Sinai diffusion and some of its descendants

GLEB OSHANIN

PHYSIQUE THÉORIQUE DE LA MATIÈRE CONDENSÉE (UMR CNRS 7600)
UNIVERSITÉ PIERRE ET MARIE CURIE (PARIS 6)
PARIS, FRANCE

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Outlook

- Sinai Diffusion. <MSD> vs Probability currents.
- Generalized Sinai Diffusion. Brownian motion on Fractional Brownian Motion.
- Melting of a Heteropolymer
- Two Stock Options at the Races
- First Passages for a couple of independent Brownian motions

Sinai diffusion (Random random walk)

- Consider a 1D, infinite in both directions chain of integers and associate with each site « x » a random variable p_x (transition probability from x to x+1) with a distribution symmetric around ½ and bounded away from 0 and 1.

  p_x and $p_{x'}$ are uncorrelated

  Sinai Diffusion: Random Walk in a Random Environment

$$P_{n+1}(x) = p_{x-1}\, P_n(x - 1) + (1 - p_{x+1})\, P_n(x+1)$$

- Continuous-space & time counterpart:

$$dx_t/dt = F(x_t) + \xi_t$$

x_t – particle trajectory

F(x) – random, position-dependent, delta-correlated in space force with zero mean

ξ_t – white noise in time

Random walk in a random potential which itself is a trajectory of a random walk

Sinai diffusion (Random random walk)

$$\overline{< x_t^2 >} \approx \ln^4(t)$$ (Sinai, 1982)

A simple argument: On a scale x a typical barrier due to a random potential is of order of √x. An (Arrhenius) time t needed to overcome such a barries solely due to thermal activation is of order

$$t \approx \exp(\sqrt{x})$$

Trajectories are strongly (logarithmically) confined

This logarithmic behavior is supported by typical realizations of disorder

Probability distribution (Kesten, 1986)

Sinai diffusion. Probability currents in finite samples

- Consider a finite Sinai chain with N+1 sites. Set the occupation of x=0 site equal to 1 and put a sink at site x=N.
- A probability current J_N for a given realization of disorder is $J_N = 1/\tau_N$

$$\tau_N = 1 + \frac{p_1}{1-p_1} + \frac{p_1}{1-p_1}\frac{p_2}{1-p_2} + \frac{p_1}{1-p_1}\frac{p_2}{1-p_2}\frac{p_3}{1-p_3} + \ldots + \frac{p_1}{1-p_1}\cdots\frac{p_N}{1-p_N}$$

- τ_N is (the so-called Kesten variable) the resistance offered by a chain to the passage of a particle through the chain
- J_N = hitting (splitting) probability = Probability that for a given environment $\{p_x\}$ a particle starting at x=1 will first hit x=N without having ever visited x=0.
- Continuous-space result, $J_L = 1/\tau_L$

$$\tau_L = \int_0^L dx \cdot \exp\left(\int_0^x dx' \cdot F(x') \right)$$

Sinai diffusion. Probability currents in finite samples

- Typical behavior $\tau_L = \int_0^L dx \cdot \exp\left(\int_0^x dx' \cdot F(x) \right) \propto \exp(\sqrt{L}), J_L \propto \exp(-\sqrt{L})$
- Typical current is very small, much smaller than the Fickian current in a homogeneous system, $J_{fick} = 1/L$ (all $p_x = 1/2$)
- Disorder-averaged current for a discrete Sinai chain

$$\frac{A}{\sqrt{N}\ln^2(N)} \leq < J_N > \leq \frac{B}{\sqrt{N}}$$ (Burlatsky, Oshanin & Moreau, 1992)

- Continuous-space & time : $\alpha \propto \sigma^2 / T^2$

$$\Phi(p) = < \exp(-p\tau_L) > = \frac{2}{\pi} \int_0^\infty dx \cdot \exp(-\frac{\alpha L x^2}{4}) \cdot \cosh(\frac{\pi x}{2}) \cdot K_{ix}(2\sqrt{p/\alpha})$$

$$< J_L > = \int_0^\infty dp \cdot \Phi(p) = \sqrt{\frac{\alpha}{\pi \cdot L}} \left(1 + \sum_{n=1} B_{2n} \cdot \frac{(\pi^2/\alpha \cdot L)^n}{n!} \right)$$ (Oshanin & Moreau, 1993)

Sinai diffusion. Probability currents in finite samples.

- Probability distribution

$$P(J) \propto \frac{1}{\sqrt{L}} \cdot \exp\left(-\frac{\ln^2(J)}{4\alpha \cdot L}\right) \cdot \frac{1}{J^{3/2}} \cdot \exp\left(-\frac{J}{J_0}\right)$$

(Oshanin, Burlatsky & Moreau, 1993; Monthus & Comtet, 1994)

- J_0 is independent of L

- Typical behavior stems from the log-normal part, disorder-average – out of the tails of the distribution

- Moments of arbitrary order $< J_L^n > \propto \frac{C_n}{\sqrt{L}}$

Sinai diffusion with global bias

- Random Force

$$\langle F(x)\rangle = F_0, \langle F(x)\, F(x')\rangle = F_0^2 + 2\, D_w\, \delta(x - x')$$

- Key parameter – a Peclet-type number:

$$\mu = D_0\, F_0 / D_w$$

- D_0 – particle diffusion coefficient in absence of disorder

- $\mu < 1$. Creep phase, $\langle x(t)\rangle \approx t^{\mu}$

- $1 < \mu < 2$. Ballistic phase with anomalous dispersion, $\langle x(t)\rangle \approx F_0\, t + t^{1/\mu}$

- $\mu \geq 2$. Ballistic phase.

(de Gennes, 1974; Kesten, Kozlov & Spitzer, 1975; Derrida & Pomeau, 1983)

Generalized Sinai diffusion

- Sinai diffusion is a random walk (Brownian motion) in a random potential W(x) which itself is a trajectory of a random walk (Brownian motion)

- Consider Brownian motion in a random quenched potential $W_H(x)$ where $W_H(x)$ is a fractional Brownian motion with the Hurst index H:

  $W_H(x)$ is a stationary Gaussian process such that

$$\langle W_H(x)\rangle = 0, \langle W_H(x)\, W_H(x')\rangle = 1/2\,(|x|^{2H} + |x'|^{2H} - |x - x'|^{2H}),\ 0 < H < 1$$

- if $H = 1/2$ then the process is in fact a Brownian motion
- if $H > 1/2$ then the increments of the process are positively correlated
- if $H < 1/2$ then the increments of the process are negatively correlated.

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Generalized Sinai diffusion

- Disorder-averaged mean-square displacement $< \overline{x_t^2} > \approx \ln^{2/H}(t)$
- Typical probability current through a finite sample $J_L \approx \exp\left(-L^H\right)$
- Disorder – averaged probability current $< J_L > \approx \dfrac{1}{L^\theta}$
- $\theta = 1 - H$, like the persistence exponent for the fBm. Is it?
- Fix t and L and vary H from 1 to 0. MSD grows while the current becomes smaller!
- Representative realizations of disorder (representative trajectories $W_H(x)$)

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Generalized Sinai diffusion. Representative disorder

- Upper bound: Let $W = \max[W_H(1), W_H(2), W_H(3), W_H(4), \ldots, W_H(N)]$

$$\tau_N = 1 + e^{W_H(1)} + e^{W_H(2)} + e^{W_H(3)} + \ldots + e^{W_H(N)} \geq e^W \Rightarrow \qquad J_N \leq e^{-W}$$

$$< J_N > \leq \int P(W) \cdot e^{-W} \cdot dW \qquad P(W) \approx \frac{const}{L^{1-H}} + o\left(\frac{1}{L^{1-H}}\right)$$

(Molchan, 1991)

- Lower bound:

$$< J_N > = E_\Omega \left\{ \frac{1}{\tau_N} \right\} \geq E_\omega \left\{ \frac{1}{\tau_N} \right\}, \omega \in \Omega$$

- ω comprises such $W_H(x)$ which obey $W_H(x) \leq f(x) = A - B \ln(x)$

$$\tau_N = 1 + e^{W_H(1)} + e^{W_H(2)} + \ldots + e^{W_H(N)} \leq 1 + e^{f(1)} + e^{f(2)} + \ldots + e^{f(N)} \to const$$

$$< J_N > \geq const \cdot E_\omega \{1\} \approx \frac{1}{L^\theta}, \theta = 1 - H$$

- Representative disorder = such trajectories $W_H(x)$ which are bounded from above by a slowly decaying function (constant current & sufficiently high measure)

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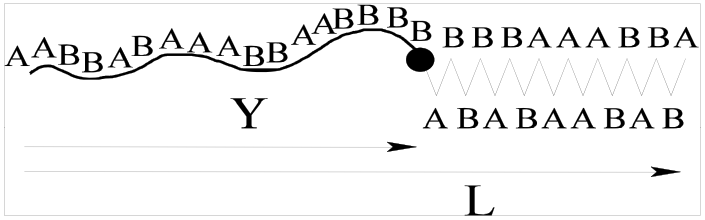
Helix or Coil? Fate of a melting heteropolymer

A heteropolymer of length L with a binary alphabet (chemical groups A and B with melting temperatures T_A and T_B).



Initially some part Y of the heteropolymer is denaturated (coil state) and the remaining part is in a helix state

Dynamics of the boundary (de Gennes, 1974): c/h boundary diffuses in presence of a random, position-dependent force F(x):

$$\dot{x}_t = F(x) + \xi(t) \qquad < F(x) >= F_0, < F(x)F(x') > -F_0^2 = \sigma^2 \delta(x - x')$$

$\xi(t)$ – white noise.

For $T=(T_A+T_B)/2$, $F_0 = 0$ – Sinai diffusion

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Helix or Coil? Fate of a melting heteropolymer

- E_+ - realization-dependent hitting probability – probability that the c/h boundary starting at x = Y hits x=L first without ever having hit x=0:

$$E_+ = \frac{J_+}{J_+ + J_-} = \frac{\tau_-}{\tau_- + \tau_+}$$

- Js are the currents from the starting point to the respective boundary, τ_+ and τ_- are familiar objects – uncorrelated random variables:

$$\tau_- = \int_0^Y dx \cdot \exp\left(-\frac{1}{T} \int_0^x dx' \cdot F(x') \right), \tau_+ = \int_Y^L dx \cdot \exp\left(-\frac{1}{T} \int_Y^x dx' \cdot F(x') \right)$$

- In absence of disorder (Brownian motion):

$$E_+ = Y/L$$

- In presence of disorder, E_+ is a random variable. Probability distribution (over quenched disorder) $P(E_+)$?

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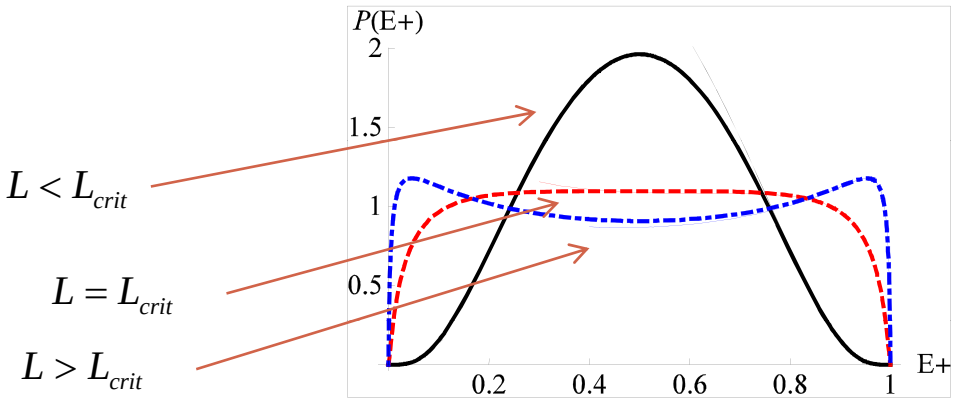
Helix or Coil? Fate of a melting heteropolymer

- Exact result:

$$P(E_+) = \frac{1}{\pi\sqrt{\alpha_+\alpha_- E_+(1-E_+)^3}} \int_{-\infty}^{\infty} du \frac{\cosh(u)}{\cosh(\eta)} \cos\left(\frac{\pi u}{\alpha_+}\right) \exp\left(\frac{\pi^2}{4\alpha_+} - \frac{u^2}{\alpha_+} - \frac{\eta^2}{\alpha_-}\right), \eta = \sinh^{-1}\left(\sqrt{\frac{E_+}{1-E_+}}\cosh(u)\right)$$

$$\alpha_- = \frac{\sigma^2}{2T^2} Y \qquad \alpha_+ = \frac{\sigma^2}{2T^2}(L-Y)$$

- For Y=L/2 (the c/h boundary starts exactly in the middle), $< E_+ > \equiv 1/2$
- Is the mean value meaningful?
- There is a critical length L_c at which the distribution changes the shape



- Mean is the least probable for sufficiently long chains. Mean does not mean much… Each realization has its unique fate.

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Two (Asian) stock options at the races: Black-Scholes forecasts

- In classical Black-Scholes model, a price S_t of a stock obeys :

$$dS_t = z \cdot S_t \, dt + \sigma \cdot S_t \cdot dW_t$$

- z is a parameter dependent on the nature of the asset, σ – volatility of the market, dW_t are the increments of a Brownian trajectory
- Stratonovich-Ito solution: $S_t = S_{t=0} \cdot \exp\left(-\frac{\sigma^2}{2} \mu \cdot t + \sigma \cdot W_t \right)$
- μ = const, which has exactly the same meaning as the Peclet-type number in the section on biased Sinai diffusion. A slope of the market. + or -.
- An Asian (path-dependent) option is a time-average

$$\tau_T = \frac{S_{t=0}}{T} \cdot \int_0^T dt \cdot \exp\left(-\frac{\sigma^2}{2} \mu \cdot t + \sigma \cdot W_t \right)$$

- Suppose one buys two very similar stocks (Freddie Mac & Fannie Mae) and is curious how much, at time T, one of them will contribute to his wealth

$$E = \frac{\tau_T^{(1)}}{\tau_T^{(1)} + \tau_T^{(2)}}$$

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Two (Asian) stock options at the races: Black-Scholes forecasts

- Distribution P(E): $$P(E) = \int_0^\infty d\tau \cdot \tau \cdot \Psi(E \cdot \tau) \cdot \Psi((1-E) \cdot \tau)$$
- Distribution of τ: $$\Psi(\tau) \propto \frac{A}{\tau^{\mu/2}} e^{-1/\alpha\tau} \sum_{0 \le n \le \mu/2} e^{-\alpha n(n-\mu)\tau} \frac{(-1)^n(\mu-2n)}{\Gamma(1+\mu-n)} L_n^{\mu-2n}(1/\alpha\tau) +$$
$$+ \frac{B}{\tau^{(\mu+1)/2}} \int_0^\infty u \cdot du \cdot \left|\Gamma(-\frac{\mu}{2} + i\frac{u}{2})\right|^2 \sinh(\pi u) \cdot W_{(\mu+1)/2, iu/2}(1/\alpha\tau)$$ (Monthus & Comtet, 1994)
- Looks scary but the overall behavior is simple: log-normal decay for large τ, exponential decay when τ → 0 and a fat power-law tail inbetween.

- Rising market: μ ≤ 0. P(E) is a delta-function at T=0. As T grows, P(E) broadens, and undergoes at a certain $T=T_c$ a continuous transition to an M-shaped form with a minimum at E=1/2 and maxima close to 0 and 1. It means that one of the options becomes a complete winner while the second is trash. The symmetry is broken.
- Decreasing market. 0 < μ ≤ 1. A transition to a U-shaped form. The symmetry is broken.
- Decreasing market. μ > 1. P(E) stays a bell-shaped function with a maximum at E = ½. Both stocks contribute proportionally.

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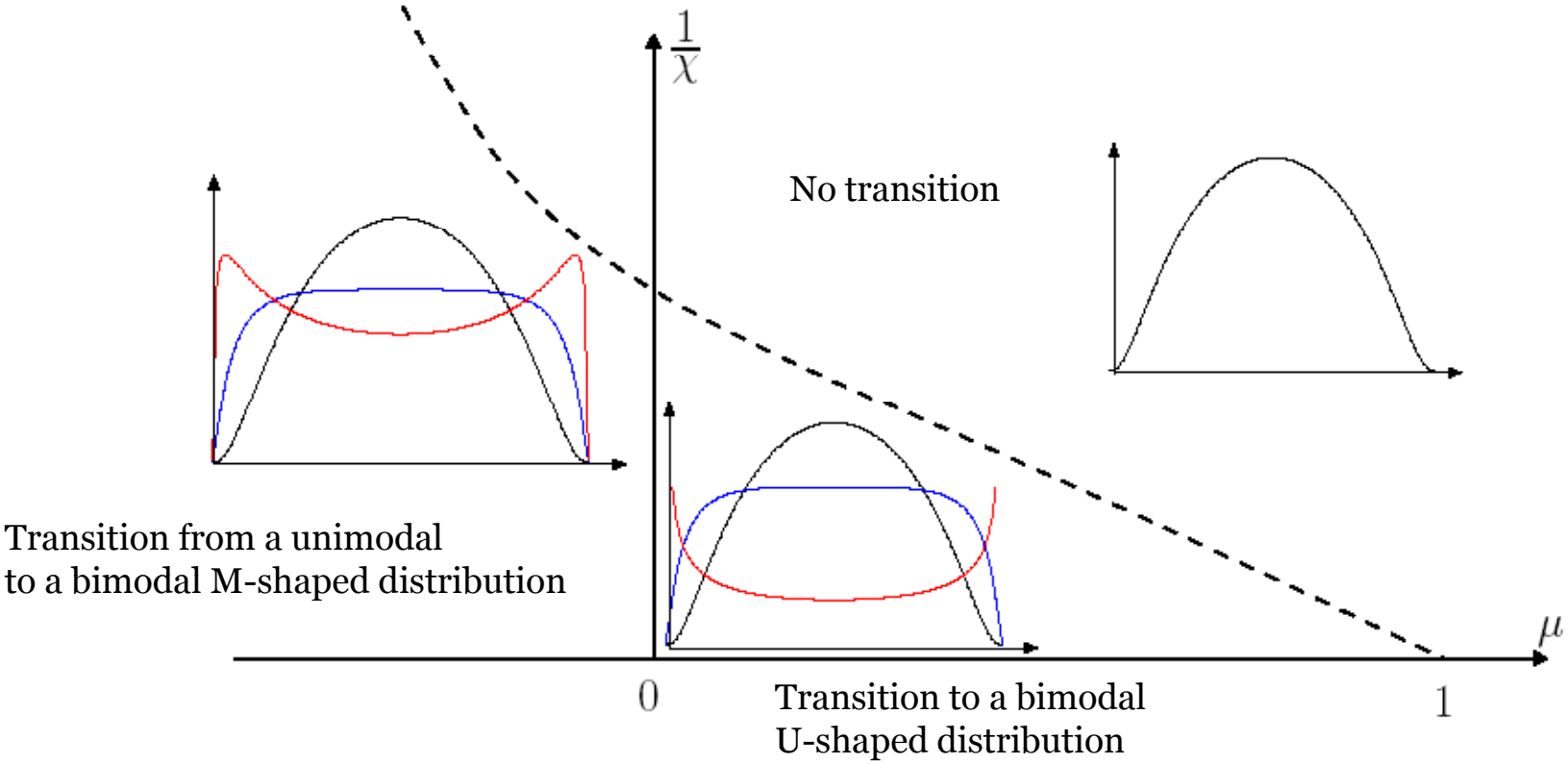
Two (Asian) stock options at the races: Black-Scholes forecasts

- Correlated τ variables. We connect the increments by a spring of rigidity χ

$$dS_t^{(1)} = z \cdot S_t^{(1)} dt + \sigma \cdot S_t^{(1)} \cdot dW_t^{(1)}$$
$$dS_t^{(2)} = z \cdot S_t^{(2)} dt + \sigma \cdot S_t^{(2)} \cdot dW_t^{(2)}$$



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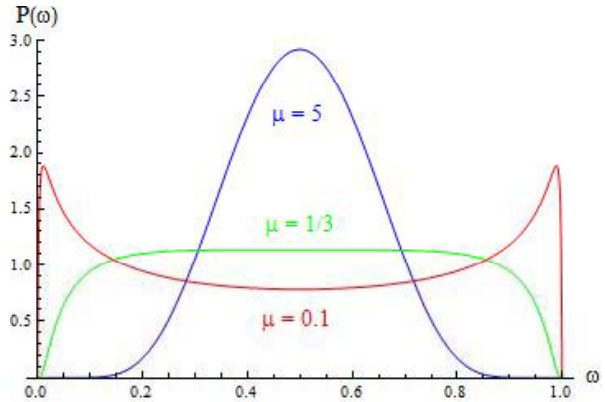
First passages for a couple of independent Brownian motions

- Consider a semi-infinite, one-dimensional system with two BMs starting simultaneously at x_0. BMs experience a constant, site-independent bias F towards the origin so that their drift velocities are = v < 0. Diffusion coefficients of both = D.
- Let t_1 and t_2 be the time moments of their first passages to the origin. The distribution $P(\omega = t_1/(t_1+t_2))$?

$$P(\omega) \propto \frac{K_1(\mu/\sqrt{\omega(1-\omega)})}{\omega(1-\omega)}$$

- $\mu = x_0\,|v|/2\,D$ – the Peclet number



- The event in which two drunken sailors (a favorite object in probability theory), appearing simultaneously at the same position on a semi-infinite street with a bar at one end (which they « smell »), will enter the bar simultaneously is
  - the most probable if $\mu > \mu_c \approx 1/3$
  - the least probable if $\mu < \mu_c \approx 1/3$

With Gregory Schehr and Carlos Mejia-Monasterio (2011)

First passages for a couple of independent Brownian motions

- Consider a one-dimensional inteval [0,1] with an adsorbing boundary at x=0 and a reflecting boundary at x = 1. Let two BMs start simultaneously at x_0. Diffusion coefficients of both = D. No bias.
- Let t_1 and t_2 be the time moments of their first passages to the origin. The distribution $P(\omega = t_1/(t_1+t_2))$?
- The event, in which two drunken sailors, appearing simultaneously at the same position on a bounded street with a bar at one end and a reflecting boundary at the other end, will enter the bar simultaneously is

  - the most probable if $x_0 > 2/\pi$
  - the least probable if $x_0 < 2/\pi$

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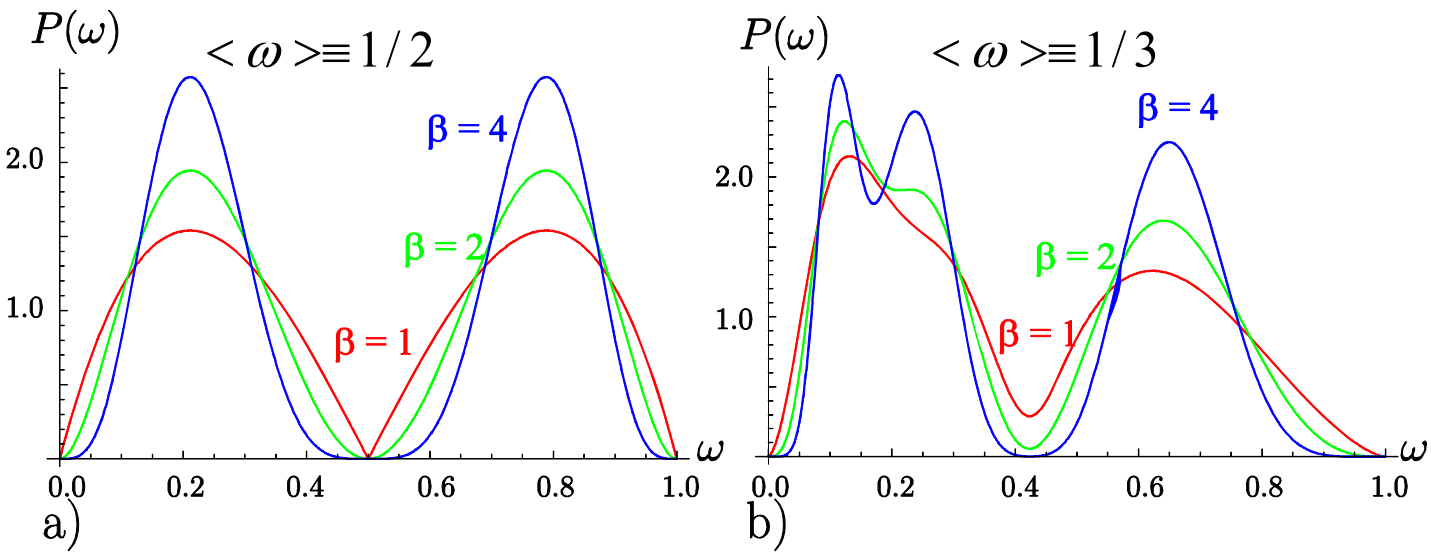
Conclusion

- The symmetry between two i.i.d random variables, whose distribution is truncated from both sides (and thus possesses all moments), is broken if the distribution has an intermediate sufficiently « fat » tail and the characteristic cut-off is less than some critical value. One has to be cautious and not confuse typical, average, most probable.
- Such distributions are ubiquituous: (in addition to mentioned examples), interspike intervals for Integrate-and-Fire model of neuron dynamics, biased motion of solids on solid surfaces, SOC, first passages in finite systems or with an inward bias, chemo- and infotaxis, and etc.
- Critical cut-off may be time, size, position, voltage, smell and etc.
- An agenda: Correlated variables (partially mentioned in the talk), e.g. Wigner proper delay times in N-channel systems (with Gregory Schehr, 2011)