron-hole liquid is strongly non-ideal and local-field effects play an essential role.

Integral characteristics of a model are determined through the pair correlation function "charge - charge", which caused by charge symmetry of model. In particular, for the free energy we have the following representation

$$F = F_0 + (2\beta V \epsilon_0)^{-1} \sum_{\mathbf{q}} V_{\mathbf{q}} \int_0^1 \mu_2^-(x, -x|\lambda) d\lambda. \quad (9.16)$$

Here F_0 is the free energy of electron's and hole's basis subsystem, $\mu_2^-(x, -x|\lambda)$ is the mentioned earlier correlation function of the model with interaction potential $\lambda v_{c_1 c_2}(\mathbf{q})$

$$\begin{aligned} \mu_2^-(x, -x|\lambda) &= \frac{1}{\beta} \langle \hat{R}_x \hat{R}_{-x} \rangle_{H_\lambda} = \frac{1}{\beta} \langle \tilde{T} \hat{S}_\lambda \rangle_{H_0}^{-1} \langle \tilde{T} \{ \hat{S}_\lambda \hat{R}_x \hat{R}_{-x} \} \rangle_{H_0}, \\ \hat{S}_\lambda &\equiv \exp \left\{ -(2\beta V \epsilon_0)^{-1} \lambda \sum_x V_{\mathbf{q}} \hat{R}_x \hat{R}_{-x} \right\}, \\ \hat{R}_x &= \sum_{\mathbf{k}, s, \nu^*} \left\{ a_{\mathbf{k}+\mathbf{q}, s}^+(\nu^* + \nu) a_{\mathbf{k}, s}(\nu^*) - b_{\mathbf{k}+\mathbf{q}, s}^+(\nu^* + \nu) b_{\mathbf{k}, s}(\nu^*) \right\}. \end{aligned} \quad (9.17)$$

In so doing $a_{\mathbf{k}, s}$ are the electron operators and $b_{\mathbf{k}, s}$ are the hole operators.

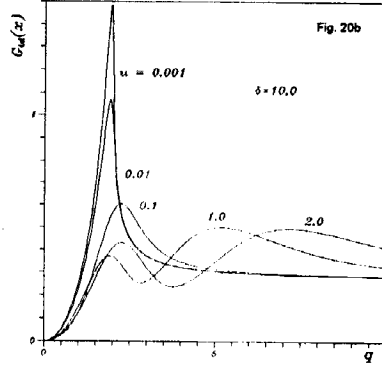
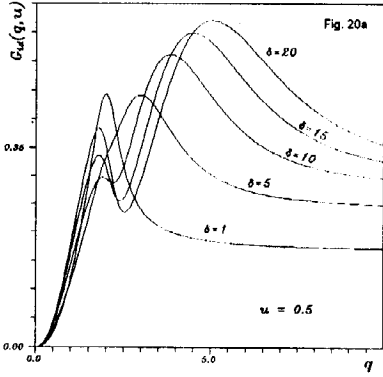
Similar to the electron-liquid model in work [51] the local-field correction function of two-species model which determined the function (9.19) are introduced

$$\begin{aligned} \mu_2^-(x, -x|1) &\equiv \mu_2^-(x, -x) = M_2(x, -x) \left\{ 1 + \frac{V_{\mathbf{q}}}{V \epsilon_0} M_2(x, -x) \right\}^{-1} = \\ \tilde{\mu}_2^0(x, -x|\mu_0) &\left\{ 1 + (V \epsilon_0)^{-1} V_{\mathbf{q}} \tilde{\mu}_2^0(x, -x|\mu_0) [1 - G_{ep}(x)] \right\}^{-1}. \end{aligned} \quad (9.18)$$

The local correction function is determined through correlation function of reference system, which are built up on "sort" functions

$$\tilde{\mu}_n^0(x_1, \dots, x_n|\mu_0) = \tilde{\mu}_n^e(x_1, \dots, x_n|\mu_e) + (-1)^n \tilde{\mu}_n^p(x_1, \dots, x_n|\mu_p). \quad (9.19)$$

Function $G_{ep}(x)$ is shown in Fig. 20a, 20b in Geldart-Taylor approximation. As one can see the dynamic correction of two-sort model is essentially different from that of an electron liquid. It has two maximum, which are caused by the effective masses of the electron and the hole ($\delta = m_p^*/m_e^*$). In Geldart-Taylor approximation $G_{ep}(x)$ depends only on effective particles



mass and is independent on coupling parameters. We have the following asymptotic at great values of wave vector

$$G_{ep}(x) \Rightarrow \frac{1}{3} [(m_p^*)^2 + (m_e^*)^2] [m_p^* + m_e^*]^{-2}, \quad (9.20)$$

and at static long wave limit

$$G_{ep}(x) \Rightarrow \frac{q^2}{4} [(m_p^*)^2 + (m_e^*)^2] [m_p^* + m_e^*]^{-2}, \quad (9.21)$$

In the work [52] the function $G_{ep}(x)$ in the approximation similar to (5.10) is investigated. The results of this calculation are shown in Fig. 21a, 21b, where $G_{ep}(x)$ is illustrated as a function of wave vectors under given frequencies or under given effective masses.

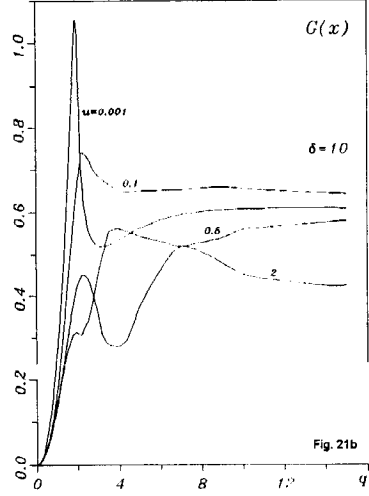
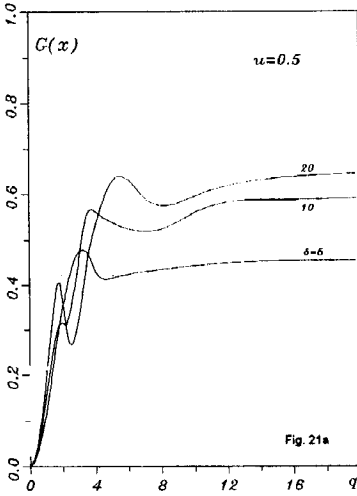
The energy of metal phase of the model at $T = 0K$ is shown in the traditional form

$$E = NE_{ex}\epsilon(r_s^*); \quad \epsilon(r_s^*) = \epsilon_0(r_s^*) + \epsilon_{HF}(r_s^*) + \epsilon_c(r_s^*). \quad (9.22)$$

Where $\epsilon_0(r_s^*) = \frac{3}{5}\eta^2(r_s^*)^{-2}$ is the reference system energy pair in Rydberg Ry ; $\epsilon_{HF}(r_s^*) = -\frac{3}{\pi}\eta(r_s^*)^{-1}$ is the Hartree-Fock energy which is independent of the particle mass, $\epsilon_c(r_s^*)$ is the correlation energy (energy of Coulomb correlations).

In the ordinary RPA we obtain the following expression for the correlation energy (when $M_2(x, -x) = \tilde{\mu}_2^0(x, -x|\mu_0)$)

$$\begin{aligned} \epsilon_c^{RPA}(r_s^*) &= -\frac{3}{2\pi}\eta^2(r_s^{**})^{-2}(1+\delta)^3\delta^{-1} \times \\ &\int_0^\infty dq q^3 \int_0^\infty du^* \left\{ \chi(q, u^*|\delta) - \ln[1 + \chi(q, u^*|\delta)] \right\}, \\ \chi(q, u^*|\delta) &= 4r_s^{**}(\pi\eta)^{-1}q^{-2}(1+\delta)^{-1}[I_{2,0}(q, u^*) + \delta I_{2,0}(q, \delta u^*)]. \end{aligned} \quad (9.23)$$



Here $q \equiv |\mathbf{q}|k_F^{-1}$, $u^* \equiv um_c^*$, $u = \nu(2\epsilon_F q)^{-1}$ are the dimensionless variables, $\epsilon_F = \hbar^2 k_F^2 / 2m_0$. The correlation energy and other characteristics in Geldart-Taylor approximation [52]

$$M_2(x, -x) = \mu_2^0(x, -x|\mu) - (2\beta V \epsilon_0)^{-1} \sum_{x_1} V_{\mathbf{q}_1} \tilde{\mu}_{4,1}^0(x, -x, x_1, -x_1|\mu_0), \tag{9.24}$$

and on the basis of the local field correction function are investigated. These results are represented in Fig. 22. (in Rydberg per pair). The dotted curve correspond to RPA, the dashed curve to approximation (9.24). The correlation energy, calculated on local correction function basis, is shown as solid line. The metal phase in this approximation exists at $\delta \geq 20$.

Conclusion

Caused by Pouli's principle, strong many-particle correlations in the degenerate Fermi systems are the physical reason, which affords the effective use of the reference system approach. Being formulated in terms of n-particle correlation functions of a reference system, this approach permit the integral, local and one-particle characteristics of the interacting many-particle system models to be computed. The principle of the correlation attenuating decreases the higher order correlation effects influence which are connected with the high group of particles and affords the expansion in the reference system correlation function convergence.

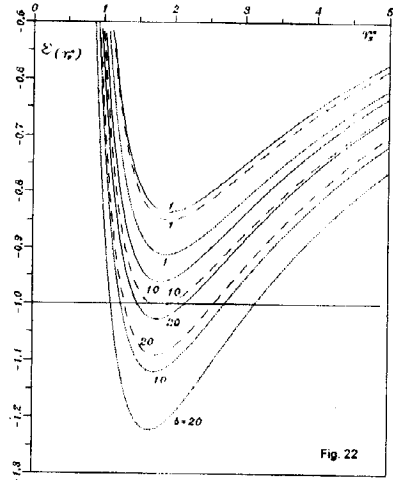
The reference system approach permits many mathematical difficulties to bypass, which are created in the interacting system characteristic by the ordinary perturbation theory having been formulated in the one-particle characteristic terms.

Having been worked out in the electron liquid theory, the reference system approach has been generalized to some models. Generally speaking, this universal method can be used for the description of interacting many-particle systems with the local interaction potential.

If the reference system is chosen, this approach immediately could be used for the description of the spatially inhomogeneous and limited system models. It gives the chance for correct obtaining of the density functional of a reference system in the case of the different phase states.

Unfortunately there are indefiniteness of a reference system choice what is disadvantage of this method.

A reference system approach, operating with the most difficult objects - many-particle correlation functions, can be used only in the symbiosis with the modern computers. In this symbiosis it is the powerful method for the solving of the vital problems of the statistical physics.



Rs	1	2	3	4	5	6	10
RPA	157.6	123.6	105.5	93.6	84.95	78.24	61.32
[19]	124.0	92.0	75.0	64.0	56.0	50.0	36.0
[20]	112.0	89.0	75.0	65.0	58.0	52.0	35.0
[21]	117.4	86.7	71.1	61.0	53.8	48.3	35.0
[22]	125.0	91.9	74.3	62.5	54.4	-	-
[23]	123.0	91.7	75.1	64.4	56.8	-	-
[24]	122.0	90.4	73.8	63.4	56.0	50.5	37.0
[18]	120.0	90.2	-	-	56.3	-	37.2
*	123.8	102.6	87.7	78.5	72.6	66.7	53.4
**	119.7	89.3	72.9	62.1	54.2	48.0	32.6
[5]	120.0	90.3	74.4	64.1	56.6	50.9	36.7

Table 1.

Rs	8	9	10	11	12	13	14	15
T K	2.4	24.8	69.1	158.4	258.0	391.0	505.0	660.0

Table 2.

Rs	$-10^3 E_{corr}$	α/α_0	F	δ_1	δ_2
0.25	190.733	0.958057	0.4227	0.0013	-
0.50	153.942	0.915262	0.3625	0.0012	-
1.00	119.956	0.827466	0.2733	0.0007	0.0041
1.50	101.697	0.737089	0.2108	0.0002	-
2.00	89.5792	0.644442	0.1639	-0.0002	0.0066
2.50	80.6987	0.549751	0.1297	-0.0004	-
3.00	73.7980	0.453219	0.1022	-0.0004	-
3.50	68.2223	0.354972	0.0816	-0.0004	-
4.00	63.5892	0.255191	0.0652	-0.0003	-
4.50	59.6575	0.153945	0.0526	-0.0002	-
5.00	56.2654	0.051345	0.0426	0.0000	0.0013
5.50	53.2996	-0.052539	0.0349	0.0003	-
6.00	50.6781	-0.157639	0.0287	0.0006	-
6.50	48.3395	-0.263870	0.0241	0.0009	-
7.00	46.2369	-0.371187	0.0204	0.0012	-
7.50	44.3338	-0.479524	0.0174	0.0015	-
8.00	42.6011	-0.588832	0.0148	0.0019	-
8.50	41.0158	-0.698954	0.0157	0.0022	-
9.00	39.5583	-0.809946	0.0109	0.0026	-
9.50	38.2138	-0.921580	0.0097	0.0030	-
10.00	36.9660	-1.033848	0.0092	0.0033	0.0068

Table 3.

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БАЗИСНИЙ ПІДХІД В ТЕОРІЇ ЕЛЕКТРОННОЇ РІДИНИ

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Викладено основи базисного підходу в теорії електронної рідини, який є ренормалізованою теорією збурень в термінах n -частинкових кореляційних функцій деякої простішої моделі (базисної системи). Приведено функціональні представлення статистичної суми, термодинамічних та кореляційних функцій, а також одночастинкових характеристик моделі електронної рідини. Подано результати розрахунків енергетичних, структурних та діелектричних характеристик у парамагнітній та надпровідних фазах. Базисний підхід застосований до опису неоднорідних електронних систем. Він узагальнений також на споріднені моделі (двосортні вироджені системи ферміонів, модель ферміонів з короткосяжною відштовхуючою взаємодією, системи частинок з багаточастинковими взаємодіями, тощо).