Condensed Matter Physics, 2011, Vol. 14, No 2, 23002: 1–14
DOI: 10.5488/CMP.14.23002
http://www.icmp.lviv.ua/journal

Generalized Fokker-Planck equation and its solution for linear non-Markovian Gaussian systems

O.Yu. Sliusarenko

Akhiezer Institute for Theoretical Physics NSC KIPT, 1 Akademichna Str., 61108 Kharkiv, Ukraine

Received February 11, 2011

In this paper we suggest a consistent approach to derivation of generalized Fokker-Planck equation (GFPE) for Gaussian non-Markovian processes with stationary increments. This approach allows us to construct the probability density function (PDF) without a need to solve the GFPE. We employ our method to obtain the GFPE and PDFs for free generalized Brownian motion and the one in harmonic potential for the case of power-law correlation function of the noise. We prove the fact that the considered systems may be described with Einstein-Smoluchowski equation at high viscosity levels and long times. We also compare the results with those obtained by other authors. At last, we calculate PDF of thermodynamical work in the stochastic system which consists of a particle embedded in a harmonic potential moving with constant velocity, and check the work fluctuation theorem for such a system.

Key words: *Fokker-Planck equation, Gaussian system, non-Markovian system, thermodynamical work, transient fluctuation relation*

PACS: *05.10.Gg, 52.65.Ff, 02.50.Ey, 05.40.-a*

1. Introduction

The theory of Markovian Brownian motion is successfully used in describing a great variety of experiments and observations [1–4]. However, it remains an inapplicable model for the majority of natural systems where a characteristic time of thermal fluctuations is comparable to that of a Brownian particle (Gaussian non-Markovian systems), or where the processes are strongly non-Gaussian (either Markovian or non-Markovian), all of which results in the fact that the long-time mean squared displacement does not grow linearly in time any more, $\left\langle x^2(t)\right\rangle \propto t^{\mu}$. This phenomenon is called anomalous diffusion, namely, when $\mu < 1$, the system is said to be subdiffusive, and when $\mu > 1$, it is superdiffusive. Evidently, when $\mu = 1$ we have an ordinary Brownian motion.

There are two paradigmatic models describing anomalous diffusion: continuous time random walk (CTRW) and fractional Brownian motion (FBM). The CTRW approach was developed by Montroll and Weiss in 1965 [5] for a description of the electric charge transport in a disordered medium (amorphous semiconductor) [6]. This model considers the independent identically distributed couples of random space-time steps whose PDFs belong to the domain of attraction of Lévy stable laws.

Recently, the Markovian Lévy processes in external fields were studied by means of Langevin and fractional kinetics technique [7–9].

The second model (FBM) was introduced by Kolmogorov in 1940 [10] and later studied by Yaglom [11]. The name "fractional Brownian motion" belongs to Mandelbrot and van Ness who suggested a stochastic integral representation of this process [12]. FBM is a continuous centered non-Markovian Gaussian process $X^{(H)}(t)$ with covariance function

$$\left\langle X^{(H)}(t)X^{(H)}(t')\right\rangle = D\left(t^{2H} + t'^{2H} - |t-t'|^{2H}\right), \tag{1.1}$$

or, at large times,

$$\left\langle X^{(H)}(t)^2\right\rangle = 2Dt^{2H}, \tag{1.2}$$

where H is Hurst index, $0 < H < 1$, and D is the generalized diffusion coefficient of the dimension $[D] = \mathrm{cm}^2/\mathrm{sec}^{2\mathrm{H}}$. Previously, the problem of particle escape from the potential well in the framework of this model was considered in paper [13] by using the method of numerical simulation of Langevin equation with fractional Gaussian noise $Y^{(H)}(t)$. The latter is a non-Markovian stationary random process which is defined as the time derivative of FBM and whose autocorrelation function exhibits a slow decay at infinity as $\langle Y^{(H)}(t)Y^{(H)}(0)\rangle \approx 2DH(2H-1)t^{2H-2}$, in contrast to white Gaussian noise, where $\langle Y^{(1/2)}(t)Y^{(1/2)}(t')\rangle = 2D\delta(t-t')$. Power spectral density for white noise does not depend on frequency, otherwise the noise is called a coloured noise.

The pioneer work of deriving a differential equation (in essence a Fokker-Planck equation, FPE) describing ordinary Brownian motion (OBM) was done by Lord Rayleigh [14], within the approach of an absence of external potential and an overdamped discrete motion of a heavy Brownian particle. A more consistent method was developed by Fokker, Smoluchowski and Planck (a detailed historical sketch may be found in [4]). However, when dealing with the coloured noise case, the above-mentioned approaches are no longer valid.

The most common example of derivation of one-dimensional Fokker-Planck equation for coloured noise may be found in paper [4]; for one-dimensional case it was done in [15]; for a particular case of a linear oscillator it was obtained and studied in [16]. The multi-dimensional case was considered in [17].

The theory of generalized Brownian motion (GBM) finds its applications in many problems of modern physics, biophysics and astronomy. Indeed, polymers [18–20], elastic chains and membranes [19, 21–24] and rough surfaces [25–27] can be described by a continuum elastic model which accounts for their general stochastic behavior; it was recently shown that the probe particle in such systems performs FBM [28, 29]. The fluctuations of magnetic field in the turbulent plasma of the Earth's magnetospheric tail turn out to have colour: in the range of frequencies $\omega \leqslant 10^{-2}$ Hz they have the properties of flicker-noise (their power spectrum is proportional to $1/\omega$). When ω is about 10^{-1} Hz, they are a brown noise with the tendency of "blackening" at lower frequencies, see, e.g., the paper [30] and works cited therein. Moreover, a similar situation is known from experiments in laboratory plasmas: it was found that the power spectra of the saturation current, electrostatic potential fluctuations, and the turbulence-induced flux measured in various plasma devices [31] have power-law dependencies. At high frequencies, an asymptotic power fall-off of the fluctuation spectra with characteristic decay indices close to 2 was denoted; at intermediate frequencies, the decay indices were about 1, gaining a weak frequency dependence at the lowest frequencies.

Another important application comes from single-molecule dynamics. In paper [32] it is shown, that the experimental data of the distance fluctuations between the two components of fluorescein-tyrosine complex can be described within the framework of the Langevin equation with harmonic potential and coloured source possessing correlation function (CF), which decays as $t^{-0.51\pm0.07}$.

Below we present a consistent method of derivation of a multi-dimensional generalized Fokker-Planck equation for linear stochastic systems driven by coloured Gaussian noise paying special attention to the case of coloured Gaussian noise with power-law correlation function.

2. Basics of the method

We use the approach to obtaining an ordinary Fokker-Planck equation for linear systems with delta-correlated noise described in monograph [33] as the basis of the suggested method for derivation of the generalized Fokker-Planck equation. The paper continues and extends the previous studies [34] where we considered the GFPE for exponential and power-law correlation function restricting ourselves only to space-homogeneous case. Here we study a more general problem for the power-law correlation function. For the integrity and clarity of presentation, we give a full description of the method, as well.

First, let us write Langevin equations in multi-dimensional form:

$$\dot{\xi}_i = -a_{ik}\xi_k + Y_i\,(t) + K_i\,. \tag{2.1}$$

Here ξ_i is the generalized coordinate, a_{ik} is the coefficient matrix, Y_i is the external noise, K_i is the regular constant force; the dot above ξ_i stands for time derivative. Let the initial conditions be $\xi_i(t=0)=\xi_i(0)$. Then, the formal solution of (2.1) is

$$\xi_i(t;\xi(0)) = \left(\mathrm{e}^{-at}\right)_{ij}\xi_j(0) + \int\limits_0^t \mathrm{d}\tau \left(\mathrm{e}^{-a(t-\tau)}\right)_{ij}(Y_j(\tau)+K_j), \tag{2.2}$$

where $a \equiv \|a_{ik}\|$ is matrix composed from the elements of a_{ik}. The probability density function (PDF) of the value ξ_i in the moment of time t with the fixed $\xi(0)$ is evidently a multi-dimensional Dirac delta-function:

$$f(\xi,t;\xi(0)) = \delta(\xi-\xi(t,\xi(0))) \equiv \prod_i \delta(\xi_i-\xi_i(t,\xi(0))). \tag{2.3}$$

In case we do not know the exact $\xi(0)$, but their initial PDF $f(\xi(0),0)$, the PDF at an arbitrary moment of time t will be of the shape:

$$f(\xi,t) = \int \mathrm{d}\xi(0) f(\xi(0),0)\langle\delta(\xi-\xi(t,\xi(0)))\rangle, \tag{2.4}$$

where $\langle\ldots\rangle$ stands for $\int \mathrm{d}\tau p_Y(\tau)\ldots$ and $p_Y(\tau)$ is the PDF of noise. By using the n-dimensional delta-function representation $\delta(\xi)=(2\pi)^{-n}\int \mathrm{d}q\exp(\mathrm{i}q\xi)$ and taking into account (2.2), we have:

$$\langle\delta(\xi-\xi(t,\xi(0)))\rangle = (2\pi)^{-n}\int \mathrm{d}q\hat{G}(q,t)\exp\left[\mathrm{i}q\left(\xi-\mathrm{e}^{-at}\xi(0)\right)\right], \tag{2.5}$$

where

$$\hat{G}(q,t) = \left\langle \exp\left\{-\mathrm{i}q\int\limits_0^t \mathrm{d}\tau\mathrm{e}^{-a(t-\tau)}(Y(\tau)+K)\right\}\right\rangle. \tag{2.6}$$

Here we should remark that we use a matrix notation and the hat indicates that the value is a Fourier image.

Expanding the PDF into Fourier integral

$$f(\xi,t) = (2\pi)^{-n}\int \mathrm{d}q\mathrm{e}^{\mathrm{i}q\xi}\hat{f}(q,t), \tag{2.7}$$

we get from (2.4) and (2.5):

$$\hat{f}(q,t) = \hat{G}(q,t)\hat{f}\left(\left(\mathrm{e}^{-at}\right)^T q, 0\right), \tag{2.8}$$

where $(\mathrm{e}^{-at})^T$ is a matrix transposed to e^{-at}.

Hereinafter we assume the random process $Y_i(t)$ to be a stationary Gaussian process, so that the following relations are true:

$$\begin{aligned} \left\langle Y_{i_1}(t_1)\ldots Y_{i_{2n+1}}(t_{2n+1})\right\rangle &= 0, \\ \left\langle Y_{i_1}(t_1)\ldots Y_{i_{2n}}(t_{2n})\right\rangle &= \sum g_{i_1i_2}(t_1-t_2)\ldots g_{i_{2n-1}i_{2n}}(t_{2n-1}-t_{2n}), \end{aligned} \tag{2.9}$$

where the summation is executed by all possible pair compositions of $i_1, t_1; i_2, t_2; \ldots; i_{2n}, t_{2n}$. The number of such pairs is $(2n-1)!! = 2n!/n!2^n$. $g_{i_1i_2}(t_1-t_2)$ is a certain function of time difference.

Using the exponential function series expansion for (2.6) and keeping in mind (2.9) we have:

$$\begin{aligned} \hat{G}(q,t) &= \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^{2n}}{(2n)!}\frac{(2n)!}{n!2^n}\left[\int\limits_0^t \mathrm{d}t_1\int\limits_0^t \mathrm{d}t_2 q_i\left(\mathrm{e}^{-at_1}\right)_{ij}q_m\left(\mathrm{e}^{-at_1}\right)_{ml}g_{jl}(t_1-t_2)\right]^n \\ &\quad\times\exp\left(-\mathrm{i}q_i\int\limits_0^t \mathrm{d}\tau\left(\mathrm{e}^{-a\tau}\right)_{ij}K_j\right). \end{aligned} \tag{2.10}$$

Here and below for simplicity we write $g_{jl}(t_1-t_2)$ instead of $g_{j_1 l_2}(t_1-t_2)$.

Introducing the value

$$\mathfrak{M}_{im}(t)=\frac{1}{2}\int_0^t \mathrm{d}t_1\int_0^t \mathrm{d}t_2\left(\mathrm{e}^{-at_1}\right)_{ij}\left(\mathrm{e}^{-at_2}\right)_{ml} g_{jl}(t_1-t_2)\,, \tag{2.11}$$

we get for $G(q,t)$:

$$\hat{G}(q,t)=\exp\left[-q_i q_m \mathfrak{M}_{im}(t)-\mathrm{i}q_i\int_0^t \mathrm{d}\tau\left(\mathrm{e}^{-a\tau}\right)_{ij}K_j\right]. \tag{2.12}$$

2.1. Fokker-Planck equation

It may be easily proven that the value $\hat{G}(q,t)$ obeys the following relation:

$$\frac{\partial\hat{G}}{\partial t}+q_i a_{ik}\frac{\partial\hat{G}}{\partial q_k}+\mathrm{i}K_i q_i\hat{G}=-q_i q_m\mathfrak{D}_{im}(t)\,\hat{G}(q,t)\,, \tag{2.13}$$

where

$$\mathfrak{D}_{im}(t)=\frac{\mathrm{d}\mathfrak{M}_{im}}{\mathrm{d}t}+a_{ik}\mathfrak{M}_{km}(t)+a_{mk}\mathfrak{M}_{ik}(t)\,. \tag{2.14}$$

On the other hand, due to an obvious equality

$$\frac{\partial}{\partial t}f\left(\left(\mathrm{e}^{-at}\right)^T q,0\right)=-qa\frac{\partial}{\partial q}f\left(\left(\mathrm{e}^{-at}\right)^T q,0\right) \tag{2.15}$$

and (2.8) we can conclude that the function $f(q,t)$ obeys the same equation as (2.13) which after the inverse Fourier transform yields:

$$\frac{\partial f(\xi,t)}{\partial t}+K_i\frac{\partial f(\xi,t)}{\partial\xi_i}=\frac{\partial}{\partial\xi_i}\left[a_{im}\xi_m f(\xi,t)\right]+\mathfrak{D}_{im}(t)\,\frac{\partial^2 f(\xi,t)}{\partial\xi_i\partial\xi_m}\,, \tag{2.16}$$

actually being the generalized Fokker-Planck equation (GFPE).

Now, let us simplify the expressions (2.14). Noticing that

$$\begin{aligned} a_{ik}\left(\mathrm{e}^{-at}\right)_{kj} &= \frac{\partial}{\partial t}\left(\mathrm{e}^{-at}\right)_{ij}\,,\\ \frac{\partial g_{jl}(t_1-t_2)}{\partial t_1} &= -\frac{\partial g_{jl}(t_1-t_2)}{\partial t_2} \end{aligned}$$

and

$$\frac{\mathrm{d}\mathfrak{M}_{im}}{\mathrm{d}t} = \frac{1}{2}\int_0^t \mathrm{d}t_1\left(\mathrm{e}^{-at_1}\right)_{ij}\left(\mathrm{e}^{-at}\right)_{ml} g_{jl}(t_1-t)+\frac{1}{2}\int_0^t \mathrm{d}t_2\left(\mathrm{e}^{-at}\right)_{ij}\left(\mathrm{e}^{-at_2}\right)_{ml} g_{jl}(t-t_2)\,,$$

we arrive at the expression

$$\mathfrak{D}_{im}(t)=\frac{1}{2}\int_0^t \mathrm{d}t_1\left[\left(\mathrm{e}^{-at_1}\right)_{ij}g_{jm}(|t_1|)\,+\,\left(\mathrm{e}^{-at_1}\right)_{mj}g_{ij}(|t_1|)\right]. \tag{2.17}$$

2.2. Probability density function

The advantage of the described method is that there is no need in solving the Fokker-Planck equation to obtain the probability density function since we have constructed it implicitly at the stage of the GFPE derivation. Indeed, according to (2.8), knowing the Fourier image of the initial PDF $\hat{f}(q,0) \equiv \mathfrak{F}\{f(\xi,0)\}$ we can easily get the PDF for an arbitrary moment of time:

$$\begin{aligned} f(\xi,t) &= \mathfrak{F}^{-1}\left\{\hat{f}(q,t)\right\} \\ &= \mathfrak{F}^{-1}\left\{\hat{G}(q,t)\,\hat{f}\left(\left(\mathrm{e}^{-at}\right)^{T} q, 0\right)\right\}, \end{aligned} \tag{2.18}$$

where $\hat{G}(q,t)$ is given with (2.12).

However, the expressions (2.11) may be rather complicated for direct calculations regarding, e.g., a power-law correlation function $g(t_1-t_2)$. By means of integration variables change we get a much more usable expression, because now the internal integral does not contain the correlation function:

$$\mathfrak{M}_{im}(t) = \frac{1}{2}\int\limits_0^t \mathrm{d}\tau g_{jl}(\tau)\int\limits_0^{t-\tau}\mathrm{d}T\left\{\left(\mathrm{e}^{-a(T+\tau)}\right)_{ij}\left(\mathrm{e}^{-aT}\right)_{ml} + \left(\mathrm{e}^{-aT}\right)_{ij}\left(\mathrm{e}^{-a(T+\tau)}\right)_{ml}\right\}.$$

3. Generalized Brownian motion

Let us now apply the derived formulae to the specific stochastic system: the generalization of the classical Brownian motion with the external random force is a stationary Gaussian noise with long memory effects.

3.1. Free generalized Brownian motion. Spatially homogenous case

First, we investigate a simple system described with the following Langevin equations:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -\gamma v + Y(t), \end{aligned} \tag{3.1}$$

where $x(t)$ is particles coordinate, γ is friction constant, $Y(t)$ is Gaussian external noise with $\langle Y(t)Y(t')\rangle \equiv g(t-t')$. The coefficient matrix a_{im} is

$$a = \begin{bmatrix} 0 & -1 \\ 0 & \gamma \end{bmatrix},$$

and $g_{ij}(t_1-t_2) = \delta_{i2}\delta_{j2}g(t_1-t_2)$. The solution of the homogenous system (3.1) yields:

$$\begin{aligned} x(t) &= x_0 + \frac{1-\mathrm{e}^{-t\gamma}}{\gamma}v_0, \\ v(t) &= v_0\mathrm{e}^{-t\gamma}. \end{aligned} \tag{3.2}$$

Comparing these expressions with (2.2) we get

$$\mathrm{e}^{-at} = \begin{bmatrix} 1 & \dfrac{1-\mathrm{e}^{-t\gamma}}{\gamma} \\ 0 & \mathrm{e}^{-t\gamma} \end{bmatrix}. \tag{3.3}$$

In what follows we restrict ourselves to the power-law noise correlation function of the form

$$g(\tau) = \frac{c}{|\tau|^{\beta}\Gamma(1-\beta)}, \tag{3.4}$$

with $0<\beta<1$, which is actually the asymptotics of the CF for fractional Gaussian noise. Note, that at $\beta\to 1$, we get the delta-function limit $g(\tau)\to c\delta(\tau)$ [35].

Now, we can write out the exact values for the coefficients $\mathfrak{D}_{ij}$ and $\mathfrak{M}_{ij}$, see equations (2.17) and (2.19), respectively:

$$\mathfrak{D}_{11}(t) = 0, \tag{3.5}$$

$$\mathfrak{D}_{12}(t) = \mathfrak{D}_{21}(t) = \frac{ct^{1-\beta}}{2\gamma\Gamma(2-\beta)}\left[1+(1-\beta)\mathrm{E}_{\beta}(t\gamma)\right]-\frac{1}{2}c\gamma^{\beta-2}, \tag{3.6}$$

$$\mathfrak{D}_{22}(t) = c\gamma^{\beta-1}-\frac{ct^{1-\beta}\mathrm{E}_{\beta}(t\gamma)}{\Gamma(1-\beta)}, \tag{3.7}$$

$$\begin{aligned}\mathfrak{M}_{11}(t) ={}& -\frac{c\left(1-\mathrm{e}^{-t\gamma}\right)t^{1-\beta}}{\gamma^{3}\Gamma(2-\beta)}+\frac{ct^{2-\beta}}{\gamma^{2}\Gamma(3-\beta)}-\frac{1}{2}c\gamma^{\beta-4}\\ &+\frac{c\left(-\mathrm{e}^{-2t\gamma}+2\mathrm{e}^{-t\gamma}\right)\mathrm{M}(1-\beta,2-\beta,t\gamma)t^{1-\beta}}{2\gamma^{3}\Gamma(2-\beta)}+\frac{c\mathrm{E}_{\beta}(t\gamma)t^{1-\beta}}{2\gamma^{3}\Gamma(1-\beta)},\end{aligned} \tag{3.8}$$

$$\mathfrak{M}_{12}(t) = \frac{c\left(1-\mathrm{e}^{-t\gamma}\right)t^{1-\beta}}{2\gamma^{2}\Gamma(2-\beta)}-\frac{c\left(-\mathrm{e}^{-2t\gamma}+\mathrm{e}^{-t\gamma}\right)t^{1-\beta}\mathrm{M}(1-\beta,2-\beta,t\gamma)}{2\gamma^{2}\Gamma(2-\beta)}, \tag{3.9}$$

$$\mathfrak{M}_{22}(t) = \frac{1}{2}c\gamma^{\beta-2}-\frac{ct^{1-\beta}\mathrm{E}_{\beta}(t\gamma)}{2\gamma\Gamma(1-\beta)}-\frac{c\mathrm{e}^{-2t\gamma}t^{1-\beta}\mathrm{M}(1-\beta,2-\beta,t\gamma)}{2\gamma\Gamma(2-\beta)}. \tag{3.10}$$

Here $\mathrm{E}_{\beta}(t)$ is an integral exponential function

$$\mathrm{E}_{\beta}(t)=\int\limits_{1}^{\infty}\mathrm{d}p\frac{\mathrm{e}^{-tp}}{p^{\beta}}$$

and $\mathrm{M}(a,b,t)$ is Kummer's confluent hypergeometric function:

$$\mathrm{M}(a,b,t)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int\limits_{0}^{1}\mathrm{d}u\mathrm{e}^{tu}u^{a-1}(1-u)^{b-a-1},$$

(see, e.g. [36]).

The generalized Fokker-Planck equation (2.16) for this case yields:

$$\frac{\partial f(v,t)}{\partial t}=\frac{\partial}{\partial v}\left[\gamma vf(v,t)\right]+D_{22}(t)\frac{\partial^{2}f(v,t)}{\partial v^{2}}. \tag{3.11}$$

When $\gamma t\gg 1$, the latter expression takes the form

$$\frac{\partial f(v,t)}{\partial t}=\frac{\partial}{\partial v}\left[\gamma vf(v,t)\right]+\frac{c}{\gamma}\left(\gamma^{\beta}-\frac{\mathrm{e}^{-t\gamma}t^{-\beta}}{\Gamma(1-\beta)}\right)\frac{\partial^{2}f(v,t)}{\partial v^{2}}. \tag{3.12}$$

According to the procedure described in section 2.2, the PDF $f(v,t)$ with the initial condition $f(v,0)=n\delta(v-v_0)$ reads

$$f(v,t)=\frac{n}{2\sqrt{\pi\sigma}}\exp\left[-\frac{\left(v-\mathrm{e}^{-t\gamma}v_0\right)^{2}}{4\sigma}\right], \tag{3.13}$$

where

$$\sigma=\frac{1}{2}c\gamma^{\beta-2}-\frac{ct^{1-\beta}\mathrm{E}_{\beta}(t\gamma)}{2\gamma\Gamma(1-\beta)}-\frac{c\mathrm{e}^{-2t\gamma}t^{1-\beta}\mathrm{M}(1-\beta,2-\beta,t\gamma)}{2\gamma\Gamma(2-\beta)}. \tag{3.14}$$

3.2. Free generalized Brownian motion. Spatially inhomogenous case

Now we examine the same system but with inhomogenous initial condition $f(x,v,0) = n\delta(x - x_0)\delta(v - v_0)$. All the values for $\mathfrak{D}_{ij}(t)$ and $\mathfrak{M}_{ij}(t)$ clearly, remain the same as in the previous section, but the GFPE and the PDF do change:

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + \gamma\frac{\partial (vf)}{\partial v} + 2\mathfrak{D}_{12}\frac{\partial^2 f}{\partial x \partial v} + \mathfrak{D}_{22}\frac{\partial^2 f}{\partial v^2}\,. \tag{3.15}$$

Again we construct the solution with the procedure explained in section 2.2:

$$f(x,v,t|\omega=0) = \frac{n}{4\pi\sqrt{\sigma}}\exp\left[-\frac{1}{4\sigma}\mathfrak{M}_{11}(t)\left(p\frac{\mathfrak{M}_{12}(t)}{\mathfrak{M}_{11}(t)} - v_0\mathrm{e}^{-t\gamma} + v\right)^2 - \frac{p^2}{4\mathfrak{M}_{11}(t)}\right], \tag{3.16}$$

where

$$p = \frac{v_0}{\gamma}\left(1 - \mathrm{e}^{-t\gamma}\right) - x + x_0 \tag{3.17}$$

and

$$\sigma = \mathfrak{M}_{11}(t)\mathfrak{M}_{22}(t) - \mathfrak{M}_{12}(t)^2. \tag{3.18}$$

3.3. Generalized Brownian motion of linear oscillator

Here we study the most general system, though restricting ourselves to the case of a harmonic potential $U(x) = \omega^2 x^2/2$. The pair of Langevin equations now have the following form:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= v,\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -\gamma v - \omega^2 x + Y(t)\,,\end{aligned} \tag{3.19}$$

where $x(t)$ is particles coordinate, γ is friction constant, ω is frequency of the linear oscillator, $Y(t)$ is the external noise. The coefficient matrix introduced in equation (2.1) is

$$a = \begin{bmatrix} 0 & -1 \\ \omega^2 & \gamma \end{bmatrix}.$$

Again, $g_{ij}(t_1 - t_2) = \delta_{i2}\delta_{j2}g(t_1 - t_2)$. The solution of the homogenous system (3.19) yields

$$\begin{aligned} v(t) &= A_1\mathrm{e}^{-\gamma t/2}\mathrm{e}^{\Omega t/2} + A_2\mathrm{e}^{-\gamma t/2}\mathrm{e}^{-\Omega t/2},\\ x(t) &= -\frac{1}{\omega^2}\left[\dot{v}(t) + \gamma v(t)\right]\\ &= -\frac{1}{2\omega^2}\Big[A_1\mathrm{e}^{-\gamma t/2}\mathrm{e}^{\Omega t/2}(\gamma + \Omega) + A_2\mathrm{e}^{-\gamma t/2}\mathrm{e}^{-\Omega t/2}(\gamma - \Omega)\Big],\end{aligned} \tag{3.20}$$

where A_1 and A_2 are constants depending on the initial conditions and here we introduce the value $\Omega \equiv +\sqrt{\gamma^2 - 4\omega^2}$. Assigning $x(0) = x_0$ and $v(0) = v_0$, we get

$$\begin{aligned} A_1 &= -\frac{2\omega^2 x_0 + v_0(\gamma - \Omega)}{2\Omega}\,,\\ A_2 &= \frac{2\omega^2 x_0 + v_0(\gamma + \Omega)}{2\Omega}\,.\end{aligned} \tag{3.21}$$

Now, substituting the latter expressions into equations (3.20) and comparing the result with the formal solution (2.2) without the integral term (since we are looking for the solution of the homogenous system), we find

$$\mathrm{e}^{-at} = \mathrm{e}^{-\gamma t/2}\begin{bmatrix} \cosh\left(\frac{\Omega t}{2}\right) + \frac{\gamma}{\Omega}\sinh\left(\frac{\Omega t}{2}\right) & \frac{2}{\Omega}\sinh\left(\frac{\Omega t}{2}\right) \\ -\frac{2\omega^2}{\Omega}\sinh\left(\frac{\Omega t}{2}\right) & \cosh\left(\frac{\Omega t}{2}\right) - \frac{\gamma}{\Omega}\sinh\left(\frac{\Omega t}{2}\right)\end{bmatrix}. \tag{3.22}$$

The final step before proceeding to the GFPE and the PDF evaluation is to obtain the exact expressions for $\mathfrak{M}_{ij}(t)$ and the generalized diffusion coefficients $\mathfrak{D}_{ij}(t)$ $(i,j=1,2)$ for our power-law correlation function (3.4).

A straightforward integration of equation (2.19) with (3.4) gives

$$\begin{aligned}
\mathfrak{M}_{11}(t) &= -\frac{c\mathrm{e}^{-2a_p t}\left(a_m\gamma - 2\mathrm{e}^{t\Omega}\omega^2\right)\mathrm{M}(1-\beta,2-\beta,a_p t)t^{1-\beta}}{2\gamma\omega^2\Omega^2\Gamma(2-\beta)} \\
&\quad - \frac{c\mathrm{e}^{-t\gamma}\left(a_p\mathrm{e}^{t\Omega}\gamma - 2\omega^2\right)\mathrm{M}(1-\beta,2-\beta,a_m t)t^{1-\beta}}{2\gamma\omega^2\Omega^2\Gamma(2-\beta)} \\
&\quad + \frac{a_m c\mathrm{E}_\beta(a_p t)t^{1-\beta}}{2\gamma\omega^2\Omega\Gamma(1-\beta)} - \frac{a_p c\mathrm{E}_\beta(a_m t)t^{1-\beta}}{2\gamma\omega^2\Omega\Gamma(1-\beta)} + \frac{\left(a_p^2 a_m^\beta - a_p^\beta a_m^2\right)c}{2a_p a_m\gamma\omega^2\Omega}\,, \qquad (3.23) \\
\mathfrak{M}_{12}(t) &= -\frac{c\mathrm{e}^{-t(2a_p+\gamma)}}{2\Omega^2\Gamma(2-\beta)}t^{1-\beta}\Big[\left(\mathrm{e}^{2a_p t} - \mathrm{e}^{t\gamma}\right)\mathrm{M}(1-\beta,2-\beta,a_p t) \\
&\quad + \left(\mathrm{e}^{2a_p t} - \mathrm{e}^{t(\gamma+2\Omega)}\right)\mathrm{M}(1-\beta,2-\beta,a_m t)\Big], \qquad (3.24) \\
\mathfrak{M}_{22}(t) &= -\frac{a_p c\mathrm{e}^{-2a_p t}\left(\gamma - 2a_m\mathrm{e}^{t\Omega}\right)\mathrm{M}(1-\beta,2-\beta,a_p t)t^{1-\beta}}{2\gamma\Omega^2\Gamma(2-\beta)} \\
&\quad - \frac{a_m c\mathrm{e}^{t(\Omega-2a_p)}\left(\mathrm{e}^{t\Omega}\gamma - 2a_p\right)\mathrm{M}(1-\beta,2-\beta,a_m t)t^{1-\beta}}{2\gamma\Omega^2\Gamma(2-\beta)} \\
&\quad - \frac{a_p c\mathrm{E}_\beta(a_p t)t^{1-\beta}}{2\gamma\Omega\Gamma(1-\beta)} + \frac{a_m c\mathrm{E}_\beta(a_m t)t^{1-\beta}}{2\gamma\Omega\Gamma(1-\beta)} + \frac{\left(a_p^\beta - a_m^\beta\right)c}{2\gamma\Omega}\,, \qquad (3.25)
\end{aligned}$$

where $\Omega = \sqrt{\gamma^2 - 4\omega^2}$, $a_p = (\gamma+\Omega)/2$, $a_m = (\gamma-\Omega)/2$.

According to equation (2.16), the GFPE for such a system reads

$$\begin{aligned}
\frac{\partial f(x,v,t)}{\partial t} &= \frac{\partial}{\partial x}\left[-vf(x,v,t)\right] + \frac{\partial}{\partial v}\left[(\omega^2 x + \gamma v)f(x,v,t)\right] \\
&\quad + \mathfrak{D}_{11}\frac{\partial^2 f}{\partial x^2} + \left(\mathfrak{D}_{12} + \mathfrak{D}_{21}\right)\frac{\partial^2 f}{\partial x\partial v} + \mathfrak{D}_{22}\frac{\partial^2 f}{\partial v^2}\,, \qquad (3.26)
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{D}_{11} &= 0, \qquad (3.27) \\
\mathfrak{D}_{12} &= \mathfrak{D}_{21} = \frac{a_m a_p c\left[\mathrm{E}_\beta(a_p t) - \mathrm{E}_\beta(a_m t)\right]t^{1-\beta}}{2\omega^2\Omega\Gamma(1-\beta)} + \frac{c\left[a_m^\beta(\gamma+\Omega) - a_p^\beta(\gamma-\Omega)\right]}{4\omega^2\Omega}\,, \qquad (3.28) \\
\mathfrak{D}_{22} &= \frac{c\left[a_m\mathrm{E}_\beta(a_m t) - a_p\mathrm{E}_\beta(a_p t)\right]t^{1-\beta}}{\Omega\Gamma(1-\beta)} + \frac{c\left(a_p^\beta - a_m^\beta\right)}{\Omega}\,. \qquad (3.29)
\end{aligned}$$

At this stage we may compare these diffusion coefficients to that obtained in the paper by Wang and Masoliver [16]. We consider only the case of the external driving noise (see section 3.2 of the mentioned paper). To establish a connection with our GFPE and equation (W29) (here the letter “W” indicates the reference to the equation from the paper [16]), let us substitute equations (W54), (W55) and (W14) into (W35). Now we see, that $\psi(t) \equiv 2\mathfrak{D}_{12}(t)$, $\phi(t) \equiv \mathfrak{D}_{22}(t)$, i.e. we get a complete coincidence between our GFPE (3.26) and Wang’s GFPE (W29).

The PDF is evaluated directly through relations (2.18) and (2.12) with $K_j \equiv 0$:

$$\begin{aligned} f(x,v,t) &= \frac{n}{4\pi\sqrt{\mathfrak{M}_{11}(t)\mathfrak{M}_{22}(t)-\mathfrak{M}_{12}(t)^2}} \exp\left(\frac{\mathrm{e}^{-\gamma t}}{4\Omega^2\left(\mathfrak{M}_{11}(t)\mathfrak{M}_{22}(t)-\mathfrak{M}_{12}(t)^2\right)}\right. \\ &\times \left\{\mathfrak{M}_{11}(t)\left[\mathrm{e}^{\gamma t/2}v\Omega - v_0\cosh(\Omega t/2)\,\Omega + \left(2x_0\omega^2+v_0\gamma\right)\sinh(\Omega t/2)\right]^2\right. \\ &+ \mathfrak{M}_{22}(t)\left[-\mathrm{e}^{\gamma t/2}x\Omega + x_0\cosh(\Omega t/2)\,\Omega + (2v_0+x_0\gamma)\sinh(\Omega t/2)\right]^2 \\ &+ 2\mathfrak{M}_{12}(t)\left[\mathrm{e}^{\gamma t/2}v\Omega - v_0\cosh(\Omega t/2)\,\Omega + \left(2x_0\omega^2+v_0\gamma\right)\sinh(\Omega t/2)\right] \\ &\left.\left.\times\left[-\mathrm{e}^{\gamma t/2}x\Omega + x_0\cosh(\Omega t/2)\,\Omega + (2v_0+x_0\gamma)\sinh(\Omega t/2)\right]\right\}\right), \end{aligned} \tag{3.30}$$

where $\mathfrak{M}_{ij}$ are given with equations (3.23–3.25).

4. Transition to Einstein-Smoluchowski equation

Now let us prove that the system considered in section 3.3 may be described with Einstein-Smoluchowski equation at high viscosity levels and at long times.

When $\omega/\gamma \ll 1$, we can neglect the time derivative of velocity and, therefore, the pair of Langevin equations (3.19) transforms into a single overdamped Langevin equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\omega^2}{\gamma}x(t) + \frac{1}{\gamma}Y(t). \tag{4.1}$$

The GFPE for such a system, according to equations (4.1) and (2.16) has the following form:

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\omega^2}{\gamma}xf(x,t)\right) + \mathfrak{D}(t)\frac{\partial^2 f(x,t)}{\partial x^2} \tag{4.2}$$

with

$$\mathfrak{D}(t) = \frac{c\omega^{2\beta-2}}{\gamma^{\beta+1}}\left(1 - \frac{\Gamma\left(1-\beta, t\omega^2/\gamma\right)}{\Gamma(1-\beta)}\right). \tag{4.3}$$

Executing the same calculations as in the previous section, for the PDF we unfold:

$$\rho(x,t) = \frac{n}{2\sqrt{\pi\mathfrak{M}(t)}}\exp\left[-\frac{\left(x - x_0\mathrm{e}^{-t\omega^2/\gamma}\right)^2}{4\mathfrak{M}(t)}\right], \tag{4.4}$$

where

$$\mathfrak{M}(t) = \frac{1}{2}c\gamma^{-\beta}\omega^{2\beta-4} - \frac{ct^{1-\beta}}{2\gamma\omega^2\Gamma(1-\beta)}\left[\mathrm{E}_\beta\left(\frac{t\omega^2}{\gamma}\right) + \frac{\mathrm{e}^{-2t\omega^2/\gamma}}{1-\beta}\mathrm{M}\left(1-\beta, 2-\beta, \frac{t\omega^2}{\gamma}\right)\right]. \tag{4.5}$$

Similar results were obtained by M. Cáceres in [15] for a stationary case [see equations (2.14) with (2.17) of the mentioned paper].

Expanding the coefficient (4.5) into a series at large (γt)'s and substituting it to the PDF (4.4) yield:

$$\begin{aligned} \rho(x,t) &\approx \frac{n\gamma^{\beta/2}\omega^{2-\beta}}{\sqrt{2\pi c}}\left(1 + \frac{t^{-\beta}\gamma^\beta\omega^{-2\beta}\mathrm{e}^{-t\omega^2/\gamma}}{\Gamma(1-\beta)}\right) \\ &\times \exp\left[-x^2\left(\frac{\gamma^\beta\omega^{4-2\beta}}{2c} + \frac{t^{-\beta}\gamma^{2\beta}\omega^{4-4\beta}\mathrm{e}^{-t\omega^2/\gamma}}{c\Gamma(1-\beta)}\right) + \frac{xx_0\gamma^\beta\omega^{4-2\beta}\mathrm{e}^{-t\omega^2/\gamma}}{c}\right]. \end{aligned} \tag{4.6}$$

Now we return to the PDF for the most general case (3.30) and also expand it into a series at $\omega/\gamma \ll 1 \ll \gamma t$:

$$
\begin{aligned}
f(x,v,t) \approx\ & A\exp\Bigg[-v^2\left(-\frac{t^{-\beta}\gamma^{2-2\beta}\mathrm{e}^{-\omega^2 t/\gamma}}{c\Gamma(1-\beta)}+\frac{t^{-\beta}\mathrm{e}^{-t\gamma}\omega^{2-2\beta}}{c\Gamma(1-\beta)}+\frac{\gamma^{-\beta}\left(\gamma^2-3\omega^2\right)}{2c}\right)\\
&-x^2\left(-\frac{t^{-2\beta}\mathrm{e}^{-t\gamma}\gamma^{\beta}\omega^{4-4\beta}}{c\Gamma(1-\beta)^2}+\frac{t^{-\beta}\gamma^{2\beta}\omega^{4-4\beta}\mathrm{e}^{-\omega^2 t/\gamma}}{c\Gamma(1-\beta)}+\frac{\omega^{4-2\beta}\left(\gamma^{\beta}-\gamma^{-\beta}\omega^{2\beta}\right)}{2c}\right)\\
&-v\left(\frac{\omega^2\gamma^{-\beta}\mathrm{e}^{-\omega^2 t/\gamma}(v_0+x_0\gamma)}{c}-\frac{v_0\mathrm{e}^{-t\gamma}\gamma^{2-\beta}}{c}\right)\\
&-x\left(\frac{v_0\mathrm{e}^{-t\gamma}\gamma^{\beta-1}\omega^{4-2\beta}}{c}-\frac{\gamma^{\beta-1}\omega^{4-2\beta}\mathrm{e}^{-\omega^2 t/\gamma}(v_0+x_0\gamma)}{c}\right)\\
&-{v_0}^2\left(\frac{\gamma^{\beta-2}\omega^{4-2\beta}\mathrm{e}^{-2\omega^2 t/\gamma}}{2c}-\frac{\mathrm{e}^{-t\gamma}\gamma^{\beta-2}\omega^{4-2\beta}}{c}\right)\\
&-{x_0}^2\left(\frac{\gamma^{\beta}\omega^{4-2\beta}\mathrm{e}^{-2\omega^2 t/\gamma}}{2c}-\frac{\mathrm{e}^{-t\gamma}\gamma^{\beta-2}\omega^{6-2\beta}}{c}\right)\\
&-v_0x_0\left(\frac{\gamma^{\beta-1}\omega^{4-2\beta}\mathrm{e}^{-2\omega^2 t/\gamma}}{c}-\frac{\mathrm{e}^{-t\gamma}\gamma^{\beta-1}\omega^{4-2\beta}}{c}\right)\Bigg], \qquad (4.7)
\end{aligned}
$$

$$
A \approx \frac{2n\gamma\omega^{2-\beta}}{4\pi c}\left(1+\frac{\omega^{-2\beta}\gamma^{\beta}t^{-\beta}\mathrm{e}^{-\omega^2 t/\gamma}}{\Gamma(1-\beta)}-\frac{\mathrm{e}^{-t\gamma}(t\omega)^{-2\beta}}{\Gamma(1-\beta)^2}\right). \qquad (4.8)
$$

Then, integrating it by v in the range of $(-\infty;\infty)$ and neglecting the terms of the higher magnitude of smallness than $\exp\left\{-\omega^2 t/\gamma\right\}$ we get:

$$
\tilde{f}(x,t) \propto \exp\left[-x^2\left(\frac{\gamma^{\beta}\omega^{4-2\beta}}{2c}+\frac{t^{-\beta}\gamma^{2\beta}\omega^{4-4\beta}\mathrm{e}^{-\omega^2 t/\gamma}}{c\Gamma(1-\beta)}\right)+\frac{xx_0\gamma^{\beta}\omega^{4-2\beta}\mathrm{e}^{-t\omega^2/\gamma}}{c}\right], \qquad (4.9)
$$

which fully corresponds to the PDF (4.6), and, therefore, proves the fact that the considered system at large times and strong friction may be described with Einstein-Smoluchowski equation.

5. GFPE for overdamped harmonic oscillator with constant drift

As a final application example of the presented technique, let us study the PDF of the thermodynamical work w in the stochastic system which consists of a particle inside a harmonic potential moving with constant velocity v_*, $U=(k/2)[x-X(t)]^2$, $X(t)=v_*t$, $x\,(t)$ is the particle's coordinate. Our aim is to get the transient fluctuation relation for such a system, which will demonstrate large-deviation symmetry properties in the PDF, and compare it to the classical case.

The work w is defined as follows:

$$
w(t)=\int \mathrm{d}X\frac{\partial U}{\partial X}=\int_0^t \mathrm{d}t'\frac{\mathrm{d}X}{\mathrm{d}t'}\frac{\partial U}{\partial X}=-kv_*\int_0^t \mathrm{d}t'(x-v_*t'). \qquad (5.1)
$$

Introducing $y\,(t)=x(t)-v_*t$, for the overdamped Langevin equation and the equation for the thermodynamical work $w\,(t)$ we have:

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{1}{\tau}y\,(t)+Y\,(t)-v_*\,,\\
\frac{\mathrm{d}w}{\mathrm{d}t} &= -kv_*y\,(t)\,, \qquad (5.2)
\end{aligned}
$$

where $\tau = m\gamma/k$. Alternatively, if we consider the plane (y, w), the coefficient matrix a of the system will be

$$a = \begin{bmatrix} 1/\tau & 0 \\ kv_* & 0 \end{bmatrix}. \tag{5.3}$$

Since $y(t) = y_0 \exp(-t/\tau)$, $y_0 = x_0$ is the initial position of the particle,

$$w(t) = w_0 + y_0 k v_* \tau \left(\mathrm{e}^{-t/\tau} - 1\right),$$

Then the evolution matrix

$$\mathrm{e}^{-at} = \begin{bmatrix} \mathrm{e}^{-t/\tau} & 0 \\ kv_*\tau\left(\mathrm{e}^{-t/\tau} - 1\right) & 1 \end{bmatrix}. \tag{5.4}$$

For the diffusion coefficients $\mathfrak{D}_{ij}(t)$ we have:

$$\mathfrak{D}_{11}(t) = c\gamma^2\tau^{1-\beta}\left(1 - \frac{\Gamma(1-\beta, t/\tau)}{\Gamma(1-\beta)}\right), \tag{5.5}$$

$$\mathfrak{D}_{12}(t) = \frac{ckv_*\gamma^2 t^{-\beta}\left(t/\tau\left[(\beta-1)\mathrm{E}_\beta(t/\tau) - 1\right] + \Gamma(2-\beta)(t/\tau)^\beta\right)}{2\Gamma(2-\beta)}, \tag{5.6}$$

$$\mathfrak{D}_{22}(t) = 0. \tag{5.7}$$

The generalized Fokker-Planck equation in this case will have the form:

$$\frac{\partial f(y, w, t)}{\partial t} = \left(\frac{y}{\tau} + v_*\right)\frac{\partial f}{\partial y} + kv_* y\frac{\partial f}{\partial w} + \mathfrak{D}_{11}\frac{\partial^2 f}{\partial y^2} + 2\mathfrak{D}_{12}(t)\frac{\partial^2 f}{\partial y \partial w}. \tag{5.8}$$

Now, considering an initial condition $f(y, w, 0) = n\delta(y - y_0)\delta(w - w_0)$, when $y_0 = 0$, $w_0 = 0$ we get for the PDF:

$$f(w, t) = \frac{n}{2\sqrt{\pi}\sqrt{\mathfrak{M}_{22}(t)}}\exp\left\{-\frac{\left[kv_*^2\tau^2\left(t/\tau + \mathrm{e}^{-t/\tau} - 1\right) + w\right]^2}{4\mathfrak{M}_{22}(t)}\right\}, \tag{5.9}$$

where

$$\begin{aligned} \mathfrak{M}_{22} = {} & \frac{cm^2v_*^2\gamma^2 t^{2-\beta}}{2\Gamma(3-\beta)}\left\{2 - \mathrm{e}^{-t/\tau}\mathrm{M}\left[1, 3-\beta, \frac{\mathrm{t}}{\tau}\right]\right. \\ & \left. - \mathrm{e}^{-t/\tau}\left(2 - \mathrm{e}^{-t/\tau}\right)\mathrm{M}\left[2-\beta, 3-\beta, \frac{\mathrm{t}}{\tau}\right]\right\}, \end{aligned} \tag{5.10}$$

which fully corresponds to the results obtained in [37].

After the relaxation stage has passed, at $t \gg \tau$ we find for the transient fluctuation relation:

$$\frac{f(w, t)}{f(-w, t)} = \exp\left[\frac{\Gamma(3-\beta) w t^{\beta-1}}{cm\gamma}\right]. \tag{5.11}$$

Thus, the fluctuation relation for the system subjected to a coloured noise with the slowly decaying power-law correlation function differs from that for ordinary Brownian motion. As we stated above, the classical case limit is revealed at $\beta \to 1$.

6. Conclusions

In this paper we suggested a consistent method for derivation of the generalized Fokker-Planck equation for linear multidimensional Gaussian non-Markovian systems. Taking the case of the Gaussian systems with slowly decaying power-law correlations, we obtained the following results:

- Firstly, we constructed the solution of generalized Fokker-Planck equation, the probability density function, without solving it directly.
- We derived generalized Fokker-Planck equation for free motion and constructed the probability density function for spatially homogeneous and inhomogeneous cases.
- For the case of the motion in a harmonic potential, the generalized Fokker-Planck equation and the probability density function were also obtained, and the results were compared to those of the other authors.
- We show the equivalence in description of generalized Brownian motion in a harmonic potential with generalized Fokker-Planck equation and generalized Einstein-Smoluchowski equation at high viscosity levels and at long times.
- Finally, we investigated the probability density function for thermodynamical work in the stochastic system which consists of a particle inside a uniformly moving harmonic potential underlining strong differences in transient fluctuation relations for the generalized Brownian motion and the ordinary Brownian motion cases.

Acknowledgements

O. S. would like to thank A.V. Chechkin for the problem setting and for the discussion of the results, S.V. Peletminskii and Yu.V. Slyusarenko for helpful comments on the paper.

References

1. Chandrasekhar S., Rev. Mod. Phys., 1943, **15**, 1; doi:10.1103/RevModPhys.15.1.
2. Van Kampen N.G., Stochastic Processes in Physics and Chemistry. North-Holland, Amsterdam, 1992.
3. Klimontovich Yu.L. Statistical Physics. Nauka, Moscow, 1982 (in Russian) [Engl. transl.: Harwood Academic Publishers, New York, 1986].
4. Hänggi P., Jung P. – In: Adv. Chem. Phys. Vol. 89, 2007, p. 239. John Wiley & Sons, Inc., Hoboken, NJ, USA; doi:10.1002/9780470141489.ch4.
5. Montroll E.W., Weiss G.H., J. Math. Phys., 1965, **6**, 167; doi:10.1063/1.1704269.
6. Scher H., Montroll E.W., Phys. Rev. B, 1975, **12**, 2455; doi:10.1103/PhysRevB.12.2455.
7. Chechkin A.V., Gonchar V.Yu., Klafter J., Metzler R. – In: Adv. Chem. Phys. Vol. 133 B, Ch. 9, 2006, p. 439. John Wiley & Sons, Inc., Hoboken, NJ, USA; doi:10.1002/0470037148.ch9.
8. Chechkin A.V., Sliusarenko O.Yu., Metzler R., Klafter J., Phys. Rev. E, 2007, **75**, 041101; doi:10.1103/PhysRevE.75.041101.
9. Chechkin A., Metzler R., Klafter J., Gonchar V., Introduction to the Theory of Lévy Flights. Anomalous Transport: Foundations and Applications. Wiley-VCH, Weinheim, 2008.
10. Kolmogorov A.N., Dokl. Akad. Nauk SSSR, 1940, **26**, 115.
11. Yaglom A.M., Am. Math. Soc. Transl. Series 2, 1958, **8**, 87.
12. Mandelbrot B.B., van Ness J.W., SIAM Rev., 1968, **10**, 422; doi:10.1137/1010093.
13. Sliusarenko O.Yu., Gonchar V.Yu., Chechkin A.V., Sokolov I.M., Metzler R., Phys. Rev. E, 2010, **81**, 041119; doi:10.1103/PhysRevE.81.041119.
14. Lord Rayleigh, Scientific Papers of Lord Rayleigh, Vol. 3. Dover, New York, 1964.
15. Cáceres M.O., Phys. Rev. E, 1999, **60**, 5208; doi:10.1103/PhysRevE.60.5208.
16. Wang Ke-Gang, Masoliver J., Physica A, 1996, **231**, 615; doi:10.1016/0378-4371(96)00125-2.
17. Adelman S.A., J. Chem. Phys., 1976, **64**(1), 124; doi:10.1063/1.431961.
18. Doi M., Edwards S.F., The Theory of Polymer Dynamics. Clarendon, Oxford, 1986.
19. Granek R., J. Phys. II, 1997, **7**, 1761; doi:10.1051/jp2:1997214.
20. Farge E., Maggs A.C., Macromolecules, 1993, **26**, 5041; doi:10.1021/ma00071a009; Caspi A., Elbaum M., Granek R., Lachish A., Zbaida D., Phys. Rev. Lett., 1998, **80**, 1106; doi:10.1103/PhysRevLett.80.1106; Amblard F., Maggs A.C., Yurke B., Pargellis A.N., Leibler S., Phys. Rev. Lett., 1996, **77**, 4470; doi:10.1103/PhysRevLett.77.4470.
21. Edwards S.F., Wilkinson D.R., Proc. R. Soc. London, Ser. A, 1982, **381**, 17; doi:10.1098/rspa.1982.0056.
22. Freyssingeas E., Roux D., Nallet F., J. Phys. II,1997, **7**, 913; doi:10.1051/jp2:1997162; Helfer E., Harlepp S., Bourdieu L., Robert J., MacKintosh F.C., Chatenay D., Phys. Rev. Lett., 2000, **85**, 457; doi:10.1103/PhysRevLett.85.457.
23. Granek R., Klafter J., EPL, 2001, **56**, 15; doi:10.1209/epl/i2001-00480-8.
24. Zilman A.G., Granek R., Chem. Phys., 2002, **284**, 195; doi:10.1016/S0301-0104(02)00548-7.
25. Majumdar S.N., Bray A.J., Phys. Rev. Lett., 2001, **86**, 3700; doi:10.1103/PhysRevLett.86.3700; Krug J., Adv. Phys., 1997, **46**, 139; doi:10.1080/00018739700101498.
26. Krug J., Kallabis H., Majumdar S.N., Cornell S.J., Bray A.J., Sire C., Phys. Rev. E, 1997, **56**, 2702; doi:10.1103/PhysRevE.56.2702.
27. Toroczkai Z., Williams E.D., Phys. Today, 1999, **52**(12), 24; doi:10.1063/1.882897.
28. Taloni A., Chechkin A., Klafter J., Phys. Rev. Lett., 2010, **104**, 160602; doi:10.1103/PhysRevLett.104.160602.
29. Taloni A., Chechkin A., Klafter J., Phys. Rev. E, 2010, **82**, 061104; doi:10.1103/PhysRevE.82.061104.
30. Zelenyi L.M., Milovanov A.V., Phys. Usp., 2004, **47**, 749-–788; doi:10.1070/PU2004v047n08ABEH001705.
31. Carreras B.A., Balbin R., van Milligen B., Pedrosa M.A., Garcia-Cortes I., Sanchez E., Hidalgo C., Bleuel J., Endler M., Thomsen H., Chankin A., Davies S., Erents K., Matthews G.F., Phys. Plasmas, 1999, **6**, 4615; doi:10.1063/1.873748.
32. Kou S.C., Xie Sunney X. Phys. Rev. Lett., 2004, **93**, 180603; doi:10.1103/PhysRevLett.93.180603.
33. Akhiezer A.I., Peletminskii S.V., Methods of Statistical Physics. Nauka, Moscow, 1977 (in Russian).
34. Sliusarenko O.Yu., Chechkin A.V. Ukr. J. Phys., 2007, **52** (12), 1193.
35. Gel'fand I.M., Shilov G.E., Generalized Functions, Vol. 1. Academic Press, New York, 1964.
36. Abramowitz M., Stegun I.A., Handbook of Mathematical Functions. National Bureau of Standards, Washington, D.C., 1972.
37. Chechkin A.V., Klages R., J. Stat. Mech., 2009, L03002; doi:10.1088/1742-5468/2009/03/L03002.

Узагальнене рівняння Фокера-Планка та його розв'язок для лінійних немарківських Ґаусових систем

О.Ю. Слюсаренко

Інститут теоретичної фізики ім. О.І. Ахієзера ННЦ ХФТІ, Україна, 61108 Харків, вул. Академічна, 1

У цій роботі ми пропонуємо послідовний підхід до виводу узагальненого рівняння Фокера-Планка (УРФП) для Ґаусових немарківських процесів із стаціонарними прирощеннями. Цей підхід дозволяє побудувати функцію розподілу (ФР) процесу без потреби безпосередньо розв'язувати УРФП. Ми застосовуємо цей метод для знаходження УРФП та ФР для вільного узагальненого броунівського руху та узагальненого броунівського руху в потенціалі для випадку степеневої кореляційної функції шуму. Ми доводимо, що розглянуті системи можуть описуватися у рамках рівняння Ейнштейна-Смолуховського за умов сильної в'язкості та великих часів. Також ми порівнюємо результати із отриманими іншими авторами. Нарешті, ми обчислюємо ФР термодинамічної роботи у стохастичній системі, що складається з частинки у гармонічному потенціалі, який рухається з постійною швидкістю, та перевіряємо флуктуаційну теорему для роботи у такій системі.

Ключові слова: *рівняння Фокера-Планка, Ґаусова система, немарківська система, термодинамічна робота, перехідне флуктуаційне співвідношення*