Condensed Matter Physics, 2004, Vol. 7, No. 2(38), pp. 361–382

CONDENSED MATTER PHYSICS

Behaviour of a three-dimensional uniaxial magnet near the critical point in an external field

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Received March 26, 2004, in final form May 24, 2004

The behaviour of a three-dimensional magnet with a one-component order parameter near the critical point in a homogeneous external field is investigated. The calculations are performed in the case when the field and temperature are dependent and related by some expression (the system tends to the critical point along some trajectory). The high- and low-temperature regions in the vicinity of $T_{\rm c}$ ($T_{\rm c}$ is the phase transition temperature in the absence of an external field) are considered. It is shown that in the weak fields the system behaviour is described in general by the temperature variable, but in the case of the strong fields the role of the temperature variable is not dominant. The corresponding expressions for the free energy, susceptibility and other characteristics of the system are obtained for each of these regions.

Key words: *three-dimensional uniaxial magnet, collective variables, critical behaviour, external field*

PACS: *05.50.+q, 05.70.Ce, 64.60.Fr, 75.10.Hk*

1. Introduction

Despite the great successes in the investigation of three-dimensional ($3D$) Ising-like systems made by means of various methods (see, for example, [1–3]) the direct calculation of basic characteristics near the critical point for a $3D$ uniaxial magnet in the external field is still an important issue. Such a calculation is our aim in this research. Mathematical description is performed within the framework of the collective variables method. This method was successfully used in investigating the critical behaviour of various magnetic and nonmagnetic systems, including its applications to the descripiton of binary displacement alloys by Gurskii [4].

This paper is the continuation of the papers series [5–7], devoted to the investigation of the effect of an external field on a critical behaviour of the lattice spin systems. Near the critical point, the long-range correlations among the particles play the key role [8]. It is appropriate to use the non-Gaussian distributions of the order parameter fluctuations for such a description [9]. Exactly this approach makes it possible to find the main characteristics of the $3D$ statistic systems near the phase transition point in the absence of an external field without using the asymptotic series of the perturbation theory, which appear when the Gaussian distributions are used [10].

In the course of calculations, we will elaborate the simplest nontrivial model of the phase transition, described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{\mathbf{l},\mathbf{j} \in \Lambda} \Phi(r_{\mathbf{lj}}) \sigma_{\mathbf{l}} \sigma_{\mathbf{j}} - h \sum_{\mathbf{l} \in \Lambda} \sigma_{\mathbf{l}}, \tag{1}$$

where the interaction potential $\Phi(r_{\mathbf{lj}})$ is the positive function that depends only on the distance $r_{\mathbf{lj}} = |\mathbf{r}_{\mathbf{l}} - \mathbf{r}_{\mathbf{j}}|$ between particles, located at the N sites of the simple cubic lattice with the period c. The quantities $\sigma_{\mathbf{l}}$ are the operators of the z-component of the spin. They have the eigenvalues ± 1, h being the normalized magnetic field. Summation is performed in the volume of the periodicity ($V = Nc^3$)

$$\Lambda = \{\mathbf{l} = (l_x, l_y, l_z) | l_i = c n_i\,, \quad n_i = 1, 2, ..., N_i\,, \quad i = x, y, z\}\,, \tag{2}$$

with periodic boundary conditions. Here $N_i^3 = N$.

We use the following Fourier transform for the potential of interaction:

$$\Phi(k) = \begin{cases} \Phi(0)(1 - 2b^2 k^2), & \mathbf{k} \in \mathcal{B}_0, \\ \Phi_0 = \Phi(0)\bar{\Phi}, & \mathbf{k} \in \mathcal{B} \backslash \mathcal{B}_0, \end{cases} \tag{3}$$

where the wave vector $\mathbf{k}$ varies in the boundaries of the first Brillouin zone

$$\mathcal{B} = \Big\{ \mathbf{k} = (k_x, k_y, k_z) | k_i = -\frac{\pi}{c} + \frac{2\pi}{c} \frac{n_i}{N_i}\,, \quad n_i = 1, 2, ..., N_i\,, \quad i = x, y, z \Big\}, \tag{4}$$

b is some constant quantity. The constant $\bar{\Phi}$ corresponds to the part of the Fourier transform for the interaction potential, that is averaged with respect to large values of the $\mathbf{k} \in \mathcal{B} \backslash \mathcal{B}_0$ [10]. Such a model potential (3) is based on the fact that large values of the wave vector are nonessential for calculating its critical characteristics. The exclusion of self-interaction, which arises in such an approach, would cause only the shift of the free energy. This is not essential for other thermodynamic functions. The region of the wave vector values $\mathcal{B}_0$ in expression (3) is assigned in the following way:

$$\mathcal{B}_0 = \Big\{ \mathbf{k} = (k_x, k_y, k_z) | k_i = -\frac{\pi}{c_0} + \frac{2\pi}{c_0} \frac{n_i}{N_{0i}}\,, \quad n_i = 1, 2, ..., N_{0i}\,, \quad i = x, y, z \Big\}, \tag{5}$$

where the effective block lattice period $c_0 = c s_0$ is defined by the parameter s_0 of the model ($s_0 \geqslant 1$), $N_0 = N_{0x} N_{0y} N_{0z}$, $N_0 = s_0^{-d} N$. The dimension of the space $d = 3$.

The description of the model (1) near the critical point in the case of the dependence of the field on the temperature is the object of the given paper. This specific case is appropriate to the phase transitions in various systems, in particular, in the pseudospin-electron model [11]. In [12] it is shown that partition function of such a model can be represented by the partition function of the Ising model with an external longitudinal field, which depends on the temperature. Carrying out our calculations, we enter some parameter Δ, which relates the external field h with the reducing temperature τ ($\tau = (T - T_c)/T_c$, where T_c is the temperature of the phase transition at $h = 0$). The values of this parameter define the small (large) values of the field in comparison with the value of the limiting field. The latter corresponds to the condition of equality of the spontaneous moment and the moment induced by the field. The Landau approach with the temperature-dependence field is presented in Appendix.

2. Basic relations

The method for calculating the free energy of the model lattice system of the one-component spins near the critical point in the presence of an external field is suggested in [5–7]. The partition function of the model (1) with the potential (3) can be introduced in the form of the N_0-multiple integral with respect to the collective variables [5,10]:

$$Z = Z_0 \sqrt{2}^{N_0 - 1} \mathrm{e}^{\tilde{a}_0 N_0} \int (\mathrm{d}\eta)^{N_0} \exp\Bigg[-a_1 \sqrt{N_0} \eta_0 - \frac{1}{2} \sum_{\mathbf{k} \in \mathcal{B}_0} \mathrm{d}(k) \eta_{\mathbf{k}} \eta_{-\mathbf{k}} - \frac{a_3}{3!} N_0^{-1/2} \sum_{\substack{\mathbf{k}_1, \ldots, \mathbf{k}_3 \\ \mathbf{k}_i \in \mathcal{B}_0}} \eta_{\mathbf{k}_1} \ldots \eta_{\mathbf{k}_3} \delta_{\mathbf{k}_1 + \ldots + \mathbf{k}_3} - \frac{a_4}{4!} N_0^{-1} \sum_{\substack{\mathbf{k}_1, \ldots, \mathbf{k}_4 \\ \mathbf{k}_i \in \mathcal{B}_0}} \eta_{\mathbf{k}_1} \ldots \eta_{\mathbf{k}_4} \delta_{\mathbf{k}_1 + \ldots + \mathbf{k}_4} \Bigg]. \quad (6)$$

For the quantity $\mathrm{d}(k)$, we have the expression

$$\begin{aligned} \mathrm{d}(k) &= \tilde{a}_2 - \beta\Phi(0) + 2b^2 \beta\Phi(0) k^2, \\ \tilde{a}_2 &= a_2 + \beta\Phi_0. \end{aligned} \quad (7)$$

Here $\beta = 1/(kT)$ is the inverse thermodynamic temperature. The coefficients $\tilde{a}_0$, a_l and the quantity Z_0 are functions of h' ($h' = \beta h$). They are calculated in [5,7].

The step-by-step integration of the expression (6) is performed based on the method developed in [9,10]. We arrive at the expression [5,7]

$$Z = Z_0 Q_0 Q_1 \ldots Q_{n_p} j_{n_p+1} \left[Q(P^{(n_p)}) \right]^{N_{n_p+1}} I_{n_p+1}, \quad (8)$$

where the partial partition functions Q_n (n is number of the block structure) are given in [5], $j_{n_p+1} = \sqrt{2}^{N_{n_p+1} - 1}$. The expression for $Q(P^{(n-1)})$ is presented in [6], and

$$I_{n_p+1} = \int (\mathrm{d}\eta)^{N_{n_p}+1} \exp\Bigg[-\tilde{a}_1^{(n_p+1)} N_{n_p+1}^{1/2} \eta_0 - \frac{1}{2} \sum_{\mathbf{k} \in \mathcal{B}_p} \mathrm{d}_{n_p+1}(k) \eta_{\mathbf{k}} \eta_{-\mathbf{k}}$$

$$
\begin{aligned}
&- \frac{1}{3!} a_3^{(n_p+1)} N_{n_p+1}^{-1/2} \sum_{\substack{\mathbf{k}_1,\ldots\mathbf{k}_3 \\ \mathbf{k}_i \in \mathcal{B}_p}} \eta_{\mathbf{k}_1} \ldots \eta_{\mathbf{k}_3} \delta_{\mathbf{k}_1+\ldots+\mathbf{k}_3} \\
&- \frac{1}{4!} a_4^{(n_p+1)} N_{n_p+1}^{-1} \sum_{\substack{\mathbf{k}_1,\ldots,\mathbf{k}_4 \\ \mathbf{k}_i \in \mathcal{B}_p}} \eta_{\mathbf{k}_1} \ldots \eta_{\mathbf{k}_4} \delta_{\mathbf{k}_1+\ldots+\mathbf{k}_4} \Bigg]. \qquad (9)
\end{aligned}
$$

The coefficients $a_l^{(n)}$ are connected with the initial values of a_l through the recurrence relations [5]. The region of the wave vector values $\mathcal{B}_p$ is defined as

$$
\mathcal{B}_n = \Big\{ \mathbf{k} = (k_x, k_y, k_z) | k_i = -\frac{\pi}{c_n} + \frac{2\pi}{c_n} \frac{n_i}{N_{n_i}}, \qquad n_i = 1, 2..., N_{n_i}, \qquad i = x, y, z \Big\}
$$

when $n = n_p + 1$. Here $c_n = c_0 s^n$, $s \geqslant 1$, $N_{nx} N_{ny} N_{nz} = N_n$. The parameter s determines the rate of increasing the effective block lattices and corresponds to the renormalization group parameter. For the coefficient $d_{n_p+1}(k)$ from (9), we have

$$
\begin{aligned}
&\mathrm{d}_{n_p+1}(k) = \mathrm{d}_{n_p+1}(0) + 2\beta\Phi(0) b^2 k^2, \\
&\mathrm{d}_{n_p+1}(0) = a_2^{(n_p+1)} - \beta\Phi(0). \qquad (10)
\end{aligned}
$$

It is convenient to pass on to the quantities w_n, r_n, v_n, u_n from the coefficients $a_l^{(n)}$ using the equations

$$
\begin{aligned}
\tilde{a}_1^{(n)} &= s^{-n} w_n, & d_n(0) &= s^{-2n} r_n, \\
a_3^{(n)} &= s^{-3n} v_n, & a_4^{(n)} &= s^{-4n} u_n. \qquad (11)
\end{aligned}
$$

The renormalized coefficients w_n, r_n, v_n and u_n satisfy the following relations [5,6], which are the solutions of the recurrence relations linearized near the fixed point:

$$
\begin{aligned}
w_n &= -c_{h1} \mathcal{M}_1(h') E_1^n - c_{h2} \mathcal{M}_1(h') T_{13}^{(0)} \left(\varphi_0^{1/2} \beta\Phi(0) \right)^{-1} E_3^n, \\
r_n &= r^* + c_{k1}^{(0)} \beta\Phi(0) \tau E_2^n + c_{k2} T_{24}^{(0)} \varphi_0^{-1/2} (\beta\Phi(0))^{-1} E_4^n, \\
v_n &= -c_{h2} \mathcal{M}_1(h') E_3^n, \\
u_n &= u^* + c_{k1}^{(0)} (\beta\Phi(0))^2 T_{42}^{(0)} \varphi_0^{1/2} \tau E_2^n + c_{k2} E_4^n. \qquad (12)
\end{aligned}
$$

These relations are valid near the critical point when $\tau < \tau^*$ and $h < h^*$ ($h^* \sim (\tau^*)^{p_0}$, where p_0 is given below in (17)). Just in this value area of variables τ and h (region of the critical regime), the variables ω_n, r_n, v_n and u_n are close to the coordinates of the fixed point $w^* = 0$, $r^* = -f_0 \beta\Phi(0)$, $v^* = 0$ and $u^* = \varphi_0 (\beta\Phi(0))^2$. The quantities E_l are the eigenvalues for the matrix $\mathcal{R}$ of the renormalization group transformation

$$
\begin{pmatrix} w_{n+1} - w^* \\ r_{n+1} - r^* \\ v_{n+1} - v^* \\ u_{n+1} - u^* \end{pmatrix} = \mathcal{R} \begin{pmatrix} w_n - w^* \\ r_n - r^* \\ v_n - v^* \\ u_n - u^* \end{pmatrix}.
$$

They have the form [5]

$$
\begin{aligned}
&E_1 = s^{(d+2)/2}, \qquad E_3 = s^{(d-2)/2},\\
&E_{2,4} = \frac{1}{2}\left\{R_{22} + R_{44} \pm \left[(R_{22} - R_{44})^2 + 4R_{24}R_{42}\right]^{1/2}\right\},
\end{aligned} \tag{13}
$$

where the R_{ij} are the matrix elements of $\mathcal{R}$. Only one of the eigenvalues E_l is smaller than unity, and the rest of them are greater than unity. For some fixed value of the parameter $s = s^*$, where $s^* = 3.3783$, we find

$$
\begin{aligned}
E_1 &= 20.977, \qquad & E_3 &= 1.838,\\
E_2 &= 7.374, \qquad & E_4 &= 0.397.
\end{aligned}
$$

For such a preferred value of s nullifying the quantities

$$
h_2^{(n)} = \sqrt{6}(r_n + q)u_n^{-1/2}, \qquad h_3^{(n)} = h_{30}v_n u_n^{-3/4}
$$

at the fixed point, our calculations become less complicated. Here $q = \bar{q}\beta\Phi(0)$, $\bar{q} = \pi^2\ (b/c)^2 s_0^{-2}(1 + s^{-2})$, $h_{30} = 24^{3/4}/6$. The constant quantities $T_{km}^{(0)}$, which are the combinations of the $R_{km}^{(0)} = R_{km}(u^*)^{-(k-m)/4}$ and E_m, when $s = s^*$ take on the values [6]

$$
T_{13}^{(0)} = -0.655, \qquad T_{24}^{(0)} = -0.535, \qquad T_{42}^{(0)} = 0.177.
$$

The coefficients c_{hl}, $c_{k1}^{(0)}$, c_{k2} appearing in (12) are given in [6], and $\mathcal{M}_1 = \tanh h'$.

We attract your attention to the behaviour of the quantities $h_2^{(n)}$ and $h_3^{(n)}$ near the critical point. Each of them takes on small values when $n < n_p$. It is easy to make sure of this using the explicit solutions of the recurrence relations (12). We have

$$
\begin{aligned}
h_2^{(n)} &= h_{22}\left[c_{k1}^{(0)}\tau E_2^n - \frac{1}{2}\varphi_0^{-1/2}T_{42}^{(0)}(c_{k1}^{(0)}\tau E_2^n)^2\right],\\
h_3^{(n)} &= h_{32}\mathcal{M}_1(h')E_3^n(1 - h_{34}c_{k1}^{(0)}\tau E_2^n),
\end{aligned} \tag{14}
$$

where $h_{22} = (6/\varphi_0)^{1/2}$, $h_{32} = -h_{30}c_{h2}(u^*)^{-3/4}$, $h_{34} = 3T_{42}^{(0)}\varphi_0^{-1/2}/4$. For all values $n < n_p$, where n_p is defined using the condition

$$
c_{k1}^{(0)}\tau E_2^{n_p+1} = f_0, \tag{15}
$$

we find that $h_2^{(n)} \ll 1$. The analogous inequality takes place for $h_3^{(n)}$, since $\mathcal{M}_1(h') \sim h' \ll 1$. Therefore, for all $n < n_p$, the quantities Q_n can be presented in the form of the series with respect to the $h_2^{(n)}$ and $h_3^{(n)}$ and, using (14), calculate their explicit expressions. For values $n > n_p$, the quantity I_{n_p+1} from (9) is calculated using the Gaussian measure. It is related with coefficients $a_3^{(n)}$ and $a_4^{(n)}$, which begin to decrease fast (in comparison with $d_n(k)$) when the number n increases.

3. Definition of the weak field and strong field regions

The evaluation of the critical behaviour for the spin model in an external field to a great extent depends on the correlation between field and temperature. When the quantities τ and h reduce to zero with some rates, then we obtain the curve in the coordinates of field and temperature. That curve defines some trajectory of tending the system to the critical point. In [6,7,10] the critical behaviour of the model of a $3D$ unaxial magnet was investigated in some limiting cases. In particular, in the case of $h = 0$, the critical behaviour of such a model is investigated in detail in [10]. The explicit analytic expressions for a number of the thermodynamic functions at temperatures above and below T_{c} are obtained. Another limiting case, which corresponds to the $T = T_{\mathrm{c}}$ and $h \neq 0$, is investigated in [6]. The behaviour of the system average spin moment and susceptibility near the critical point as a function of the field is found. The existence of the limiting value of the field variable $\tilde{h}_{\mathrm{c}}$ for each value of the temperature $\tilde{\tau}$ has been established in [7]. These two variables are related by the expression

$$\tilde{h}_{\mathrm{c}} = \tilde{\tau}^{p_0}, \tag{16}$$

where the exponent p_0 is the universal quantity:

$$p_0 = \frac{d+2}{2}\nu. \tag{17}$$

Here ν is the critical exponent of the correlation length. The new designations are introduced as

$$\tilde{\tau} = \tau c_{k1}^{(0)}/f_0, \qquad \tilde{h} = h'/f_0. \tag{18}$$

The condition (16) corresponds to the case when the field and the temperature effects on the system near the critical point are equivalent. The value of the exponent p_0 from (16) has been obtained in [7] under the condition that magnitudes of the critical regime regions by the effect of temperature and field are equal. Then

$$n_p = m_\tau = n_h. \tag{19}$$

The quantity m_τ characterizes the exit point from the critical regime by the temperature in the absence of an external field [10]. For the temperatures above and below T_{c}, we obtain

$$m_\tau = -\frac{\ln\tilde{\tau}}{\ln E_2} - 1. \tag{20}$$

In the region $T < T_{\mathrm{c}}$, the value m_τ is denoted as μ_τ for convenience. Then

$$\mu_\tau = -\frac{\ln\tilde{\tau}_1}{\ln E_2} - 1, \tag{21}$$

where

$$\tilde{\tau}_1 = -\tilde{\tau}. \tag{22}$$

In the presence of an external field in addition to the quantities m_τ and μ_τ we have the exit point n_h from the critical regime by the field variable, which has the form [6]

$$n_h = -\frac{\ln \tilde{h}}{\ln E_1} - 1. \tag{23}$$

Exit of the system from the critical region with the given $\tilde{h}$ and $\tilde{\tau}$ is characterized by one of the quantities m_τ (or μ_τ) and n_h, or by their combination. This depends on the ratio of the quantities $\tilde{h}_c$ and $\tilde{h}$. In figure 1 the diagram of the regions, defined by different trajectories of tending the system to the critical point ($\tilde{\tau} = 0, \tilde{h} = 0$), is shown. The curve 1 corresponds to the limit values of the field (16).

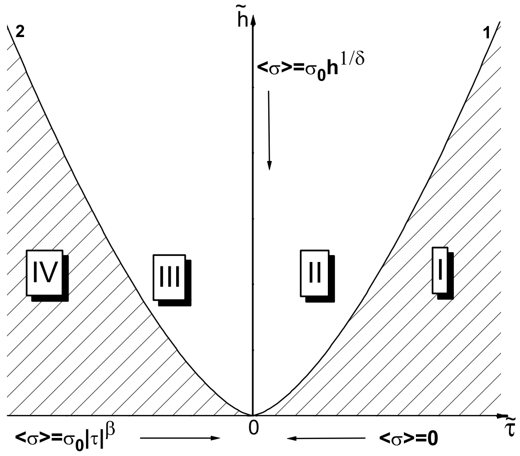

Figure 1. Schematic sketch of the regions of the possible location of the trajectories of tending the system to the critical point. The curves 1 and 2 correspond to the limiting value of the field when $T > T_c$ and $T < T_c$, respectively. The regions I and IV correspond to the small values of the field when $T > T_c$ and $T < T_c$, and regions II and III are characterized by the large values of the fields.

In a general case, the sizes of the critical regime regions effected by the field and temperature are different in magnitude. Then, the equality (19) is not valid and we consider two cases:

$$n_h > m_\tau \tag{24}$$

and

$$m_\tau > n_h. \tag{25}$$

In the present paper, we consider some specific case when the field h depends on the temperature τ. In general, we can represent the function $h(\tau)$ at $\tau \ll 1$ as the following series:

$$h(\tau) = a\tau + b\tau^2 + \dots$$

Depending on which term is more essential, different cases can be considered. If only the linear term in τ is present, then the function $h(\tau)$ will be represented by a straight line, which lies inside the region of strong fields in the vicinity of the critical point. This corresponds to the inequality (25). On the contrary, for the purely quadratic dependence of h on τ, we get the parabola in the weak field region. For this case, the inequality (24) is valid. In the most general case, one might use the following relation:

$$\tilde{h} = \tilde{\tau}^{p_0(1+\Delta)}. \tag{26}$$

The weak field region is determined by $\Delta > 0$. The case of $-1 < \Delta < 0$ corresponds to strong fields. At $\Delta \gg 1$ we come to the case $h \to 0$, described in [10]. In view of (20) and (23) from (26), we find

$$n_h = m_\tau(1+\Delta) + \Delta. \tag{27}$$

In the limit $\Delta \to -1$, we obtain the case of a strong external field for all values of the variable $\tilde{\tau}$ when $\tilde{\tau} < \tilde{\tau^*}$.

The conditions (24) and (25) define the different ways of calculating the free energy of the system. At first, let us consider the case of $n_h > m_\tau$ (or $n_h > \mu_\tau$ when $T < T_c$), which corresponds to the weak field region.

4. Free energy of the system near the critical point at the weak fields in the case of $T > T_c$

Let us calculate the free energy of the spin system when the external field decreases with the temperature coming down to T_c according to the law (26). We use the expression (8), in which we assign

$$n_p = m_\tau = -\frac{\ln \tilde{\tau}}{\ln E_2} - 1. \tag{28}$$

According to the results of [7,14], we present the free energy of the system near the critical point in the form of a few terms:

$$F = F_0 + F_{\mathrm{CR}} + F_{m_\tau+1} + F_p + F_I. \tag{29}$$

The quantity

$$F_0 = -kTN\left(\ln 2 + \ln\cosh h' + \frac{1}{2}\beta\Phi(0)\bar{\Phi}\right) \tag{30}$$

corresponds to the contribution from the noninteracting spins when $\bar{\Phi} = 0$. The method of calculating the F_{CR} (the contribution from short-wave modes of spin-density oscillations), as well as $F_{m_\tau+1}$, F_p and F_I (the contributions from long-wave oscillation modes) is described in detail in [14]. We obtain the following final formulas:

$$F_{\mathrm{CR}} = -kTNs_0^{-3}\left(\mathrm{e}_{0p} + \mathrm{e}_{1p}\tilde{\tau} + \mathrm{e}_{2p}\tilde{\tau}^2 + \mathrm{e}_{3p}\tilde{h}^2 + \mathrm{e}_{4p}\tilde{\tau}^{3\nu}\right),$$

$$
\begin{aligned}
F_{m_\tau+1} &= -kTN_0 f_{mp}\tilde{\tau}^{3\nu},\\
F_p &= -kTN_0 s^{-3}\tilde{\tau}^{3\nu}\left[f_{p2} - (m_\tau+1)\ln s\right],\\
F_I &= -kTN_0 s^{-3}\tilde{\tau}^{3\nu}\left(\frac{1}{2}\ln\pi + (m_\tau+2)\ln s - \frac{1}{2}I_0''\right) - kTNl_{11T}\tilde{h}^2\tilde{\tau}^{-2\nu}.
\end{aligned}
\tag{31}
$$

The expressions for the coefficients of the relations (31) are given in [14]. The quantities $F_{m_\tau+1}$ and F_p form the contribution to the free energy $F_{\mathrm{TR}} = F_{m_\tau+1} + F_p$, related with the transition region (TR). We separated this region when calculating the expression for the free energy at $T > T_{\mathrm{c}}$ in the case of $h = 0$ (see [10,15]). The case of the weak fields ($\tilde{h} < \tilde{h}_{\mathrm{c}}$) at $T > T_{\mathrm{c}}$ requires to separate the TR as well.

Based on equation (29), the summation free energy gets the form

$$
F = -kTN\left[\ln\cosh h' + l_0 + l_{10T}\tilde{\tau}^{3\nu} + l_{2T}\tilde{h}^2 + l_{3T}\tilde{\tau} + l_{4T}\tilde{\tau}^2 + l_{11T}\tilde{h}^2\tilde{\tau}^{-2\nu}\right], \tag{32}
$$

where

$$
\begin{aligned}
l_0 &= \ln 2 + \frac{1}{2}\beta_{\mathrm{c}}\Phi(0)\bar{\Phi} + s_0^{-3}\mathrm{e}_{0p},\\
l_{10T} &= s_0^{-3}\left[\mathrm{e}_{4p} + f_{mp} + s^{-3}\left(f_{p2} + \frac{1}{2}\ln\pi + \ln s - \frac{1}{2}I_0''\right)\right],\\
l_{2T} &= s_0^{-3}\mathrm{e}_{3p}, \qquad l_{3T} = s_0^{-3}\mathrm{e}_{1p} - \frac{1}{2}\beta_{\mathrm{c}}\Phi(0)\bar{\Phi}f_0/c_{k1}^{(0)},\\
l_{4T} &= s_0^{-3}e_{2p} + \frac{1}{2}\beta_{\mathrm{c}}\Phi(0)\bar{\Phi}f_0^2/(c_{k1}^{(0)})^2, \qquad l_{11T} = \frac{1}{4}a_{mp}^2,\\
a_{mp} &= \frac{\mathcal{M}_{20}}{\mathcal{M}_2}f_0 s\left(2/r_{m_\tau+2}\right)^{1/2}, \qquad r_{m_\tau+2} = f_0\beta\Phi(0)(E_2-1).
\end{aligned}
\tag{33}
$$

Here $\mathcal{M}_2(h')$ is the cumulant of the second order [9,10], and the expression for $\mathcal{M}_{20}$ is given in [6,7].

The formula (32) describes the free energy of the lattice system of spins in the weak external field ($\tilde{h} < \tilde{h}_{\mathrm{c}}$, where the limit field $\tilde{h}_{\mathrm{c}}$ is defined in (16)). It is valid for all $T > T_{\mathrm{c}}$ near the critical point.

Let us transform the last term of the expression (32). Taking into account (26), it can be introduced in the form

$$
F_\eta = -kTNl_{11T}\tilde{\tau}^{\mathrm{d}\nu + (\mathrm{d}+2)\nu\Delta}. \tag{34}
$$

The contribution to the free energy (34) enables one to find the value of the parameter Δ, when the exponent of the $\tilde{\tau}$ exceeds number two. The exponent of $\tilde{\tau}$ is equal to two when Δ' satisfies the following equality:

$$
\mathrm{d}\nu + (d+2)\Delta'\nu = 2, \qquad \Delta' = \frac{2 - \mathrm{d}\nu}{(d+2)\nu}. \tag{35}
$$

For all $\Delta > \Delta'$, the second derivative of the quantity F_η with respect to the temperature comes down to zero when $\tilde{\tau} \to 0$. However, in the region of the values

$$0 \leqslant \Delta < \Delta'$$

it will tend to the infinity when the system tends to the critical point. Then, the system trajectory lies between the following trajectories

$$\tilde{h} = \tilde{\tau}^{p_0(1+\Delta')}, \qquad \tilde{h}_c = \tilde{\tau}^{p_0}.$$

They define some transition region, where the effect of the terms like (34) will be essential.

Let us consider the expression for the free energy (34)in terms of the field variable $\tilde{h}$. Taking into account (26), we have

$$F_\eta = -kTNl_{11T}\tilde{h}^{\eta'}. \tag{36}$$

Here

$$\eta' = \frac{2d + 2(d+2)\Delta}{d+2+(d+2)\Delta}. \tag{37}$$

It should be noted that the expression (36) corresponds to the system with the condition (26). Here, we introduce the dependence of the field value on the temperature. This condition permits us to choose the weak field region, but requires that the value of the field should decrease when the system tends to the critical point. The free energy term of the type (36) permits us to define the susceptibility of the system in the direction of the curve (26). Note, that for η', the limit relations take place

$$\lim_{\Delta\to 0} \eta' = \frac{2d}{d+2}, \qquad \lim_{\Delta\to\infty} \eta' = 2. \tag{38}$$

Let us calculate the quantity

$$\sigma_+ = -\frac{1}{N}\frac{\partial F}{\partial h}.$$

We will use the equation (32) as F, where the last term has the form of (36). We obtain

$$\sigma_+ = \tanh h' + 2l_{2T}f_0^{-2}h' + \eta' l_{11T}f_0^{-1}\tilde{h}^{1/\delta_\Delta}, \tag{39}$$

where

$$\delta_\Delta = \frac{d+2+(d+2)\Delta}{d-2+(d+2)\Delta}. \tag{40}$$

In the case of $\Delta = 0$, we have $\delta_0 = 5$, and when $\Delta \to \infty$, we find $\delta_\infty = 1$.

The quantity of σ_+ (39) decreases to zero when the system tends to the critical point. The rate of decreasing depends on the value of Δ, i.e., on the system trajectory. If $\Delta = 0$ (the case of $\tilde{h} = \tilde{h}_c$), we have

$$\sigma_+ = \frac{2d}{d+2}l_{11T}f_0^{-1}\tilde{h}^{1/\delta_0}. \tag{41}$$

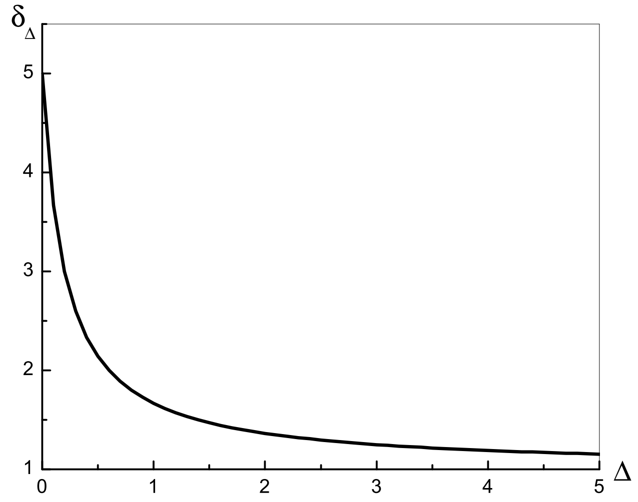

Figure 2. Dependence of quantity δ_Δ (40) on parameter Δ when $d = 3$.

When parameter Δ rises, the amplitude σ_+ increases a little (see (38), (39)), and the quantity δ_Δ tends to one. The limit $\Delta \to \infty$ corresponds to the absence of the magnetic field, $h' = f_0 \lim\limits_{\Delta\to\infty} \tilde{\tau}^{p_0(1+\Delta)} = 0$, and then the quantity σ_+ becomes zero. The dependence of δ_Δ on the parameter Δ is shown in figure 2. The Δ-dependence curve for δ_Δ is compared in Appendix with the curve for the analogous quantity $\delta_{0\Delta}$ obtained using the Landau approach.

Let us find the quantity

$$\chi_+ = -\frac{1}{N}\frac{\partial^2 F}{\partial h^2}.$$

Using (32), it is defined by the epxression

$$\chi_+ = \beta\left[\mathcal{M}_2(h') + 2l_{2T}f_0^{-2} + \chi_{01}\tilde{h}^{\eta'-2}\right], \tag{42}$$

where

$$\chi_{01} = \eta'(\eta'-1)l_{11T}f_0^{-2}.$$

The exponent $\eta' - 2$ satisfies the relation

$$\eta' - 2 = -\frac{4}{(d+2)(1+\Delta)}. \tag{43}$$

In view of (26), we find

$$\tilde{h}^{\eta'-2} = \tilde{\tau}^{-\frac{4}{(d+2)(1+\Delta)}\frac{d+2}{2}\nu(1+\Delta)} = \tilde{\tau}^{-2\nu}.$$

Thus, the asymptote of the expression (42) by the temperature does not depend on the parameter Δ and for all trajectories, which belong to the weak field region

$(\tilde{h} \leqslant \tilde{h}_c)$, the system's susceptibility has the form

$$\chi_+ = \beta\left[\chi_0 + \chi_{01}\tilde{\tau}^{-\gamma}\right]. \tag{44}$$

Here $\gamma = 2\nu$, and

$$\chi_0 = \mathcal{M}_2(h') + 2l_{2T}f_0^{-2}.$$

5. Critical behaviour of the system at the weak fields in the case of $T < T_c$

Let us perform the calculation of the free energy of the system in the low-temperature region in the case of the weak fields. As before, we will assume that the applied field decreases when the system tends to the critical point according to the law

$$\tilde{h} = \tilde{\tau}_1^{p_0(1+\Delta)}, \tag{45}$$

where $\Delta \geqslant 0$. For the convenience of calculations, we introduce the quantity

$$\tilde{\tau}_1 = -\tilde{\tau} = \frac{T_c - T}{T_c}\frac{c_{k1}^{(0)}}{f_0},$$

which is positive when $T < T_c$. In order not to use the new notices in the case of the temperatures $T < T_c$, we will set the tilde above the corresponding values. If these values satisfy the analogous expressions at $T > T_c$ and $T < T_c$, then we will save the notices, as for $T > T_c$, without the tilde.

We can represent the free energy at $T < T_c$ according to (8) as a sum of the following components:

$$\tilde{F} = F_0 + \tilde{F}_{\mathrm{CR}} + \tilde{F}_p + \tilde{F}_I. \tag{46}$$

The quantity F_0 is defined in (30). The coefficients of the components

$$\begin{aligned}
\tilde{F}_{\mathrm{CR}} &= -kTN_0\left(\mathrm{e}_{0p} + \mathrm{e}_{1p}\tilde{\tau} + \mathrm{e}_{2p}\tilde{\tau}^2 + \mathrm{e}_{3p}\tilde{h}^2 + \tilde{\mathrm{e}}_{4p}\tilde{\tau}_1^{3\nu}\right), \\
\tilde{F}_p &= -kTN_0\tilde{\tau}_1^{3\nu}\left[\tilde{f}_{p1} - \mu_\tau \ln s\right], \\
\tilde{F}_I &= -kTN\left[\left(\tilde{E}_{02} + s_0^{-3}(\ln s - \frac{1}{2}I_0')\right)\tilde{\tau}_1^{3\nu} + a_{1m}\tilde{h}\tilde{\sigma} + l_{11\mu}\tilde{h}^2\tilde{\tau}_1^{-2\nu}\right] \\
&\quad -kTNs_0^{-3}\tilde{\tau}_1^{3\nu}\mu_\tau \ln s
\end{aligned} \tag{47}$$

are given in [14].

Let us write down the final expression for the free energy near the critical point when $T < T_c$ according to (46). It has the following final form:

$$\tilde{F} = -kTN\left[\ln\cosh h' + l_0 + l_{1m}\tilde{\tau}_1^{3\nu} + a_{1m}\tilde{h}\tilde{\sigma} + l_{2m}\tilde{h}^2 + l_{3m}\tilde{\tau} + l_{4m}\tilde{\tau}^2 + l_{11\mu}\tilde{h}^2\tilde{\tau}_1^{-2\nu}\right]. \tag{48}$$

Here

$$
\begin{aligned}
&l_{1m} = \tilde{E}_{022} + s_0^{-3}(\tilde{e}_{4p} + \tilde{f}_{p1}), \qquad \tilde{E}_{022} = \tilde{E}_{02} + s_0^{-3}\left(\ln s - \frac{1}{2}I_0'\right),\\
&l_{2m} = s_0^{-3}e_{3p}, \qquad l_{3m} = s_0^{-3}e_{1p} - \frac{1}{2}\beta_{\mathrm{c}}\Phi(0)\bar{\Phi}f_0/c_{k1}^{(0)},\\
&l_{4m} = s_0^{-3}e_{2p} + \frac{1}{2}\beta_{\mathrm{c}}\Phi(0)\bar{\Phi}f_0^2/(c_{k1}^{(0)})^2, \qquad l_{11\mu} = \frac{1}{8}\left(\frac{\mathcal{M}_{20}}{\mathcal{M}_2}\right)^2 f_0/\beta\Phi(0).
\end{aligned}
\tag{49}
$$

The quantity l_0 is given in (33).

The expression (48) corresponds to the free energy of the system near the critical point in the weak fields when $T < T_{\mathrm{c}}$.

As in the case $T > T_{\mathrm{c}}$, we find the quantity

$$\sigma_- = -\frac{1}{N}\frac{\partial \tilde{F}}{\partial h}.$$

Taking into account (48), it is represented by the expression

$$\sigma_- = \tanh h' + 2l_{2m}f_0^{-2}h' + a_{1m}\tilde{\sigma}f_0^{-1} + \eta' l_{11\mu}f_0^{-1}\tilde{h}^{1/\delta_\Delta}, \tag{50}$$

which differs from the analogous expression at $T > T_{\mathrm{c}}$ (39) by the presence of the term

$$a_{1m}\tilde{\sigma}f_0^{-1} = a_{1m}\tilde{\sigma}_0\tilde{\tau}_1^{\nu/2}f_0^{-1}, \tag{51}$$

where

$$a_{1m} = f_0\mathcal{M}_{20}/\mathcal{M}_2, \qquad \tilde{\sigma}_0 = \left(\frac{12f_0\beta\Phi(0)}{u_{\mu_\tau+1}}s_0^{-3}\right)^{1/2}, \qquad u_{\mu_\tau+1} = u^*\left(1 - f_0\varphi_0^{-1/2}T_{42}^{(0)}\right).$$

Then $l_{2m} = l_{2T}$ (see (33)), however $l_{11\mu}$ from (50) is essentially smaller than l_{11T} from (39). We have

$$
\begin{aligned}
l_{11T} &= \left(\frac{\mathcal{M}_{20}}{\mathcal{M}_2}\right)^2 f_0\frac{s^2}{2(E_2-1)\beta\Phi(0)},\\
l_{11\mu} &= \left(\frac{\mathcal{M}_{20}}{\mathcal{M}_2}\right)^2 f_0\frac{1}{8\beta\Phi(0)}.
\end{aligned}
\tag{52}
$$

The relation

$$\frac{l_{11T}}{l_{11\mu}} = \frac{4s^2}{E_2-1} \tag{53}$$

is the universal quantity. As for the σ_+ at $T > T_{\mathrm{c}}$, weak fields ($\Delta \to \infty$) cause the linear-field dependence of σ_-, and the fields, which are close to the limit value $\tilde{h}_{\mathrm{c}}$ ($\Delta = 0$), give $\sigma_- \sim \tilde{h}^{1/5}$.

Let us estimate the quantity

$$\chi_- = -\frac{1}{N}\frac{\partial^2\tilde{F}}{\partial h^2}.$$

We obtain

$$\chi_- = \beta \left[\mathcal{M}_2(h') + 2l_{2m} f_0^{-2} + \chi_{02} \tilde{h}^{\eta'-2} \right], \tag{54}$$

where

$$\chi_{02} = \eta'(\eta'-1) l_{11\mu} f_0^{-2}. \tag{55}$$

The exponent $\eta' - 2$ is defined in (43) and for all values of Δ it takes on the negative values. Therefore, the value χ_- will tend to the infinity in the case of $h \to 0$.

In view of (45), the temperature dependence of the quantity χ_- has the form

$$\chi_- \approx \beta \chi_{02} \tilde{\tau}_1^{-\gamma} \tag{56}$$

when $\tilde{\tau}_1 \to 0$. Here, as at $T > T_c$,

$$\gamma = 1.219. \tag{57}$$

The ratio of the quantities χ_+ (42) and χ_- (54) at $h \to 0$ is defined, in general, by the singular components and according to (53) is equal to

$$\frac{\chi_+}{\chi_-} \approx \frac{l_{11T}}{l_{11\mu}} = 7.16. \tag{58}$$

6. Region of strong fields

We consider the trajectories of the system tending to the critical point, which belong to the regions II and III in figure 1 (for the Landau approach, see Appendix). In the case of strong fields at $T > T_c$, the field is related with a reduced temperature by the expression

$$\tilde{h} = \tilde{\tau}^{p_0(1-\Delta_1)}, \tag{59}$$

where $0 < \Delta_1 < 1$. For the following calculations, we will use the relations (20) and (23). When the condition (59) is satisfied, we find

$$n_h = m_\tau - \Delta_1(m_\tau + 1). \tag{60}$$

Comparing (60) with the corresponding relation (27), which takes place in the case of the weak fields, we discover an essential difference. For the weak fields, we have (24), and in the case of strong fields, the inequality $n_h < m_\tau$ is performed. This means that the quantity n_p should be identified with n_h, and not with m_τ (as this takes place in the case of the weak fields):

$$n_p = n_h = -\frac{\ln \tilde{h}}{\ln E_1} - 1. \tag{61}$$

Let us calculate the free energy of the system in the case of $T > T_c$ with large values of the field. We have

$$F_h = F_0 + F_{\mathrm{CR},h} + F_{p,h} + F_{I,h}\,. \tag{62}$$

The quantity F_0 is given in (30). Using the formulas

$$
\begin{aligned}
F_{\mathrm{CR},h} &= -kT\ln Q_0 + F'_{\mathrm{CR},h},\\
F'_{\mathrm{CR},h} &= -kTN_0\Big(f^{(0)}_{\mathrm{CR}} + f^{(1)}_{\mathrm{CR}}\tilde{\tau} + f^{(2)}_{\mathrm{CR}}\tilde{\tau}^2 + f^{(3)}_{\mathrm{CR}}\tilde{h}^2\\
&\qquad - F_{10}\tilde{h}^{6/5} - F_{11}\tilde{h}^{6/5+\Delta_{11}} - F_{12}\tilde{h}^{6/5+2\Delta_{11}}\Big),\\
\Delta_{11} &= \frac{\Delta_1}{p_0(1-\Delta_1)},\\
F_{p,h} &= -kTN_0\left[f_{p1c} - n_h\ln s\right]\tilde{h}^{6/5},\\
F_{I,h} &= -kTN\tilde{E}_{02}\tilde{h}^{6/5} - kTN_0\tilde{h}^{6/5}n_h\ln s
\end{aligned}
\tag{63}
$$

and the expressions for their coefficients [14], one can calculate the free energy F_h. We get

$$
\begin{aligned}
F_h &= -kTN\Big[\ln\cosh h' + l_0 + l_1\tilde{h}^{6/5} + l^{(+)}_{11}\tilde{h}^{6/5+\Delta_{11}}\\
&\qquad + l_{12}\tilde{h}^{6/5+2\Delta_{11}} + l_2\tilde{h}^2 + l_3\tilde{\tau} + l_4\tilde{\tau}^2\Big],
\end{aligned}
\tag{64}
$$

where exponent Δ_{11} is given in (63), and

$$
\begin{aligned}
&l_0 = \ln 2 + \frac{1}{2}\beta_{\mathrm{c}}\Phi(0)\bar{\Phi} + s_0^{-3}\mathrm{e}_{0p}, \qquad l_1 = \tilde{E}_{02} + s_0^{-3}(f_{p1c} - F_{10}),\\
&l^{(+)}_{11} = -s_0^{-3}F_{11}, \qquad l_{12} = -s_0^{-3}F_{12},\\
&l_2 = s_0^{-3}\mathrm{e}_{3p}, \qquad l_3 = s_0^{-3}\mathrm{e}_{1p} - \frac{1}{2}\beta_{\mathrm{c}}\Phi(0)\bar{\Phi}f_0/c^{(0)}_{k1},\\
&l_4 = s_0^{-3}\mathrm{e}_{2p} + \frac{1}{2}\beta_{\mathrm{c}}\Phi(0)\bar{\Phi}f_0^2/(c^{(0)}_{k1})^2.
\end{aligned}
\tag{65}
$$

The quantities F_{10}, F_{11}, F_{12} appear in the expression for the contribution to the free energy $F'_{\mathrm{CR},h}$. They are represented in [14].

As in the case of the weak fields, we calculate the quantity

$$
\sigma^{(h)}_{+} = -\frac{1}{N}\frac{\partial F_h}{\partial h}.
$$

Taking into account (64), we have

$$
\begin{aligned}
\sigma^{(h)}_{+} &= \tanh h' + \frac{6}{5}l_1 f_0^{-6/5}(h')^{1/\delta} + \left(\frac{6}{5}+\Delta_{11}\right)l^{(+)}_{11}f_0^{-6/5-\Delta_{11}}(h')^{\frac{1}{\delta}+\Delta_{11}}\\
&\quad + \left(\frac{6}{5}+2\Delta_{11}\right)l_{12}f_0^{-6/5-2\Delta_{11}}(h')^{\frac{1}{\delta}+2\Delta_{11}} + 2l_2 f_0^{-2}h'.
\end{aligned}
\tag{66}
$$

Here $\delta = 5$. In the case of $\Delta_1 \to 1$ (then $\Delta_{11} \to \infty$), this result corresponds to the analogous formula of the paper [6]. For another limiting case of $\Delta_1 = 0$ (then $\Delta_{11} = 0$), the relation (66) reduces to the corresponding expression, obtained in [7].

Let us find the susceptibility of the system

$$\chi_+^{(h)} = -\frac{1}{N}\frac{\partial^2 F_h}{\partial h^2}.$$

Using (64), we have

$$\chi_+^{(h)} = \beta\left[\mathcal{M}_2(h') + 2l_2 f_0^{-2} + \chi_{01}^{(h)}\tilde{h}^{\eta_0-2} + \chi_{02}^{(h)}\tilde{h}^{\eta_0-2+\Delta_{11}} + \chi_{03}^{(h)}\tilde{h}^{\eta_0-2+2\Delta_{11}}\right]. \quad (67)$$

Here

$$\begin{aligned}
\eta_0 &= \frac{2d}{d+2},\\
\chi_{01}^{(h)} &= \eta_0(\eta_0-1)l_1 f_0^{-2},\\
\chi_{02}^{(h)} &= (\eta_0+\Delta_{11})(\eta_0+\Delta_{11}-1)l_{11}^{(+)} f_0^{-2},\\
\chi_{03}^{(h)} &= (\eta_0+2\Delta_{11})(\eta_0+2\Delta_{11}-1)l_{12} f_0^{-2}.
\end{aligned} \quad (68)$$

The formula (67) at $\Delta_1 \to 1$ corresponds to the result of [6], and at $\Delta_1 = 0$ reduces to the result of [7].

The research in the region of strong fields at $T < T_\mathrm{c}$ is similar to the case of $T > T_\mathrm{c}$. Like in section 4, we will use the quantity $\tilde{\tau}_1 = -\tilde{\tau}$. The field can be written as

$$\tilde{h} = \tilde{\tau}_1^{p_0(1-\Delta_1)}. \quad (69)$$

The quantity n_h, which characterizes the exit point of the system from the critical regime of the fluctuations, in the case of the strong fields has the form (23). Comparing the magnitudes of the regions of the critical regime by the temperature and field, we find

$$n_h = \mu_\tau - \Delta_1(\mu_\tau + 1), \quad (70)$$

where $0 < \Delta_1 < 1$, and μ_τ is determined by the expression (21). We come to the conclusion that $n_h < \mu_\tau$, and, thus, the expression $n_p = n_h$ is valid. We have already used such a condition in this section while calculating the free energy at $T > T_\mathrm{c}$.

Let us represent the free energy for the case of $T < T_\mathrm{c}$ in the form

$$\tilde{F}_h = F_0 + \tilde{F}_{\mathrm{CR},h} + \tilde{F}_{p,h} + \tilde{F}_{I,h}. \quad (71)$$

Here F_0 coincides with (30). The quantity $\tilde{F}_{\mathrm{CR},h}$ has the following form:

$$\begin{aligned}
\tilde{F}_{\mathrm{CR},h} = -kTN_0\Big[&\mathrm{e}_{0p} + \mathrm{e}_{1p}\tilde{\tau} + \mathrm{e}_{2p}\tilde{\tau}^2 + \mathrm{e}_{3p}\tilde{h}^2\\
&- F_{10}\tilde{h}^{6/5} + F_{11}\tilde{h}^{6/5+\Delta_{11}} - F_{12}\tilde{h}^{6/5+2\Delta_{11}}\Big].
\end{aligned} \quad (72)$$

Comparing this expression with the corresponding expression at $T > T_\mathrm{c}$, we can see that their difference is related with the sign of the term near the coefficient F_{11}:

$$F_\Delta^{(\pm)} = \pm kTN_0 F_{11}\tilde{h}^{6/5+\Delta_{11}}. \quad (73)$$

The "+" and "−" signs correspond to temperatures above and below T_{c}, respectively.

The expression $\tilde{F}_{p,h}$ coincides with the analogous expression at $T > T_{\mathrm{c}}$ (see (63)). The expression for $\tilde{F}_{I,h}$ does not change either. As in the case of $T > T_{\mathrm{c}}$, it is given by the formula, which is represented in (63).

In this way, the free energy of the system at $T < T_{\mathrm{c}}$ for the large values of the field near the critical point has the form (64), where $l_{11}^{(+)}$ should be replaced by the coefficient $l_{11}^{(-)}$, for which, we have

$$l_{11}^{(-)} = -l_{11}^{(+)}. \tag{74}$$

Thus, the description of the critical behaviour of a $3D$ Ising-like system in an external field is performed for the case when the system tends to the critical point along some trajectory in the coordinates of the field $\tilde{h}$ and temperature $\tilde{\tau}$ (the values of the field and temperature are related by some expression). The description of system properties can be performed in terms of independent variables temperature and field (as done in our preprint [16]). This is important from the point of view of comparing the results with the conclusions, which directly follow from the hypothesis of the two-parametric scaling. Then, fixing one of the variables and differentiating thermodynamic functions with respect to another variable, from the scaling hypothesis one can obtain the known relations among the critical exponents. Based on these relations when two critical exponents are available, other exponents can be obtained. The results of investigation, presented in the paper when $\Delta = 0$, correspond to the case, which also takes place in the analysis using independent variables $\tilde{h}$ and $\tilde{\tau}$. Just when $\Delta = 0$, the system is in the external field $\tilde{h} = \tilde{h}_{\mathrm{c}}$. Then, all preceding considerations for independent variables turn out to be valid.

7. Conclusions

The method proposed for describing $3D$ one-component magnet near the critical point takes into account the simultaneous effect of the temperature and field on the behaviour of the system. The consideration is carried out in the ensemble when the field and temperature are not independent variables. They are related by some expression. The system tends to the critical point along some trajectory in coordinates of the field $\tilde{h}$ and temperature $\tilde{\tau}$. The field is specific since it decreases with $\tilde{\tau} \to 0$. Such problems appear with describing the properties of a series of statistical systems (see, for example, [11,12]).

It is established that the behaviour of the system varies depending on the rate of the field decreasing. The weak field and strong field regions are defined. For the former, the expression for free energy has the form (32) at $T > T_{\mathrm{c}}$ and (48) $T < T_{\mathrm{c}}$. The susceptibility in both cases is characterized by the temperature critical exponent γ (the formula (44) at $T > T_{\mathrm{c}}$ and (56) at $T < T_{\mathrm{c}}$). It is shown that this exponent does not depend on the parameter Δ, which defines the trajectory of the system. It is identical for each trajectory in the weak field region. For the strong field, the behavior of the system is defined by the field variable $\tilde{h}$. The expression for free energy (64) has the same functional form for the region $T > T_{\rm c}$ and $T < T_{\rm c}$. There is a difference for one of the coefficients just in the sign (see (74)). The Landau approach adapted to our research is considered in addition to the proposed analytic method based on the use of a non-Gaussian measure density.

Appendix

Let us consider the effect of the external field, which depends on the temperature, on the behaviour of the magnetic system with one-component order parameter, using the Landau approach. In this approach for the system near the critical point, the free energy is represented as a series with respect to the order parameter σ.

$$F = F_0' + a\tau\sigma^2 + b\sigma^4 + g(\nabla\sigma)^2 - h\sigma. \tag{A.1}$$

Here a, b, g are quantities, which do not depend on temperature τ and field h.

We assume that the field h depends on τ, i.e., it is changed with the system tending to the critical point. In the region of temperatures $\tau < 0$, the spontaneous magnetic moment

$$\sigma_S \sim |\tau|^{\beta_0}, \tag{A.2}$$

where the critical exponent $\beta_0 = 0.5$, as well as the moment induced by the field

$$\sigma_I = \chi h \sim |\tau|^{-\gamma_0} h \tag{A.3}$$

exist in the system. Here $\gamma_0 = 1$ is the critical exponent of the susceptibility χ.

Let us find the value of the field $h = h_{c0}$ when the condition

$$\sigma_S = \sigma_I \tag{A.4}$$

is satisfied. According to (A.2) and (A.3) we obtain

$$h_{c0} \sim |\tau|^{p_0}, \tag{A.5}$$

where

$$p_0 = \gamma_0 + \beta_0 = 1.5. \tag{A.6}$$

The value h_{c0} will be called the limiting field.

The inverse proposition is valid too. When the field h from (A.1) is changed according to the law (A.5) then at $T < T_{\rm c}$ the relation (A.4) takes place for all τ ($\tau \ll 1$).

It is known [13] that under the condition (A.5) one can express each thermodynamic quantity via temperature τ or field h. Therefore, using the (A.5) from (A.3), we find

$$\sigma_I \sim h_{c0}^{1/\delta_0}, \tag{A.7}$$

where $\delta_0 = p_0/\beta_0 = 3$, as this takes place in the Landau theory.

Let us consider the field part of the Landau free energy

$$F_h = -h\sigma. \tag{A.8}$$

In the case of $h = h_{c0}$ it can be represented in two equivalent forms. Taking into account (A.2) and (A.5) from (A.8), we get

$$F_h \sim -|\tau|^{p_0+\beta_0}, \tag{A.9}$$

which is the evidence of the absence of divergence of the second derivative with respect to the temperature in the critical point. At $h = h_{c0}$ one can express the quantity F_h via field variable h. Using (A.7), we find

$$F_h \sim -h^{1+1/\delta_0}. \tag{A.10}$$

The dependencies (A.9) and (A.10) are valid only in the case of performing the (A.5). Let us consider a general case when the field h decreases with greater or smaller rate in comparison with h_{c0}. We assume

$$h = |\tau|^{p_0(1+\Delta)}, \tag{A.11}$$

where Δ is some parameter. At $\Delta > 0$ the field h decreases with a smaller rate than h_{c0}, and at $-1 < \Delta < 0$ it decrease with a greater rate than h_{c0}.

We consider the case of $\Delta > 0$. Then the order parameter is defined, in general, by the contribution of the spontaneous moment σ_S. For (A.8), we obtain the dependence

$$F_h \sim -|\tau|^{2+p_0\Delta}. \tag{A.12}$$

Thus, for the dependence (A.11) the role of the field contribution is much smaller than at $h = h_{c0}$.

In the case of $\Delta > 0$ we write down the field contribution of the type (A.10). It has the form

$$F_h \sim -h^{\eta_0}, \tag{A.13}$$

where for the exponent η_0, we have

$$\eta_0 = \frac{2 + p_0\Delta}{p_0(1+\Delta)}. \tag{A.14}$$

In the limiting cases, we obtain

$$\eta_0 = \begin{cases} 4/3, & \Delta \to 0, \\ 1, & \Delta \to \infty. \end{cases}$$

Let us find the quantity

$$\sigma = -\frac{\partial F}{\partial h}, \tag{A.15}$$

which we will interpret as some "order parameter". According to (A.13), we have

$$\sigma \sim h^{1/\delta_{0\Delta}}, \tag{A.16}$$

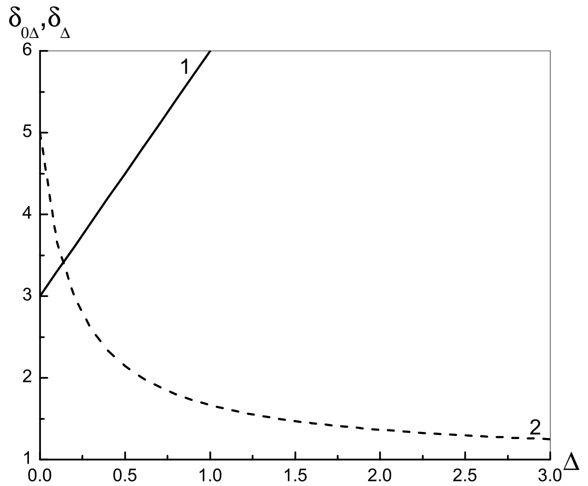

Figure 3. Dependence of quantities $\delta_{0\Delta}$ from (A.17) and δ_{Δ} from (40) on the parameter Δ. The solid line corresponds to $\delta_{0\Delta}$ (Landau approach), and dash line corresponds to δ_{Δ} (non-Gaussian approximation).

where

$$\delta_{0\Delta} = 3(1+\Delta). \tag{A.17}$$

The dependence of this quantity on Δ is shown by the line 1 in figure 3. As one can see, this dependence has the inverse direction in comparison with curve 2, which is obtained taking into account the contributions from long-range fluctuations of the order parameter. At $\Delta = 2/15$, both quantities have the same value $\delta_0' = 3.4$. Thus, when the external field decreases with the smaller rate as the limiting field h_{c0} with the system tending to the critical point, then the order parameter is proportional to its spontaneous part (A.2). On the other hand, using (A.11), we find (A.16). The latter equality represents the dependence of the order parameter on the external field, which depends on the temperature according to the (A.11). In some particular cases the equality (A.16) reduces to the well-known dependencies. The first of them corresponds to the value $\Delta = 0$ when $\delta_{0\Delta} = \delta_0 = 3$. Another case $\Delta \to \infty$ on account of (A.11) corresponds to the absence of the field. In this case, $\delta_{0\Delta} \to \infty$, however, $h \to 0$. Taking this limit we obtain the dependence (A.2).

According to the field h at $\Delta > 0$, the susceptibility of the system has the form

$$\chi \sim h^{\eta_0 - 2}, \tag{A.18}$$

where

$$\eta_0 - 2 = \frac{\beta_0 - p_0 - p_0\Delta}{p_0(1+\Delta)}. \tag{A.19}$$

Let us consider the case of strong fields. In this case, it is convenient to write down the relation (A.11) in the form

$$h = |\tau|^{p_0(1-\Delta_1)}, \tag{A.20}$$

where $0 \leqslant \Delta_1 \leqslant 1$. For all $\Delta_1 > 0$, the order parameter of the system is defined, in general, by the contribution of the induced magnetic moment

$$\sigma_I \sim h^{1/\delta_0}. \tag{A.21}$$

The field part of the Landau free energy has the form

$$F_h \sim -h^{1+1/\delta_0}. \tag{A.22}$$

For the derivative of the type (A.15), we get the dependence (A.21), and for the susceptibility of the system along the direction of the curve for (A.20), we have

$$\chi \sim h^{-2/3}. \tag{A.23}$$

The latter coincides with (A.18) at $\Delta = 0$. Thus, when the external field h decreases more slowly than h_{c0} with the system tending to the critical point, the field part of the expression for the Landau free energy is a homogenous function of the field with the exponent of homogeneity $1 + 1/\delta_0$. The susceptibility of the system along the direction (A.20) is proportional to the h with the critical exponent $-2/3$. These both exponents do not depend on the parameter Δ_1 in contrast to the case of the weak fields.

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Поведінка тривимірного одновісного магнетика поблизу критичної точки в зовнішньому полі

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Отримано 26 березня 2003 р., в остаточному вигляді – 24 травня 2004 р.

Досліджена поведінка тривимірного магнетика з однокомпонентним параметром порядку поблизу критичної точки у випадку наявності однорідного зовнішнього поля. Розрахунки виконані у випадку, коли величини поля та температури є залежними та зв'язаними між собою певним співвідношенням (система прямує до критичної точки по певній траєкторії). Розглянуті області температур вищих та нижчих за T_c (T_c – температура фазового переходу при відсутності поля). Показано, що при слабких значеннях поля поведінка системи описується в основному температурною змінною, а для випадку сильних полів роль температурної змінної не є домінуючою. Для кожної з цих областей отримані відповідні вирази для вільної енергії, сприйнятливості та інших характеристик системи в околі критичної точки.

Ключові слова: *тривимірний одновісний магнетик, колективні змінні, критична поведінка, зовнішнє поле*

PACS: *05.50.+q, 05.70.Ce, 64.60.Fr, 75.10.Hk*