

# Relevance of the fixed dimension perturbative approach to frustrated magnets in two and three dimensions

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We show that the critical behavior of two- and three-dimensional frustrated magnets cannot reliably be described from the known five- and six-loop perturbative renormalization-group results. Our conclusions are based on a careful reanalysis of the resummed perturbative series obtained within the zero-momentum massive scheme. In three dimensions, the critical exponents for XY and Heisenberg spins display strong dependences on the parameters of the resummation procedure and on the loop order. This behavior strongly suggests that the fixed points found are in fact spurious. In two dimensions, we find, as in the  $O(N)$  case, that there is apparent convergence of the critical exponents but toward erroneous values. As a consequence, the interesting question of the description of the crossover/transition induced by  $Z_2$  topological defects in two-dimensional frustrated Heisenberg spins remains open.

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## I. INTRODUCTION

After more than 30 years of intensive studies, the critical behavior of frustrated magnets is still controversial (see Ref. 1 and references therein). At the root of the problem is the competition between the interactions among neighboring spins that gives rise to a canted ground state, and thus, to a symmetry-breaking scheme where the rotational group is fully broken. This is, for instance, the case in the paradigmatic example of frustrated magnets, the stacked triangular antiferromagnets (STAs), where the three spins on an elementary cell display a planar  $120^\circ$  structure in the ground state. As a consequence, the order parameter is a matrix instead of a vector [a  $SO(3)$  matrix for Heisenberg spins and a  $2 \times 2$  matrix for XY spins] and the critical properties are therefore entirely different from those of unfrustrated systems.

For instance, in space dimension  $d=2$ , the first homotopy group of  $SO(3)$  being nontrivial— $\Pi_1[SO(3)]=Z_2$ —one expects for Heisenberg spins a deconfinement of topological excitations<sup>2</sup> that could give rise to a Kosterlitz-Thouless (KT)-type transition<sup>3,4</sup> or, at least, to a crossover behavior—see below. Numerous experimental<sup>5–9</sup> and numerical<sup>2,10–17</sup> studies have indeed shown indications of a nontrivial phenomenon occurring at finite temperature. For XY spins, the order parameter space is given by  $SO(2) \times Z_2$ . In this case coexist Ising degrees of freedom, topological excitations— $\Pi_1[SO(2)]=Z$ —and spin waves. A very debated issue has been the nature of the phase transition(s) occurring in this system as the temperature is varied: either two separate Ising and KT transitions or a unique one (see Ref. 18 and references therein).

In  $d=3$ , the question of the criticality has been extremely controversial (see Ref. 1). On the one hand, many experiments display scaling behaviors for XY and Heisenberg spins with critical exponents differing from those of the  $O(N)$  universality class (see Ref. 1 for a review). On the other hand, other experiments as well as extensive Monte

Carlo simulations performed on STAs or on similar models have exhibited weak first-order behavior.<sup>19–27</sup> Two main explanations have been proposed to describe these contradicting results.

The first one is based on a perturbative renormalization-group (RG) approach performed at fixed dimension (FD) either within the minimal-subtraction (MS) scheme without  $\epsilon$  expansion<sup>28</sup> or within the zero-momentum massive scheme,<sup>29–32</sup> at five- and six-loop order, respectively. Within these approaches, stable RG fixed points were found for  $N=2$  and  $N=3$  leading to the prediction of second-order phase transitions in  $d=3$  for frustrated magnets. Note that, within these FD approaches, one also finds a fixed point in  $d=2$  with nontrivial critical exponents in the  $N=2$  and  $N=3$  cases.<sup>31,33,34</sup> This fact has led to the hypothesis of a Kosterlitz-Thouless-type behavior induced by  $Z_2$  topological defects for Heisenberg spins.<sup>33</sup>

The second explanation is based on both the  $\epsilon=4-d$  (or pseudo- $\epsilon$ ) expansion<sup>35–37</sup> and the nonperturbative RG (NPRG) approaches.<sup>1,38–41</sup> In these approaches, one finds that there exists, within the  $(d,N)$  plane, a line  $N_c(d)$  that separates a second-order region for  $N > N_c$  from a first-order region for  $N < N_c$ . Within the  $\epsilon$  expansion, one finds  $N_c(d=3) \approx 5.3$ ,<sup>37</sup> pseudo- $\epsilon$  expansion gives  $N_c(d=3) \approx 6.23$  (Ref. 36) and within the NPRG approach  $N_c(d=3) \approx 5.1$  (Ref. 1) so that the transition for  $N=2$  and  $N=3$  are predicted to be of the first order. A thorough analysis performed within the NPRG approach<sup>1,38,40</sup> has shown that even if there is, strictly speaking, no fixed point below  $N_c(d=3)$  the RG flow is very slow for  $N=2$  and  $3$  in a whole region of the coupling constant space so that there is pseudoscaling without universality on a large range of temperature, in agreement with the numerical and experimental data.

Although the NPRG approach very likely explains the whole body of known data (see Ref. 1 for details), several points of the physics of frustrated magnets remain controversial. The first one is that, although the occurrence of (weak) first-order transitions is by now well established in several

three-dimensional systems,<sup>19–27</sup> this does not imply that all systems, sharing the same order parameter and the same symmetries, undergo first-order phase transitions (see, for instance, Ref. 28 for a recent numerical computation where a second-order phase transition is observed). In other words, the very existence of a parameter domain where the frustrated systems would undergo second-order transitions is still debated. A second point that should be understood is the origin of the discrepancy between the two scenarii above and, in particular, between the results obtained within the different perturbative schemes:  $\epsilon$  (or pseudo- $\epsilon$ ) expansion on one hand and the FD approaches on the other hand. A last and important question is the nature of the transition that occurs in  $d=2$  for Heisenberg spins: phase transition or simple crossover behavior between a low-temperature—spin-wave—phase and a high-temperature phase with both spin waves and vortices ?

It is clear that answering to these questions amounts to answering to the question of the existence of a genuine attractive fixed point in the RG flow of frustrated magnets in  $d=3$  and 2 for  $N=2$  and 3. In the perturbative framework, this is essentially equivalent to proving (or disproving) the reliability and convergence properties of the resummation procedures necessary to obtain sensible results out of the perturbative series. This work has been initiated in  $d=3$  in a previous publication where the five-loop perturbative series obtained in the  $\overline{\text{MS}}$  scheme *without*  $\epsilon$  expansion have been carefully reexamined.<sup>42</sup> Studying (i) the convergence properties of the critical exponents with the order  $L$  of the expansion (number of loops) and with respect to the variations in the parameters involved in the resummation procedure, (ii) the properties of the fixed-point coordinates  $[u^*(d,N), v^*(d,N)]$  considered as functions of  $d$  and  $N$ , and (iii) the continuation of the fixed point found in  $d=3$  for  $N=2$  and  $N=3$  up to  $d=4$ , the authors of Ref. 42 have provided strong arguments in favor of the spurious character of the fixed points found in  $d=3$ , i.e., that they are artifacts of the perturbative expansion in the  $\overline{\text{MS}}$  scheme.

In the present paper, we extend the previous analysis to the series obtained in the zero-momentum massive scheme in  $d=3$  at six loops<sup>30</sup> and in  $d=2$  at five loops.<sup>34</sup> In  $d=3$ , we apply the criteria used in Refs. 42 and 43—principle of minimal sensitivity (PMS) and principle of fastest apparent convergence (PFAC)—and confirm that the fixed points found in  $d=3$  for  $N=2$  and  $N=3$  are most likely spurious. In  $d=2$ , the situation is more delicate. We recall that, already for the (nonfrustrated)  $O(N)$  models, the critical exponents found from the  $(\phi^2)^2$  field theory are quantitatively wrong although apparently converged for all  $N$ . This striking phenomenon, already mentioned in Ref. 44 relies on the presence of nonanalytic contributions to the  $\beta$  function at the fixed point.<sup>45</sup> The same kind of problem has been mentioned in the case of frustrated magnets<sup>33,34</sup> but it was assumed to leave unaffected the qualitative predictions, in particular, the existence of a nontrivial fixed point in the Heisenberg case. We show here, on the contrary, that the phenomenon of apparent convergence toward erroneous values, together with the presence of instabilities of some critical exponents with respect to the resummation parameters, leads to seriously question the conclusions drawn in the past as for the critical behavior of these systems.

Our study altogether shows clear evidences that the FD perturbative approaches to three-dimensional frustrated magnets are not reliable, at least at the orders studied, and that there is no convincing evidence of a genuine phase transition induced by vortices in two-dimensional Heisenberg spin systems. Finally, for the same reasons as in the Heisenberg case, we show that the behavior of XY spins in  $d=2$  cannot be elucidated from the five-loop perturbative data.

The paper is organized as follows. In Sec. II, we study in detail the  $O(N)$  case in  $d=2$  and  $d=3$ . In  $d=3$ , this allows us to illustrate on a well-known example the kind of stability (respectively, instability) properties expected for a genuine (respectively, spurious) fixed point. In  $d=2$ , this allows us to illustrate the fact that there can be fast apparent convergence of the critical exponents but toward erroneous values due to nonanalytic contributions. In Sec. III, frustrated magnets are studied in  $d=3$  and in  $d=2$ . In  $d=3$ , we confirm the spurious character of the fixed points found for  $N=2$  and  $N=3$ . In  $d=2$ , we show the unreliability of the conclusions—phase transition controlled by a fixed point—deduced from the results obtained perturbatively at five loops in the  $N=3$  case. We then analyze the  $N=2$  case and reach the same conclusions as in the  $N=3$  case.

## II. $O(N)$ MODELS IN TWO AND THREE DIMENSIONS

In the following, we study the convergence of the resummed perturbative series obtained for the frustrated models. Since we need to determine criteria to decide whether the perturbative results are (or are not) converged, we illustrate briefly how convergence of the resummed series shows up for the  $O(N)$  models in  $d=3$ . We show that the behavior of the correction to scaling exponent  $\omega$  as a function either of the loop order or of the resummation parameters is a good indicator of the numerical convergence of the perturbative results. The exponent  $\eta$ , when available, is also studied. By analyzing the two-dimensional  $O(N)$  case we also show that, contrarily to common belief, the five-loop results for the critical exponents are not converged. The reason of this behavior is however rather subtle since there is, in fact, apparent convergence but toward erroneous values, a phenomenon that we call *anomalous apparent convergence*.

### A. Resummation procedure

As well known, the perturbative series obtained in the  $O(N)$  models for the  $\beta$  function describing the running of the coupling constant with the scale are not convergent.<sup>46,47</sup> They are asymptotic series which, in the case of the zero-momentum massive scheme, are Borel summable.<sup>48</sup> Powerful resummation methods have been used in the past that, thanks to the knowledge of the behavior of the series at large order, lead to converged and accurate results (see Refs. 47 and 49 for reviews). We recall in the following the kind of resummation procedure that we use throughout this paper.

Let us consider a series

$$f(u) = \sum_n a_n u^n, \quad (1)$$

where the coefficients  $a_n$  are supposed to grow as  $n!$ .

The Borel-Leroy sum associated with  $f(u)$  is given by

$$B(u) = \sum_n \frac{a_n}{\Gamma[n+b+1]} u^n, \quad (2)$$

where  $b$  is a parameter whose meaning will become clear later.

The resulting series is now supposed to converge, in the complex plane, inside a circle of radius  $1/a$ , where  $u = -1/a$  is the singularity of  $B(u)$  closest to the origin. Then, using this definition as well as  $\Gamma[n+b+1] = \int_0^\infty t^{n+b} e^{-t} dt$ , one can rewrite

$$f(u) = \sum_n \frac{a_n}{\Gamma[n+b+1]} u^n \int_0^\infty dt e^{-t} t^{n+b}. \quad (3)$$

Interchanging summation and integration, one can now define the Borel transform of  $f$  as

$$f_B(u) = \int_0^\infty dt e^{-t} t^b B(ut). \quad (4)$$

In order to perform the integral [Eq. (4)] on the whole real positive semiaxis, one has to find an analytic continuation of  $B(u)$ . Several methods can be used, Padé approximants constitute one possibility.<sup>50-52</sup> However, it is generally believed that the use of a conformal mapping<sup>53,54</sup> is more efficient since it makes use of the convergence properties of the Borel sum. Under the assumption that all the singularities of  $B(u)$  lie on the negative real axis and that the Borel-Leroy sum is analytic in the whole complex plane except for the cut extending from  $-1/a$  to  $-\infty$ , one can perform the change in variable,

$$\omega(u) = \frac{\sqrt{1+au}-1}{\sqrt{1+au}+1} \Leftrightarrow u(\omega) = \frac{4}{a} \frac{\omega}{(1-\omega)^2} \quad (5)$$

that maps the complex  $u$ -plane cut from  $u = -1/a$  to  $-\infty$  onto the unit circle in the  $\omega$  plane such that the singularities of  $B(u)$  lying on the negative axis now lie on the boundary of the circle  $|\omega| = 1$ . The resulting expression  $B[u(\omega)]$  has a convergent Taylor expansion within the unit circle  $|\omega| < 1$  and can be rewritten as

$$B[u(\omega)] = \sum_n d_n(a,b) [\omega(u)]^n, \quad (6)$$

where the coefficients  $d_n(a,b)$  are computed so that the re-expansion of the right-hand side (rhs) of Eq. (6) in powers of  $u$  coincides with that of Eq. (1). One obtains through Eq. (6) an analytic continuation of  $B(u)$  in the whole  $u$  cut plane so that a resummed expression of the series  $f$  can be written as

$$f_R(u) = \sum_n d_n(a,b) \int_0^\infty dt e^{-t} t^b [\omega(ut)]^n. \quad (7)$$

In practice, it is interesting to generalize the expression (7) by introducing<sup>55</sup> the expression

$$f_R(u) = \sum_n d_n(\alpha,a,b) \int_0^\infty dt e^{-t} t^b \frac{[\omega(ut)]^n}{[1-\omega(ut)]^\alpha} \quad (8)$$

whose meaning will be explained just below.

If an infinite number of terms of the series  $f_R(u)$  were known, expression (8) would be independent of the parameters  $a$ ,  $b$ , and  $\alpha$ . However, when only a finite number of terms are known,  $f_R(u)$  acquires a dependence on them. In principle, the parameters  $a$  and  $b$  are fixed by the large-order behavior of the series,

$$a_{n \rightarrow \infty} \sim (-a_{10})^n n! n^{b_{10}} \quad (9)$$

which leads to  $a = a_{10}$  and  $b \geq b_{10} + 3/2$ ,<sup>54</sup> where  $a_{10}$  and  $b_{10}$  denote the large-order value of  $a$  and  $b$ . As for  $\alpha$ , it is determined by the strong-coupling behavior of the initial series,

$$f(u \rightarrow \infty) \sim u^{\alpha_0/2} \quad (10)$$

which can be imposed at any order of the expansion by choosing  $\alpha = \alpha_0$ . The common assumption is that the above choice of  $a$ ,  $b$ , and  $\alpha$  improves the convergence of the resummation procedure since it encodes exact results.

Let us however emphasize that, often, only  $a$  is known and that the other parameters,  $\alpha$  and  $b$ , must be considered either as free (as, for instance, in Ref. 28) or variational (as, for instance, in Refs. 42 and 43 where  $\alpha$  is determined by optimizing the apparent convergence of the series). In any case, the choice of value of  $a$ ,  $\alpha$ , and  $b$  must be validated *a posteriori*.

### B. $O(N)$ models in three dimensions and principles of convergence

The dependence of the critical exponents upon the parameters  $a$ ,  $b$ , and  $\alpha$  is an indicator of the (non) convergence of the perturbative series. Indeed, in principle, any converged physical quantity  $Q$  should be independent of these parameters. However, in practice, at a given order  $L$  of approximation (loop order), all physical quantities depend (artificially) on them:  $Q \rightarrow Q^{(L)}(a,b,\alpha)$ . Even if  $a$  is fixed at the value obtained from the large-order behavior, all physical quantities remain dependent on  $b$  and  $\alpha$  at finite order. We consider that the optimal result for  $Q$  at order  $L$  corresponds to the values  $(b_{\text{opt}}^{(L)}, \alpha_{\text{opt}}^{(L)})$  of  $(b, \alpha)$  for which  $Q$  depends most weakly on  $b$  and  $\alpha$ , i.e., for which it is stationary,

$$Q_{\text{opt}}^{(L)} = Q^{(L)}(b_{\text{opt}}^{(L)}, \alpha_{\text{opt}}^{(L)}) \quad \text{with} \quad \left. \frac{\partial Q^{(L)}(b, \alpha)}{\partial b} \right|_{b_{\text{opt}}^{(L)}, \alpha_{\text{opt}}^{(L)}} = \left. \frac{\partial Q^{(L)}(b, \alpha)}{\partial \alpha} \right|_{b_{\text{opt}}^{(L)}, \alpha_{\text{opt}}^{(L)}} = 0, \quad (11)$$

where, of course,  $b_{\text{opt}}^{(L)}$  and  $\alpha_{\text{opt}}^{(L)}$  are functions of the order  $L$ .

The validity of this procedure, known as the PMS, requires that there is a unique pair  $(b_{\text{opt}}^{(L)}, \alpha_{\text{opt}}^{(L)})$  such that  $Q^{(L)}$  is stationary. This is generically not the case: several stationary points are often found. A second principle allows us to “optimize” the results even in the case where there are several “optimal” values of  $b$  and  $\alpha$  at a given order  $L$ : this is the so-called PFAC. The idea underlying this principle is that when the numerical value of  $Q^{(L)}$  is almost converged (that is,  $L$  is sufficiently large to achieve a prescribed accuracy) then the next order of approximation must consist only in small change in this value:  $Q^{(L+1)} \simeq Q^{(L)}$ . Thus, the preferred values of  $b$  and  $\alpha$  should be the ones

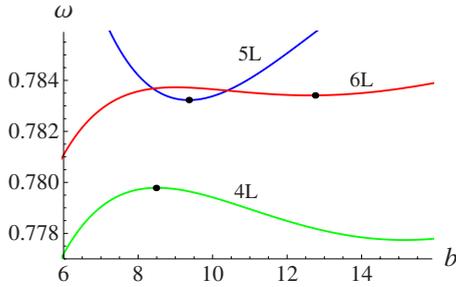


FIG. 1. (Color online) The exponent  $\omega$  of the three-dimensional  $O(4)$  model as a function of the resummation parameter  $b$  at four-, five-, and six-loop orders. The dot on each curve corresponds to a stationary value of  $\omega = \omega(\alpha, b)$  in both  $\alpha$  and  $b$  directions with  $a$  fixed to its large-order value  $a_{l_0} \approx 0.1108$ . One has  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (3.2, 8.5)$  at four loops,  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (5.2, 9.5)$  at five loops, and  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (5, 13)$  at six loops.

for which the difference between two successive orders  $Q^{(L+1)}(b^{(L+1)}, \alpha^{(L+1)}) - Q^{(L)}(b^{(L)}, \alpha^{(L)})$  is minimal. In practice, the two principles should be used together for consistency and, if there are several solutions to Eq. (11) at order  $L$  and/or  $L+1$ , one should choose the couples  $(b_{\text{opt}}^{(L)}, \alpha_{\text{opt}}^{(L)})$  and  $(b_{\text{opt}}^{(L+1)}, \alpha_{\text{opt}}^{(L+1)})$  for which the stationary values  $Q^{(L)}(b_{\text{opt}}^{(L)}, \alpha_{\text{opt}}^{(L)})$  and  $Q^{(L+1)}(b_{\text{opt}}^{(L+1)}, \alpha_{\text{opt}}^{(L+1)})$  are the closest, that is for which there is fastest apparent convergence. These principles have been developed and used in Refs. 42, 43, and 54, see also Ref. 56.

Nice examples where these two principles work very well and indeed lead to optimized values of the critical exponents are the  $O(N)$  models in  $d=3$  computed perturbatively at four-, five- and six-loop orders (within the zero-momentum massive scheme). The series for the  $\beta$  function of the coupling constant<sup>57</sup> are resummed thanks to a conformal Borel transform. Subsequently, one obtains the fixed-point coordinate  $u^*$ , its stability being defined by the correction to scaling exponent  $\omega$ ,

$$\omega = \left. \frac{\partial \beta(u)}{\partial u} \right|_{u=u^*}. \quad (12)$$

A positive value of  $\omega$  (or a positive real part if it is complex) corresponds to a stable fixed point. We show in Fig. 1 the exponent  $\omega$  of the  $O(4)$  model in  $d=3$  as a function of the parameter  $b$  for the values of  $\alpha$  for which stationarity is found for both  $b$  and  $\alpha$ . As expected, the dependence of  $\omega$  upon the resummation parameters becomes smaller as the order in the loop expansion increases as illustrated by the curves  $\omega(b)$  that flatten between four and six loops, see Fig. 1. At this order, one finds  $\omega \approx 0.783$ . As an indicator of the quality of the convergence we give the difference between the fifth and the sixth order for the exponent  $\omega$ :  $\omega(L=6) - \omega(L=5) \approx 2 \times 10^{-4}$ . Note that this case also illustrates the situation where—at six-loop order—several stationary points occur and where the PFAC allows us to select a single solution, see Fig. 1. Our results are comparable with results obtained by Guida and Zinn-Justin<sup>58</sup> for  $N=4$ :  $\omega = 0.774 \pm 0.020$  (six/seven loops in  $d=3$ ),  $\omega = 0.795 \pm 0.030$  (within the  $\epsilon^5$  expansion).

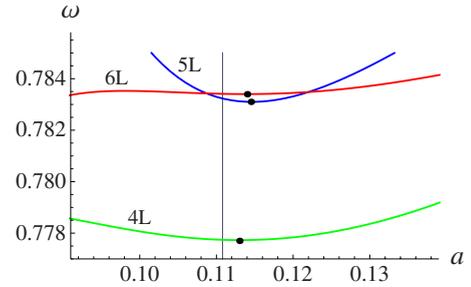


FIG. 2. (Color online) The exponent  $\omega$  of the three-dimensional  $O(4)$  model as a function of the resummation parameter  $a$  at four-, five-, and six loops. The vertical line corresponds to  $a = a_{l_0} \approx 0.1108$ . The values chosen for  $\alpha$  and  $b$  are such that  $\omega$  is stationary with respect to  $a$  and  $b$  when  $a = a_{l_0}$  (see Fig. 1). Note that  $\omega(a_{\text{opt}}) \approx \omega(a_{l_0})$ .

It is remarkable that the same study performed on other critical exponents, other values of  $N$  and even with other perturbative series (obtained from the MS scheme, for instance) always leads to the same kind of results with values of the exponents found that are very close to the best known values obtained from Monte Carlo simulations. This proves that the above methodology is indeed efficient.

Finally note that the same argument can also be applied to the determination of an optimal value of  $a$ ,  $a_{\text{opt}}$ , from the PMS applied to this parameter: if there is convergence of the resummed series, we expect that  $a_{\text{opt}}$  almost coincides with the value determined by the large-order analysis, Eq. (9),  $a_{\text{opt}} \approx a_{l_0}$ . The difference between these quantities is a measure of the convergence level of the series. We show in Fig. 2 on the example of the  $O(4)$  model in  $d=3$  that, as expected, the value  $a_{\text{opt}}$  is very close to  $a_{l_0}$  and the difference between  $\omega(a_{\text{opt}})$  and  $\omega(a_{l_0})$  is extremely small.

The criteria given above are of crucial importance, especially when considering FD approaches. Indeed, generically the (nonresummed) series obtained at  $L$  loops for the  $\beta$  function are polynomials of order  $L+1$  in the coupling constant  $u$ . Thus, the fixed-point equation  $\beta(u^*) = 0$  admits  $L+1$  roots  $u^*$  that are either real or complex. Contrary to what occurs within the  $\epsilon$  expansion, where the coupling constant is by definition of order  $\epsilon$ , in the FD approach no real root can be *a priori* selected or, reciprocally, discarded. As a result, the generic situation is that the number of fixed points as well as their stability vary with the order  $L$ : at a given order, there can exist several real and stable fixed points or none instead of a single one. In principle, the resummation procedure allows both to restore the nontrivial Wilson-Fisher fixed point and to eradicate the nonphysical, spurious, roots. In particular, we expect that spurious solutions should satisfy neither the PMS nor the PFAC criteria. We show in Fig. 3 on the example of the  $3d$   $O(4)$  model that there exists, beside the usual Wilson-Fisher stable fixed point, a spurious (unstable) fixed point. As expected, the exponents computed at this fixed point are very unstable with respect to variations in the resummation parameters  $a$  (the same behavior occurs when variations in  $b$  are considered), a behavior which, according to our criteria, is sufficient to discard it.<sup>59</sup>

A potential difficulty with this procedure is that for more involved models, bringing into play several coupling con-

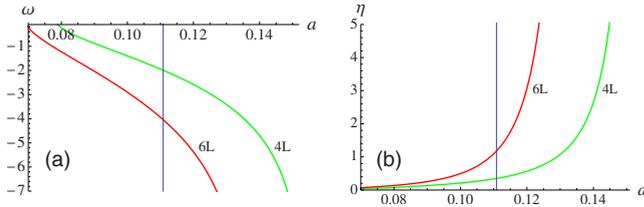


FIG. 3. (Color online) The exponents  $\omega$  and  $\eta$  as functions of  $a$  at the spurious fixed point of the  $O(4)$  model in  $d=3$  at four and six loops. The vertical line corresponds to  $a=a_{10}\approx 0.1108$ . The values considered for  $\alpha$  and  $b$  are, respectively, equal to 6 and 4. Other values give similar results. The exponent  $\omega$  is negative since the spurious fixed point is repulsive.

stants, the resummation procedure is very likely less efficient than for the  $O(N)$  models since it is performed with respect to one coupling constant only—see below. In this case, the instability displayed by the spurious fixed points can be much weaker than for  $O(N)$  models. However, some of the present authors have previously shown<sup>42</sup> that the criterion of (in)stability given above still remains reliable in these more general and ambiguous situations. More precisely they have shown, on the example of frustrated magnets (and on the model with cubic anisotropy) that fixed points suspected to be spurious from a stability analysis, have been confirmed to be so from additional independent arguments. These arguments are: (i) persistence of the fixed point as a nontrivial, non-Gaussian one up to (and above) the upper critical dimension  $d=4$ , a fact which is forbidden for a  $\phi^4$ -like theory (see Refs. 60 and 61, and reference therein), (ii) existence of a topological singularity in the mapping between  $(N, d)$  and the fixed-point coordinates that makes these last quantities multivalued functions of  $(d, N)$  that is manifestly a pathology.

From the discussion above it appears that a necessary condition for a fixed point to be considered as a genuine fixed point is that it satisfies both the PMS and the PFAC. We however now show on the example of the  $2d$   $O(N)$  models that, although necessary, these conditions are not sufficient.

**C.  $O(N)$  models in two dimensions: Anomalous apparent convergence**

The same kind of analysis of the perturbative results obtained from the  $\phi^4$  model in three dimensions can be performed for all  $N$  in two dimensions.<sup>57</sup> We show in Fig. 4 the exponent  $\omega$  of the two-dimensional  $O(4)$  model obtained at three, four, and five loops in the zero-momentum massive scheme and the anomalous dimension  $\eta$  at four and five loops (the three-loop results does not lead to a clear stationary behavior). A conformal Borel resummation method has been used.

For this model, because of Mermin-Wagner’s theorem,<sup>62</sup> the correlation length is infinite at zero temperature only and the critical exponents are exactly known:  $\eta=0$  and  $\omega=2$ .<sup>63,64</sup> We can see in Fig. 4 that although the values obtained for these exponents seem well converged, they are erroneous since using both the PMS and PFAC one finds:  $\eta\approx 0.12$  and  $\omega\approx 1.37$ . It is important to emphasize that  $N=4$  is not an

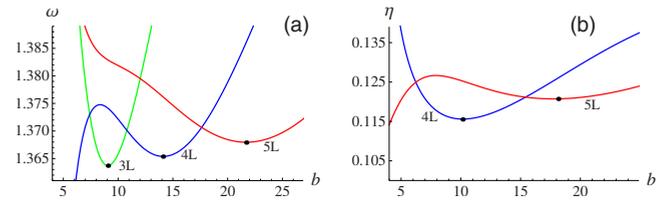


FIG. 4. (Color online) The exponents  $\omega$  and  $\eta$  of the two-dimensional  $O(4)$  model as functions of the resummation parameter  $b$  at three-, four-, and five-loop order (the result for  $\eta$  at three loops is not displayed since there is no clear stationarity for  $\eta$  in this case). The parameter  $a$  has been fixed at the value obtained from the large-order behavior:  $a=a_{10}\approx 0.1789$ . For  $\omega$  one has  $(\alpha_{\text{opt}}, b_{\text{opt}})=(3.1, 9)$  at three loops,  $(\alpha_{\text{opt}}, b_{\text{opt}})=(3.1, 14)$  at four loops, and  $(\alpha_{\text{opt}}, b_{\text{opt}})=(3.1, 21.5)$  at five loops. For  $\eta$ , one has  $(\alpha_{\text{opt}}, b_{\text{opt}})=(4.4, 10)$  at four loops and  $(\alpha_{\text{opt}}, b_{\text{opt}})=(4.6, 18)$  at five loops. The dot on each curve corresponds to stationary values of  $\omega=\omega(\alpha, b)$  and  $\eta=\eta(\alpha, b)$  in both directions.

isolated case in this respect. For all two-dimensional  $O(N)$ —with  $N\geq 1$ —models, the critical exponents seem to be converged at five-loop order but toward erroneous values. For instance, in the Ising model, and at five loops, Orlov and Sokolov<sup>44</sup> have found  $\eta=0.146$ , and Pogorelov and Suslov<sup>65</sup>  $\eta=0.145(14)$  whereas the exact result is  $\eta=0.25$ .

We have also studied the  $a$  dependence of the critical exponents. We have found here again that  $a_{\text{opt}}\approx a_{10}=0.1789$ , see Fig. 5. This means that the  $a$  dependence is not either a good indicator of an anomalous apparent convergence in this case.

An analysis of the underlying reasons of this anomalous convergence has been performed in the Ising case in Ref. 45 (see also Refs. 66 and 67). The explanation is that there very likely exist, in the  $\beta$  function, terms such as  $1-(1-u/u^*)^e$  with  $u^*$  the fixed-point value of  $u$  and  $e$  a small number (probably  $1/7$  in the Ising case).<sup>45,68</sup> In the perturbative expansion performed around  $u=0$ , such terms lead to small contributions to the  $\beta$  function that seems to be under control. However, they play an important role in the vicinity of  $u^*$ ; they are even nonanalytic at this point since their derivatives with respect to  $u$  are singular at  $u^*$ . Reconstructing such terms from a perturbative expansion is thus difficult and, as a consequence, the perturbative results are doomed to failure although they look converged. Thus we are lead to the con-

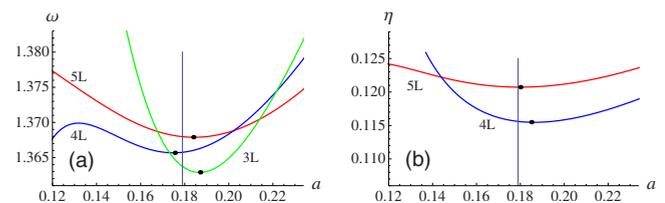


FIG. 5. (Color online) The exponents  $\omega$  and  $\eta$  of the two-dimensional  $O(4)$  model as functions of the resummation parameter  $a$ , at three, four, and five loops (the result for  $\eta$  at three loops is not displayed since there is no clear stationarity in this case). The vertical line corresponds to  $a_{10}\approx 0.1789$ . The values chosen for  $\alpha$  and  $b$  are such that the exponents are at their stationary value when  $a=a_{10}$  (see Fig. 4). Note that the stationary values  $\omega(a)$  and  $\eta(a)$  indicated by dots are very close to their values at  $a=a_{10}$ .

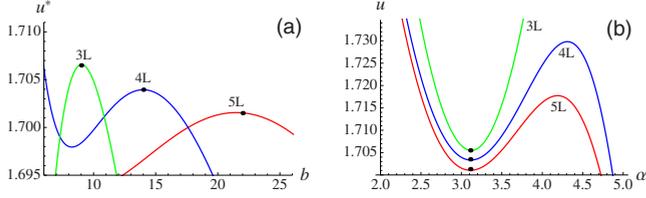


FIG. 6. (Color online) The fixed-point coupling constant  $u^*$  of the  $O(4)$  model in  $d=2$ : (a) as a function of  $b$  at fixed  $\alpha=3.1$ , (b) as a function of  $\alpha$  at fixed  $b=b_{\text{opt}}=9, 14, 22$  at three, four, and five loops, respectively. The parameter  $a$  has been fixed at  $a_{10}=0.1789$ .

clusion that PMS and PFAC are necessary conditions for convergence but are not sufficient.

Let us now make a remark specific to  $d=2$ . In this dimension, the existence of a non trivial root  $u^*$  of the  $\beta$  function, stable with respect to the resummation parameters  $b$  and  $\alpha$  and displaying good convergence properties, see Fig. 6, is not sufficient in itself to know whether the transition is trivial (taking place at zero temperature) or not since  $u$  is not directly related to the temperature. In principle, the triviality (for  $N \geq 3$ ) or nontriviality (for  $N=1$  or 2) of the critical exponents should be sufficient to conclude. However, as previously emphasized, the presence of strong nonanalyticities in the two-dimensional  $\beta$  functions of the Ising and  $O(N)$  models prevent us to do so since they completely spoil the determination of the critical exponents.

### III. FRUSTRATED $O(N) \times O(2)$ MODELS

Let us now come to the frustrated models we are directly interested in. A first analysis of the convergence of the  $\overline{\text{MS}}$  series obtained at five loops in these models<sup>28</sup> was done in Ref. 42. In the following, we study these models by analyzing the perturbative series obtained in the massive zero-momentum scheme at six loops in three dimensions<sup>73</sup> and at five loops in two dimensions,<sup>34</sup> a case that the  $\overline{\text{MS}}$  series do not allow to satisfactorily study since the values of the coupling constants at the fixed point are out of the region of Borel summability.

The Hamiltonian relevant for frustrated systems is given by<sup>69–72</sup>

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} [(\partial \phi_1)^2 + (\partial \phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)] + \frac{u}{4!} [\phi_1^2 + \phi_2^2]^2 + \frac{v}{12} [(\phi_1 \cdot \phi_2)^2 - \phi_1^2 \phi_2^2] \right\}, \quad (13)$$

where  $\phi_i$ ,  $i=1, 2$  are  $N$ -component vector fields. The resummation procedure outlined above can be generalized to the case where there are several coupling constants as it is the case for frustrated systems. For a function  $f$  of the two variables  $u$  and  $v$  known through its series expansion in powers of  $u$  and  $v$ , the resummation procedure used in Refs. 30, 33, and 73 consists in assuming that  $f$  can be considered as a function of  $u$  and of the ratio  $z=v/u$ ,

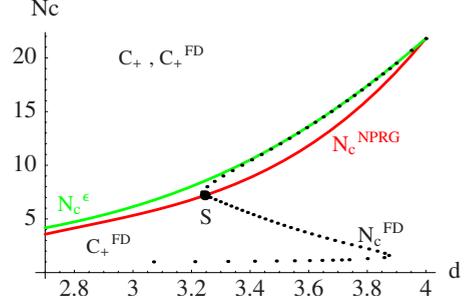


FIG. 7. (Color online) Curves  $N_c(d)$  obtained within the  $\epsilon$ -expansion ( $N_c^\epsilon$ ), the  $\overline{\text{MS}}$  scheme without  $\epsilon$  expansion ( $N_c^{\text{FD}}$ ) and the NPRG approach ( $N_c^{\text{NPRG}}$ ). The resummation parameters for the  $\overline{\text{MS}}$  curve are  $a=0.5$ ,  $b=10$ , and  $\alpha=1$ . The part of the curve  $N_c^{\text{FD}}$  below S corresponds to a regime of non-Borel summability. The attractive fixed point is called  $C_+$  and  $C_+^{\text{FD}}$  when it is found in the fixed dimension ( $\overline{\text{MS}}$ ) scheme.

$$f(u, z) = \sum_n a_n(z) u^n \quad (14)$$

and in resumming with respect to  $u$  only. Under this hypothesis, the resummed expression associated with  $f$  reads

$$f_R(u, z) = \sum_n d_n[\alpha, a(z), b; z] \int_0^\infty dt e^{-t} t^b \frac{[\omega(ut; z)]^n}{[1 - \omega(ut; z)]^\alpha} \quad (15)$$

with

$$\omega(u; z) = \frac{\sqrt{1 + a(z)u} - 1}{\sqrt{1 + a(z)u} + 1}, \quad (16)$$

where, as above, the coefficients  $d_n[\alpha, a(z), b; z]$  in Eq. (15) are computed so that the reexpansion of the rhs of Eq. (15) in powers of  $u$  coincides with that of Eq. (14). Of course, since the resummation is performed in only one variable, we cannot expect in this case a convergence of the resummed quantities as good as in the  $O(N)$  case.

#### A. Frustrated models in $d=3$

We recall in Fig. 7 the results obtained using different approaches. In the  $(d, N)$  plane, a line  $N_c(d)$  is found in all approaches such that the stable fixed point exists for  $N > N_c(d)$  and disappears for  $N < N_c(d)$ . This result is interpreted as the occurrence of a second-order transition for values of  $N$  above  $N_c(d)$  and a first-order transition for values of  $N$  below  $N_c(d)$ . In the  $\epsilon$  expansion<sup>35–37</sup> and within the NPRG,<sup>1,38–41</sup> the lines  $N_c(d)$  are both monotonic and are very similar (see Fig. 7). They lead to the fact that  $N_c(d=3) > 3$  and the transition is thus found to be of first order for  $N=2$  and 3 in three dimensions. On the contrary, in the  $\overline{\text{MS}}$  scheme without  $\epsilon$  expansion,<sup>28</sup> the curve  $N_c(d)$  is found to have a S shape, see Fig. 7, and thus, at  $d=3$ , fixed points exist for  $N=2$  and 3. In the massive scheme also fixed points are found for these values of  $N$ .<sup>30</sup>

The  $\overline{\text{MS}}$  scheme perturbative series at five loops were reexamined in Ref. 42. The bad convergence of the re-

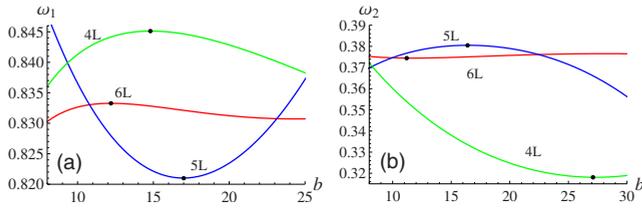


FIG. 8. (Color online) The exponents  $\omega_1$  and  $\omega_2$  of the three-dimensional frustrated model for  $N=8$  as functions of the resummation parameter  $b$  at four-, five-, and six-loop orders. The dot on each curve corresponds to a stationary value of  $\omega = \omega(\alpha, b)$  in both  $\alpha$  and  $b$  directions with  $a$  fixed to its large-order value  $a_{l_0} \approx 0.0554$ . For  $\omega_1$  one has  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (5.1, 14.8)$  at four loops,  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (7.8, 17)$  at five loops, and  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (7, 12.2)$  at six loops. For  $\omega_2$  one has  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (6.5, 27.1)$  at four loops,  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (7.9, 16.4)$  at five loops, and  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (7.5, 11.7)$  at six loops.

summed series, the analytic properties of the coordinates of the fixed points  $(u^*, v^*)$  considered as functions of  $(d, N)$  [presence of a topological singularity  $S$ , see Fig. 7, in the  $(d, N)$  plane] and the fact that the fixed points found at  $N=2$  and  $N=3$  in  $d=3$  do not become Gaussian when they are followed continuously in  $d$  up to  $d=4$  led the authors of Ref. 42 to conclude that these fixed points were either spurious or the results nonconverged. By reanalyzing the resummed series obtained at six loops in  $d=3$  in the massive scheme we show in the following (i) that for sufficiently large values of  $N$  (typically  $N > 7$ ) the resummed series for the exponents converge well, (ii) that for  $N=2$  and 3 these series do not lead to converged results. The situation is thus similar to what has already been obtained in the  $\overline{\text{MS}}$  scheme.

### 1. $N=8$ frustrated model in $d=3$

Let us start our analysis by the  $N=8$  case to show how the results obtained at large and small values of  $N$  for frustrated systems display completely different convergence properties. Since the model involves two coupling constants  $u$  and  $v$ , there are two eigenvalues of the stability matrix of the RG flow at the fixed point that we call  $\omega_1$  and  $\omega_2$ . They represent the generalization of the exponent  $\omega$  of the  $O(N)$  models and they rule the stability of the fixed point: it is attractive when  $\omega_1$  and  $\omega_2$  have both positive real parts.

We take for  $a$  the value obtained from the large-order analysis:  $a_{l_0} = 0.0554$ . We find that the PMS is satisfied at four-, five- and six-loop orders: for suitable values of the parameters  $b$  and  $\alpha$  the two exponents  $\omega_1$  and  $\omega_2$  depend weakly on these parameters and are reasonably well converged. This is clear from Fig. 8 where we show the  $b$  dependence of  $\omega_1$  and  $\omega_2$ . Moreover the difference between the values at five and six loops of, for instance,  $\omega_1$  is small:  $\omega_1(L=6) - \omega_1(L=5) \approx 0.012$ . Note that our values of  $\omega_1$  and  $\omega_2$  in this case are fully compatible with those obtained in Ref. 32.

We have also studied the  $a$  dependence of these quantities and find that the optimal value of  $a$  is close to its large-order value ( $a_{\text{opt}} \approx a_{l_0} = 0.0554$ ) as it is the case in the  $O(N)$  models, see Fig. 9.

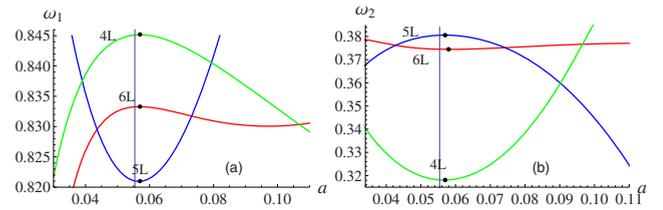


FIG. 9. (Color online) The exponents  $\omega_1$  and  $\omega_2$  of the three-dimensional frustrated model for  $N=8$  as functions of the parameter  $a$  at four-, five-, and six-loop orders. The vertical lines corresponds to the large-order value of  $a_{l_0} = 0.0554$ . The values chosen for  $\alpha$  and  $b$  are such that the exponents are at their stationary value when  $a = a_{l_0}$  (see Fig. 8).

These results indicate that the convergence properties of the  $N=8$  frustrated model are globally similar to those of the  $O(N)$  models although less accurate very likely because in the latter case the resummation is less efficient due to the presence of two coupling constants.

### 2. $N=2$ and $N=3$ frustrated models in $d=3$

We now analyze the physical values of  $N$ , that is  $N=2$  and 3. In Ref. 42, it has been found in the  $\overline{\text{MS}}$  scheme that, because of the presence of the singularity  $S$  (see Fig. 7) which exists in this scheme for  $N \approx 7$  and  $d \approx 3.2$ , the results obtained from the resummed series above and below  $N \approx 7$  are very different. In the massive scheme, the series are known in integer space dimensions only and it is thus not possible to know whether this perturbative scheme leads also to the existence of a singularity. We nevertheless show that, within this scheme, the results obtained for  $N=2$  and 3 are very different from those obtained for  $N \geq 8$  and are fully compatible with those obtained with the  $\overline{\text{MS}}$  scheme.<sup>42</sup>

Let us first recall that for  $N=2$  and  $N=3$ , the fixed point is (in most cases but not systematically, in particular, for values of  $\alpha$  different from 1,2,3) an attractive focus, that is  $\omega_1$  and  $\omega_2$  are complex conjugate and  $\text{Re}(\omega_1) = \text{Re}(\omega_2) > 0$ . We, again, take for  $a$  the value obtained from the large-order analysis:  $a_{l_0} = 0.1108$  for  $N=2$  and  $a_{l_0} = 0.095$  for  $N=3$ . For these values of  $a$  and for  $L=4, 5$ , and 6, we find that  $\text{Re}(\omega_1)$  [or equivalently  $\text{Re}(\omega_2)$ ] considered as a function of  $\alpha$  and  $b$  is *nowhere* stationary, even approximately, see Fig. 10. Moreover, at fixed  $\alpha$  and  $b$ , the gap between the values of  $\text{Re}(\omega_1)$  at two successive loop orders:  $\text{Re}(\omega_1)(L+1)$

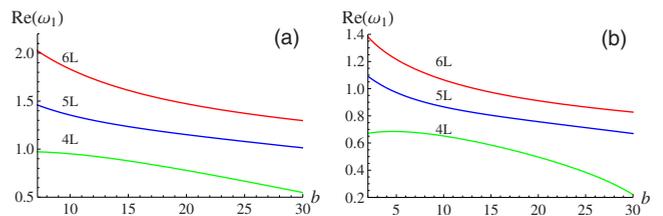


FIG. 10. (Color online) The (real part of the) exponent  $\omega_1$  of the three-dimensional frustrated model (a) for  $N=2$  and (b) for  $N=3$  as a function of  $b$  for  $\alpha=6$  at four, five, and six loops. The parameter  $a$  has been fixed at the value obtained from the large-order behavior:  $a_{l_0} = 0.1108$  for  $N=2$  and  $a_{l_0} = 0.095$  for  $N=3$ . Other values of the parameter  $\alpha$  give similar results.

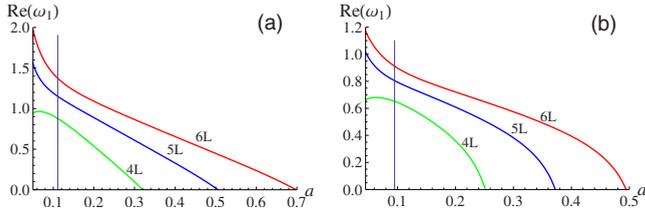


FIG. 11. (Color online) The (real part of the) exponent  $\omega_1$  of the three-dimensional frustrated model (a) for  $N=2$  and (b) for  $N=3$  as a function of  $a$ . The vertical lines corresponds to  $a_{10}=0.1108$  for  $N=2$  and  $a_{10}=0.095$  for  $N=3$ . One has taken  $\alpha=6$  and  $b=25, 20$ , and  $15$  at six, five, and four loops for  $N=2$  and  $b=20, 15$ , and  $10$  at six, five, and four loops for  $N=3$ . Other values of the parameters  $\alpha$  and  $b$  give similar results.

$-\text{Re}(\omega_1)(L)$ , is always large, of order  $0.5$  for  $N=2$  and  $0.2$  in the  $N=3$  case, see Fig. 10. Thus neither the PMS nor the PFAC are satisfied for these values of  $N$ .

We have also studied the stability of our results with respect to variations in  $a$  for characteristic values of  $\alpha$  and  $b$ , see Fig. 11. Here again we find no stationarity.

From these results it clearly appears that the critical exponents deduced from the resummed series obtained in the massive scheme in  $d=3$  display both lack of convergence and of stability for the values  $N=2$  and  $3$ . It seems therefore very likely that the existence of fixed points for these values of  $N$  is an artifact either of the fixed dimension schemes or of the resummation method. In any case, we confirm by this study that there is no reason coming from the fixed dimension approaches to question the results obtained either within the  $\epsilon$  expansion or the NPRG and that, very probably, the transitions found in  $d=3$  and  $N=2$  and  $3$  are always of first order. Let us now perform the same kind of analysis for the two-dimensional models.

### B. Frustrated models in $d=2$

As already emphasized, the two-dimensional case is particularly interesting because of the presence of topological excitations in the Heisenberg case.<sup>2</sup> Because of the homotopy properties of the symmetry group  $SO(3)$  of these systems, the topological excitations are different from the  $O(2)$  vortices encountered in the ferromagnetic XY system. It is still an open question to know whether the deconfinement of these defects could trigger a genuine phase transition, as in the Kosterlitz-Thouless case. Note that such a phenomenon would be surprising since one knows from the spin-wave—low-temperature—approach<sup>74–76</sup> that, contrary to the  $O(2)$  model, the spin-spin correlation length of  $O(3)$  frustrated models is finite at low—but nonvanishing—temperature and that vortices tend to further disorganize the system. We let aside the delicate question of the very mechanism underlying a hypothetical genuine phase transition in these systems and focus on the question of the existence of a finite-temperature fixed point within the FD formalism.

As for the XY case, the question is to know whether there is a unique or two separate (Ising and KT) phase transitions. From the most recent Monte Carlo simulations, it has been argued that there are two distinct but very close phase tran-

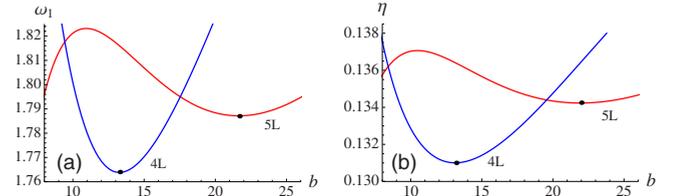


FIG. 12. (Color online) The exponents  $\omega_1$  and  $\eta$  in the two-dimensional frustrated  $N=8$  case as a function of  $b$  at four- and five-loop orders. The dots at each curve correspond to a stationary value of  $\omega = \omega(\alpha, b)$  in both  $\alpha$  and  $b$  directions. For  $\omega_1$  one has:  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (4.7, 13.3)$  at four loops and  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (4.7, 21.7)$  at five loops. For  $\eta$  one has  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (4.55, 13.2)$  at four loops and  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (4.55, 22)$  at five loops.

sitions, the Ising one taking place at the highest temperature. Accordingly one could expect the transition to be characterized by Ising critical exponents.

In  $d=2$  and for the values of  $N \geq 4$ , there is no topological defects. As a consequence, there cannot be any other fixed point but the zero-temperature one. Thus, for these values of  $N$  and because of Mermin-Wagner's theorem, the correlation length diverges at zero temperature only and with an exponent  $\nu$  which is infinite (exponential divergence of the correlation length). Moreover the anomalous dimension  $\eta$  is always vanishing at a zero-temperature fixed point as can be checked on the low-temperature expansion performed within the nonlinear sigma model.<sup>74–76</sup> Thus, as in the two-dimensional  $O(N)$  case, any nonvanishing  $\eta$  for  $N \geq 4$  must be considered as an artifact and, for the perturbation theory, as a signal of an anomalous apparent convergence, see above. We start our analysis by the  $N=8$  case and then study the physically relevant cases:  $N=2, 3$ .

#### 1. $N=8$ case in $d=2$

For  $N=8$  and being given the previous discussion, one should obtain only trivial results as for the critical exponents:  $\eta=0$  and  $\omega_1=\omega_2=2$ . We have computed  $\eta$  and  $\omega_1$  (the largest eigenvalue) as functions of the resummation parameters  $\alpha$  and  $b$  by taking for  $a$  the value computed from the large-order behavior:  $a_{10} \approx 0.0895$ . We find a stationary solution for the two exponents studied, see Fig. 12. However, as in the  $O(N)$  models, we find that the value of  $\eta$  thus obtained:  $\eta \approx 0.13$  is unphysical since it should be zero. We find  $\omega_1 \approx 1.79$  which is far from the expected physical value  $\omega_1=2$ .

We have also studied the  $a$  dependence of these exponents. Here again, we find good convergence properties with an extremum around the value  $a_{10}$ , see Fig. 13.

It thus appears that there very likely exist in frustrated models, as in  $O(N)$  models, nonanalytic terms in the  $\beta$  functions that spoil the convergence of the resummed perturbative expansion of the critical exponents. We can already assert that this dramatically alters the relevance of the perturbative  $\phi^4$  approach for the study of the two-dimensional frustrated systems.

#### 2. $N=2$ and $N=3$ cases in $d=2$

We now perform the same analysis as above for the physically relevant values of  $N$ , that is  $N=2$  and  $3$ . Let us first

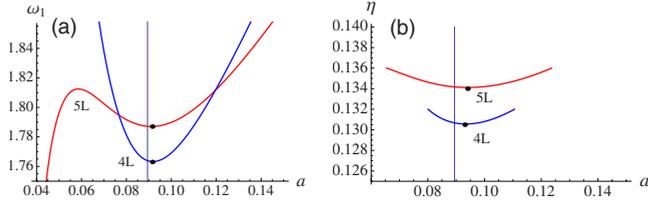


FIG. 13. (Color online) The exponents  $\omega_1$  and  $\eta$  in the two-dimensional frustrated model for  $N=8$  as functions of  $a$  at four and five loops. The vertical lines corresponds to  $a=a_{\text{opt}}=0.0895$ . The values of  $a$  and  $b$  are such that the exponents are at their stationary values when  $a=a_{10}$  (see Fig. 12).

notice that, for these values of  $N$ , the fixed point starts to exist beyond three loop order only. We fix  $a$  at its large-order value:  $a \approx 0.1790$  for  $N=2$  and  $a \approx 0.1534$  for  $N=3$ .

Let us first discuss the  $N=2$  case. We find that the correction to scaling exponent  $\omega_1$  (and thus  $\omega_2$ ) is complex for a large range of parameters  $\alpha$  and  $b$  which means that the fixed point is a focus. We show in Fig. 14 that there is no value of  $\alpha$  and  $b$  where  $\omega_1$  is stationary with respect to both parameters. Moreover, at fixed  $\alpha$  and  $b$ , the difference between the four- and five-loop results is large. This is a clear signal of the nonconvergence of the value of  $\omega_1$ . In Ref. 34, an average value for this critical exponent has been proposed:  $\omega_1 = 2.05(35) \pm i0.80(55)$  at five loops. According to our stability and convergence principles, this value does not really make sense.

The situation is a little bit different for the exponent  $\eta$ . At four loops,  $\eta$  is nowhere stationary in the  $\alpha$  direction as can be seen in Fig. 15(a) whereas there is an almost stationary value  $\eta \approx 0.275$  at five loops in both  $\alpha$  and  $b$  directions for  $\alpha \approx 4.2$  and  $b \approx 11$ , see Fig. 15(b) [which is compatible with the value given in Ref. 34 where  $\eta=0.28(8)$ ]. We have performed the analysis of the stability of our results for  $\eta$  when  $a$  is varied around  $a_{10}$  at fixed  $\alpha$  and  $b$ . We find that indeed the five-loop results do not vary much with  $a$  and that the optimal value of  $a$  is close to  $a_{10}$ .

We conclude that the results for the  $N=2$  case show no convergence with the loop order and a poor stability with respect to variations in  $\alpha$  and  $b$  but perhaps for the exponent  $\eta$  at five loops. Let us notice that the value found  $\eta \approx 0.275$  is relatively close to the exact value expected for an Ising transition ( $\eta=0.25$ ). However, at the same time, it is far from the five-loop value  $\eta=0.146$  (Ref. 44) obtained directly with the  $\phi^4$  field theory. We shall develop on this below.

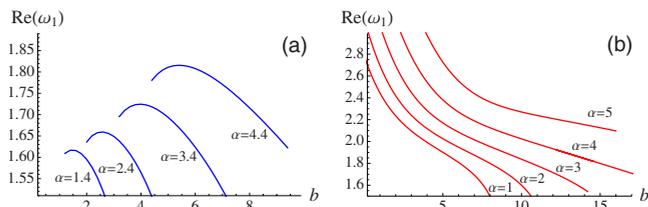


FIG. 14. (Color online) The (real part of the) exponent  $\omega_1$  of the two-dimensional frustrated model for  $N=2$  as a function of  $b$  for different values of  $\alpha$  (a) at four loops and for  $\alpha=1.4, 2.4, 3.4, 4.4$  (b) at five loops and for  $\alpha=1, 2, 3, 4, 5$ . We have chosen  $a \approx 0.1790$ .

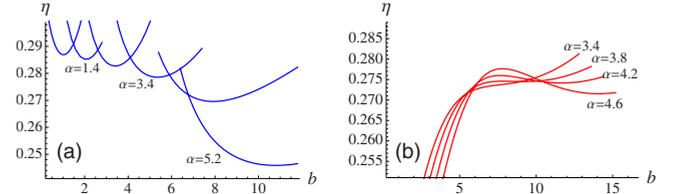


FIG. 15. (Color online) The exponent  $\eta$  of the two-dimensional frustrated model for  $N=2$  as a function of  $b$  for different values of  $\alpha$  (a) at four loops and for  $\alpha=0.3, 1.4, 2.4, 3.4, 4.4, 5.2$  (b) at five loops and for  $\alpha=3.4, 3.8, 4.2, 4.6$ . We have chosen  $a \approx 0.1790$ .

Let us now examine the  $N=3$  case, Fig. 16. The fixed point is again a focus. The difference with the  $N=2$  case is that there now exists a stationary point for  $\text{Re}(\omega_1)$  at five loops for  $\alpha \approx 5.95$  and  $b=10.25$ , but not at four loop order, see Fig. 16(a), where there is no stationarity with respect to  $\alpha$ . At this stationary point, one has  $\text{Re}(\omega_1) \approx 1.78$  [which is compatible with the result found in Ref. 34:  $\text{Re}(\omega_1) = 1.55(25)$  that anyway displays a large error bar]. We find stationary points for  $\eta$  at four- and five-loop orders, see Fig. 16(b). At five loops the value of  $\eta$  at the stationary point is  $\eta=0.23$  [that compares well with the value  $\eta=0.23(5)$  of Ref. 34]. The convergence seems better in this  $N=3$  case than in the  $N=2$  case since now both  $\omega_1$  and  $\eta$  display stationary values.

From the discussion above one could be tempted to conclude, in the  $N=3$  case, that the value  $\eta=0.23$ , although affected by a large error bar ( $\delta\eta=0.05$  according to Ref. 34), is sufficiently large to ensure that  $\eta$  does not vanish, as claimed in Ref. 34. In this case the transition would be nontrivial, that is, would occur at finite temperature. We now argue that the results obtained at five loops are not sufficiently accurate to support this conclusion. The reason is that the error on  $\eta$  is, in fact, underestimated. To see this we have computed  $\eta(N)$  (according to our two principles) for all values of  $N$  between 2 and 8, see Fig. 17. As already emphasized there cannot exist nontrivial fixed points and, thus, nonvanishing anomalous dimensions  $\eta$  for any value of  $N \geq 4$ . As seen in Fig. 17 this is violated by the perturbative results at five loops. This implies that the error  $\delta\eta$  on  $\eta$  at five loops is of order  $\eta$  itself, that is, in the  $N=4$  case, of order 0.20. Being given that  $\eta(N)$  is monotonically decreasing, the error bar increases as  $N$  decreases. In the  $N=3$  case,

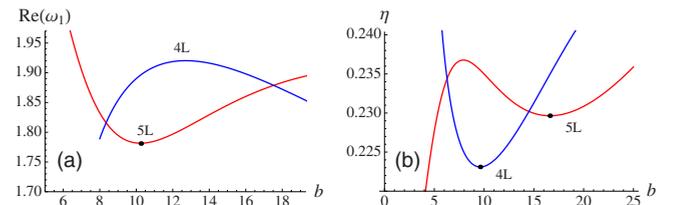


FIG. 16. (Color online) The (real part of the) exponent  $\omega_1$  and  $\eta$  of the two-dimensional frustrated model for  $N=3$  as functions of  $b$  at four and five loops. (a) For  $\omega_1$  there is a stationary point only at five loops for  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (5.95, 10.1)$ . (b) For  $\eta$  there are stationary points at four loops at  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (4.28, 9.6)$  and at five loops at  $(\alpha_{\text{opt}}, b_{\text{opt}}) = (4.45, 16.6)$ . The value of  $a$  has been taken equal to its large-order value  $a \approx 0.1534$ .

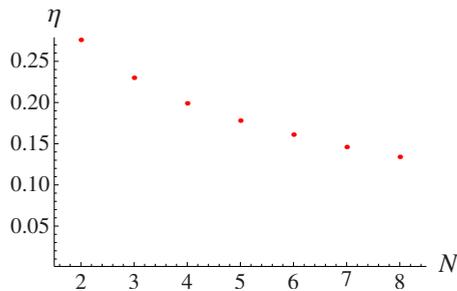


FIG. 17. (Color online) Exponent  $\eta$  of the two-dimensional frustrated model as a function of the number of spin components  $N$ .

the error bar is thus at least equal to 0.20 and since  $\eta(N=3)$  is found to be equal to 0.23 it is impossible to conclude that  $\eta$  is nonvanishing in this case. While our considerations extend also very likely to the  $N=2$  case as for the existence of a large error on the result, this case is particular. Indeed for  $N=2$  one expects  $\eta=0.25$  since the transition likely belongs to the Ising universality class. At first sight, the value found at five loops ( $\eta=0.275$ ) could seem encouraging. However, let us recall that one finds  $\eta=0.146$  from the one-component  $\phi^4$  model in  $d=2$  (Refs. 44 and 65) which is very far from the expected result. As can be seen from the  $N \geq 4$  results, the series for frustrated magnets do not exhibit better convergence properties than those of the  $\phi^4$  model and thus the value of  $\eta$  found in the  $N=2$  frustrated case should very likely be interpreted as a numerical coincidence. This conclusion is reinforced by the fact that, as explained previously, the stability properties of  $\eta$  in the  $N=2$  case are also unsatisfactory, see Fig. 15.

#### IV. CONCLUSION

We have investigated the series obtained from FD perturbative approaches in zero-momentum massive RG scheme for  $O(N)$  models and frustrated magnets both in  $d=2$  and in  $d=3$  at five- and six-loop orders, respectively. From a general point of view, the result of our study is that only the  $O(N)$  models in  $d=3$  provides unambiguous and precise results. For frustrated magnets, our results in  $d=3$ , that show an absence of stationarity of the exponents considered as functions of the resummation parameters  $\alpha$  and  $b$  and a bad convergence with the number of loops, provide strong support to the spurious character of the fixed points found for  $N=2$  and  $N=3$ . Without providing a definitive answer to the question of the nature of the phase transition that frustrated

magnets undergo in  $d=3$  our results weaken a lot the predictions of a second-order behavior. Since all other studies than the FD approaches ( $\epsilon$  expansion, NPRG) predict first-order behavior (see Ref. 1), we are naturally led to the conclusion that three-dimensional frustrated magnets should always exhibit first-order behaviors.

In  $d=2$ , the situation is more ambiguous since the critical exponents satisfy the PMS in some cases (at some orders and for some critical exponents). At first sight, one could deduce from these results the existence, for Heisenberg spins, of a finite-temperature phase transition triggered by the deconfinement of topological excitations. However, a careful comparative study between the  $O(N)$  and frustrated models shows that the presence of nonanalyticities spoils the determination of the critical exponents and forbids to conclude.

It is not clear that only a few higher orders of the perturbative expansion would be sufficient to clarify the situation and one has to think about another approach in both  $d=2$  and  $d=3$ . From this point of view, the NPRG seems to be able to circumvent the main difficulties. Indeed being not based on a perturbative expansion (in the traditional sense), it does not suffer from some of the problems encountered in the weak-coupling approaches. In particular, it seems to be unaffected by the problems of nonanalyticities since the value found for  $\eta$  in the  $d=2$  Ising case is, within this approach, found equal to 0.254,<sup>77</sup> in excellent agreement with the exact result. The  $d=2$  case for both frustrated Heisenberg and XY systems is under investigation.<sup>78</sup>

Let us finally emphasize that the methodology put forward in this paper could be relevant for any system analyzed within the FD perturbative approach. Indeed, in this case, the existence of spurious fixed points is the generic case and one has to be especially careful when fixed points that have no counterpart within the  $\epsilon$ -expansion approach occur. In such circumstances, the principles employed here—PMS and PFAC—can be of great interest to reject or to accept the fixed points as physical solutions.

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