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## Biconical critical dynamics

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**Abstract** – A complete two-loop renormalization group calculation of the multicritical dynamics at a tetracritical or bicritical point in three-dimensional anisotropic antiferromagnets in an external magnetic field is performed. Although strong scaling for the two order parameters (OPs) perpendicular and parallel to the field is restored as found earlier, in the experimentally accessible region the effective dynamical exponents for the relaxation of the OPs remain different since their equal asymptotic values are not reached.

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Systems with more than one order parameter (OP) exhibit a rich variety of phases separated by transition lines which might meet in multicritical points. The interaction might favor simultaneously ordering of two OPs. Such a doubled-ordered phase is known as supersolid phase [1] and is under investigation since its possible observance in <sup>4</sup>He [2]. The two OPs might describe the order in physically different phases: *e.g.* from the normal fluid to the superfluid or to the normal solid phase. If both orders appear one might have transitions from the superfluid to the supersolid phase and from the normal solid phase to the supersolid phase. Another example would be a system with transitions to a superconducting and a magnetically ordered phase and a phase where both orderings appear. In purely magnetic systems the phases are characterized by different orderings in spin space. There is a correspondence between the quantum liquid system and magnetic systems where the supersolid phase corresponds to the biconical phase [3]. The existence of a biconical phase leads to the occurrence of a tetracritical point where four second order phase transition lines meet and which belongs to a new universality class [4].

In the case of the three-component ( $n=3$ ) three-dimensional ( $d=3$ ) anisotropic antiferromagnets in an external magnetic field in  $z$ -direction the disordered

(paramagnetic) phase is separated from the ordered phases by two second-order phase transition lines: i) one to the spin flop phase (ordering in the spin space perpendicular to the external magnetic field) and ii) one to the antiferromagnetic phase (ordering parallel to the external field). The point where these two lines meet is a multicritical point which turned out to be either tetracritical or bicritical depending on whether the ordered phases are separated by an intermediate biconical phase or not. The static phase transitions on each of the phase transition lines belong for i) to an XY-model with  $n=2$  and for ii) to an Ising model with  $n=1$  [4]. Whether a bicritical or tetracritical point is realized depends on the specific fourth-order couplings [4,5]. Several antiferromagnets have been suggested for observing the theoretically proposed phase diagrams, a review on the experimental situation can be found in [6–8].

Concerning the dynamical universality classes they might be different for the systems mentioned above depending on the possible reversible and non-reversible terms in the equations of motion and the conservation properties. In the magnetic system considered here the transition i) belongs to the class described by model-F and ii) belongs to the model-C class (for the definitions of the models see [9]). At the multicritical point the critical behavior is described by a new universality class both in statics and dynamics characterized by the biconical fixed

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point [5]. The advantageous feature of these systems is that all the different OPs characterizing the ordered phase are experimentally accessible. This is most important for the dynamical behavior since the only other example belonging to model F is the superfluid transition in  $^4\text{He}$  where the OP is not directly measurable. Here the OPs are the components of the staggered magnetization. Their correlations (static and dynamical) are experimentally accessible by neutron scattering. Realistic models might be more complicated (see, *e.g.*, [10]) but the behavior near the multicritical point is well described by the renormalization group (RG) theory.

The dynamical model we analyze goes beyond the pure relaxational dynamics [11] and has been considered by means of the field-theoretical RG approach in [12–14] replacing earlier mode coupling theories [15]. It was argued that due to non-analytic terms in  $\epsilon = 4 - d$  a dynamical fixed point (FP) in two-loop order (which was calculated only partly) qualitatively different from the one-loop FP is found. In one-loop order the relaxation times of the components of the staggered magnetization parallel and perpendicular to the external magnetic field scale differently whereas in two-loop order they would scale similarly if the new FP would be stable. In addition it turned out that the FP value of the time scale ratio of the two OPs cannot be found by  $\epsilon$  expansion and might be very small at  $d=3$ , namely of  $\mathcal{O}(10^{-86})$ . A basic assumption of the above analysis was that within statics the Heisenberg FP is stable. However, it turned out in two-loop statics using resummation techniques that in  $d=3$  the Heisenberg FP interchanges its stability with the biconical FP [5]. Here we calculate the complete functions in two-loop order which allows us to consider the non-asymptotic behavior near the multicritical point.

The non-conserved OP in an isotropic antiferromagnet is given by the three-component vector  $\vec{\phi}_0$  of the staggered magnetization, which is the difference of two sublattice magnetizations. In an external magnetic field applied to the anisotropic antiferromagnet the OP splits into two OPs,  $\vec{\phi}_{\perp 0} = (\phi_0^x, \phi_0^y)$  perpendicular to the field, and  $\phi_{\parallel 0} = \phi_0^z$  parallel to the external field. In addition to the two OPs the  $z$ -component of the magnetization, which is the sum of the two sublattice magnetizations, has to be considered as conserved secondary density  $m_0$ . The static critical behavior of the system is described by the functional

$$\begin{aligned} \mathcal{H} = \int d^d x \left\{ \frac{1}{2} \dot{r}_{\perp} \vec{\phi}_{\perp 0} \cdot \vec{\phi}_{\perp 0} + \frac{1}{2} \sum_{i=1}^d \nabla_i \vec{\phi}_{\perp 0} \cdot \nabla_i \vec{\phi}_{\perp 0} \right. \\ + \frac{1}{2} \dot{r}_{\parallel} \phi_{\parallel 0} \phi_{\parallel 0} + \frac{1}{2} \sum_{i=1}^d \nabla_i \phi_{\parallel 0} \nabla_i \phi_{\parallel 0} + \frac{\dot{u}_{\perp}}{4!} (\vec{\phi}_{\perp 0} \cdot \vec{\phi}_{\perp 0})^2 \\ + \frac{\dot{u}_{\parallel}}{4!} (\phi_{\parallel 0} \phi_{\parallel 0})^2 + \frac{2\dot{u}_{\times}}{4!} (\vec{\phi}_{\perp 0} \cdot \vec{\phi}_{\perp 0}) (\phi_{\parallel 0} \phi_{\parallel 0}) \left. \right\} \\ + \frac{1}{2} m_0^2 + \frac{1}{2} \dot{\gamma}_{\perp} m_0 \vec{\phi}_{\perp 0} \cdot \vec{\phi}_{\perp 0} + \frac{1}{2} \dot{\gamma}_{\parallel} m_0 \phi_{\parallel 0} \phi_{\parallel 0} - \dot{h} m_0 \left. \right\}, \end{aligned} \quad (1)$$

with familiar notations for bare couplings  $\{\dot{u}, \dot{\gamma}\}$ , masses  $\{\dot{r}\}$  and field  $\dot{h}$  [5,11]. One may switch from the description of real OP components  $\vec{\phi}_{\perp 0}$  to a complex OP, a macroscopic wave function, as it appears in a superfluid or superconductor by defining  $\psi_0 = \phi_0^x - i\phi_0^y$ . Apart from demonstrating that all these systems belong to the same the static universality class it also is of practical advantage in the dynamic calculation.

The critical dynamics of relaxing OPs coupled to a diffusing secondary density is governed by the following equations of motion [12] (there the complex OP  $\psi_0$  was used):

$$\begin{aligned} \frac{\partial \phi_{\perp 0}^{\alpha}}{\partial t} = -\dot{\Gamma}_{\perp}' \frac{\delta \mathcal{H}}{\delta \phi_{\perp 0}^{\alpha}} + \dot{\Gamma}_{\perp}'' \epsilon^{\alpha\beta z} \frac{\delta \mathcal{H}}{\delta \phi_{\perp 0}^{\beta}} \\ + \dot{g} \epsilon^{\alpha\beta z} \phi_{\perp 0}^{\beta} \frac{\delta \mathcal{H}}{\delta m_0} + \theta_{\phi_{\perp}}^{\alpha}, \end{aligned} \quad (2)$$

$$\frac{\partial \phi_{\parallel 0}}{\partial t} = -\dot{\Gamma}_{\parallel}' \frac{\delta \mathcal{H}}{\delta \phi_{\parallel 0}} + \theta_{\phi_{\parallel}}, \quad (3)$$

$$\frac{\partial m_0}{\partial t} = \dot{\lambda} \nabla^2 \frac{\delta \mathcal{H}}{\delta m_0} + \dot{g} \epsilon^{z\alpha\beta} \phi_{\perp 0}^{\alpha} \frac{\delta \mathcal{H}}{\delta \phi_{\perp 0}^{\beta}} + \theta_m, \quad (4)$$

with the Levi-Civita symbol  $\epsilon^{ijk}$ . Here  $\alpha, \beta = x, y$  and the sum over repeated indices is implied. Combining the kinetic coefficients of the OP to a complex quantity,  $\dot{\Gamma}_{\perp} = \dot{\Gamma}_{\perp}' + i\dot{\Gamma}_{\perp}''$ , the imaginary part constitutes a precession term created by the renormalization procedure even if it is absent in the background. The kinetic coefficient  $\dot{\lambda}$  and the mode coupling  $\dot{g}$  are real. The stochastic forces  $\vec{\theta}_{\phi_{\perp}}$ ,  $\vec{\theta}_{\phi_{\parallel}}$  and  $\theta_m$  fulfill Einstein relations

$$\langle \theta_{\phi_{\perp}}^{\alpha}(x, t) \theta_{\phi_{\perp}}^{\beta}(x', t') \rangle = 2\dot{\Gamma}_{\perp}' \delta(x - x') \delta(t - t') \delta^{\alpha\beta}, \quad (5)$$

$$\langle \theta_{\phi_{\parallel}}(x, t) \theta_{\phi_{\parallel}}(x', t') \rangle = 2\dot{\Gamma}_{\parallel}' \delta(x - x') \delta(t - t'), \quad (6)$$

$$\langle \theta_m(x, t) \theta_m(x', t') \rangle = -2\dot{\lambda} \nabla^2 \delta(x - x') \delta(t - t'). \quad (7)$$

The reversible terms in these dynamic equations for the OP components and the conserved density have been derived by using generalized Poisson brackets for the spin components defining the staggered magnetization and the  $z$ -component of the magnetization (for more details see [9] and the literature cited there). An exception constitutes the term with  $\dot{\Gamma}_{\perp}''$  which appears due to the renormalization procedure and the non-zero asymmetric coupling  $\dot{\gamma}_{\perp}$ . Similar dynamic equations may also appear in the other systems where biconical phases are observed. The superfluid transition is described by model F and also for the superconducting transition this universality class has been suggested [16,17] although no explicit derivation based on the methods used here has been performed.

Applying the renormalization procedure using minimal subtraction scheme [18] we find the flow equations for the time scale ratios of the renormalized kinetic coefficients and the mode coupling between the perpendicular OP components and the magnetization. We define time scale

ratios by the ratios of the kinetic coefficients of the OPs and the secondary density  $w_\perp \equiv \frac{\Gamma_\perp}{\lambda}$ ,  $w_\parallel \equiv \frac{\Gamma_\parallel}{\lambda}$ , as well as the ratios between the relaxation rates of the two OPs  $v \equiv \frac{\Gamma_\parallel}{\Gamma_\perp} = \frac{w_\parallel}{w_\perp}$ ,  $v_\perp \equiv \frac{\Gamma_\perp}{\Gamma_\perp} = \frac{w_\perp}{w_\perp}$ , and the mode coupling parameters  $f_\perp \equiv g/\sqrt{\Gamma_\perp \lambda}$  or  $F = g/\lambda$ . For these dynamic parameters we obtain the flow equations

$$l \frac{dw_\perp}{dl} = w_\perp (\zeta_{\Gamma_\perp} - \zeta_\lambda), \quad l \frac{dw_\parallel}{dl} = w_\parallel (\zeta_{\Gamma_\parallel} - \zeta_\lambda), \quad (8)$$

$$l \frac{df_\perp}{dl} = -\frac{f_\perp}{2} \left( \epsilon + \zeta_\lambda - 2\zeta_m + \Re \left[ \frac{w_\perp}{w'_\perp} \zeta_{\Gamma_\perp} \right] \right), \quad (9)$$

where  $l$  is the RG flow parameter and the  $\zeta_\lambda$ -function is obtained by the renormalization procedure as

$$\zeta_\lambda = \frac{1}{2} \gamma_\perp^2 + \frac{1}{4} \gamma_\parallel^2 - \frac{f_\perp^2}{2} (1 + \mathcal{Q}). \quad (10)$$

The function  $\mathcal{Q} \equiv \mathcal{Q}(\gamma_\perp, w_\perp, F)$  contains all higher-order contributions beginning with two-loop order and is identical to the corresponding function in model F (see (A.28) and (A.29) in [9]). We obtain the  $\zeta$ -function for the perpendicular kinetic coefficient  $\Gamma_\perp$  as

$$\begin{aligned} \zeta_{\Gamma_\perp} &= \zeta_{\Gamma_\perp}^{(A)}(\{u\}, v_\perp, v) + \frac{D_\perp^2}{w_\perp(1+w_\perp)} \\ &\quad - \frac{2}{3} \frac{u_\perp D_\perp}{w_\perp(1+w_\perp)} A_\perp - \frac{1}{2} \frac{D_\perp^2}{w_\perp^2(1+w_\perp)^2} B_\perp \\ &\quad - \frac{1}{2} \frac{\gamma_\parallel D_\perp}{1+w_\perp} \left( \frac{u_\times}{3} + \frac{1}{2} \frac{\gamma_\parallel D_\perp}{1+w_\perp} \right) X_\perp, \end{aligned} \quad (11)$$

where we have introduced the coupling  $D_\perp \equiv w_\perp \gamma_\perp - iF$ . The functions  $A_\perp \equiv A_\perp(\gamma_\perp, \Gamma_\perp, w_\perp, F)$ ,  $B_\perp \equiv B_\perp(\gamma_\perp, \Gamma_\perp, w_\perp, F)$  are identical to eqs. (A.25), (A.26) in [9].  $X_\perp$  is defined as

$$X_\perp \equiv 1 + \ln \frac{2v}{1+v} - \left( 1 + \frac{2}{v} \right) \ln \frac{2(1+v)}{2+v}, \quad (12)$$

$\zeta_{\Gamma_\perp}^{(A)}(\{u\}, v_\perp, v)$  is the  $\zeta$ -function of the perpendicular relaxation  $\Gamma_\perp$  in the biconical model A, but now with a complex kinetic coefficient  $\Gamma_\perp$

$$\begin{aligned} \zeta_{\Gamma_\perp}^{(A)}(\{u\}, v_\perp, v) &= \frac{u_\perp^2}{9} \left( 2 \ln \frac{2}{1 + \frac{1}{v_\perp}} \right. \\ &\quad \left. + (2 + v_\perp) \ln \left( \frac{1 + \frac{1}{v_\perp}}{1 + 2\frac{1}{v_\perp}} - \frac{1}{2} \right) \right) \\ &\quad + \frac{u_\times^2}{36} \left( \ln \frac{(1+v)^2}{v(2+v)} + \frac{2}{v} \ln \frac{2(1+v)}{2+v} - \frac{1}{2} \right). \end{aligned} \quad (13)$$

The dynamic  $\zeta$ -function of the parallel relaxation kinetic coefficient  $\Gamma_\parallel$  is obtained as

$$\begin{aligned} \zeta_{\Gamma_\parallel} &= \zeta_{\Gamma_\parallel}^{(C)}(u_\parallel, \gamma_\parallel, w_\parallel) - \frac{1}{2} \frac{w_\parallel \gamma_\parallel}{1+w_\parallel} \left[ \left( \frac{2}{3} u_\times + \frac{w_\parallel \gamma_\parallel}{1+w_\parallel} \gamma_\perp \right) \right. \\ &\quad \left. \times \Re \left[ \frac{T_1}{w'_\perp} \right] - \frac{\gamma_\parallel F}{2(1+w_\parallel)} \Im \left[ \frac{T_2}{w'_\perp} \right] \right] + \zeta_{\Gamma_\parallel}^{(A)}(\{u\}, v_\perp, v). \end{aligned} \quad (14)$$

$\zeta_{\Gamma_\parallel}^{(C)}(u_\parallel, \gamma_\parallel, w_\parallel) = \zeta_\Gamma(u_\parallel, \gamma_\parallel, \Gamma_\parallel, w_\parallel) - \zeta_\Gamma^{(A^*)}(u_\parallel, \Gamma_\parallel)$ , where the functions on the right-hand side are defined by (A.8) and (A.9) for  $n=1$  in [9]. The functions  $T_1$  and  $T_2$  are defined as

$$\begin{aligned} T_1 &\equiv D_\perp \left[ 1 + \ln \frac{1 + \frac{1}{v_\perp}}{1+v} \right. \\ &\quad \left. - \left( v + \frac{1}{v_\perp} (1+v) \right) \ln \frac{(1+v) \left( 1 + \frac{1}{v_\perp} \right)}{v + \frac{1}{v_\perp} (1+v)} \right], \end{aligned} \quad (15)$$

$$\begin{aligned} T_2 &\equiv w_\perp^+ D_\perp \left[ (1+v_\perp)v - \ln \frac{1 + \frac{1}{v_\perp}}{1+v} \right. \\ &\quad \left. - \left( v + \frac{1}{v_\perp} (1+v) \right) (v+v_\perp(1+v)) \ln \frac{(1+v) \left( 1 + \frac{1}{v_\perp} \right)}{v + \frac{1}{v_\perp} (1+v)} \right], \end{aligned} \quad (16)$$

and  $\zeta_{\Gamma_\parallel}^{(A)}(\{u\}, v_\perp, v)$  is the  $\zeta$ -function of the kinetic coefficient of the parallel relaxation in the biconical model A. With a complex  $\Gamma_\perp$  it reads

$$\begin{aligned} \zeta_{\Gamma_\parallel}^{(A)}(\{u\}, v_\perp, v) &= \frac{u_\parallel^2}{4} \left( \ln \frac{4}{3} - \frac{1}{6} \right) \\ &\quad + \frac{u_\times^2}{18} \left( \ln \frac{(1+v) \left( \frac{1}{v_\perp} + v \right)}{v + \frac{1}{v_\perp} (1+v)} + vv_\perp \ln \frac{\left( 1 + \frac{1}{v_\perp} \right) \left( \frac{1}{v_\perp} + v \right)}{v + \frac{1}{v_\perp} (1+v)} \right. \\ &\quad \left. + v \ln \left( \frac{1 + \frac{1}{v_\perp}}{v + \frac{1}{v_\perp} (1+v)} - \frac{1}{2} \right) \right). \end{aligned} \quad (17)$$

In order to find the FP values of the time scale ratios and the mode coupling the right-hand sides of eqs. (8), (9) have to be zero. If the FP value of the mode coupling  $f_\perp$  were zero, one would obtain the FP values of the time ratios of model C discussed in [19]. However this FP is unstable. If the FP value of  $f_\perp$  is non-zero then due to the logarithmic terms in  $v$  in the  $\zeta$ -functions both OPs have to have the same time scales *i.e.* a finite non-zero FP value  $v^*$ . This is only possible either for non-zero finite FP values of  $w_\perp$  and  $w_\parallel$  or when both of these FP values are zero. No finite FP values for  $w_\perp$  and  $w_\parallel$  have been found. In the other case the *approach* to zero of both time



Table 1: Two-loop FP values of the mode coupling  $f_{\perp}$ , the ratios  $q = w_{\parallel}/w'_{\perp}$ ,  $s = w'_{\parallel}/w'_{\perp}$  and the dynamic exponents in the subspace  $w_{\parallel} = 0$ ,  $w_{\perp} = 0$  with finite value of  $v = q/(1 + is)$  for the static biconical  $\mathcal{B}$  and Heisenberg  $\mathcal{H}$  FPs. For comparison we add the FP values for the exponents that govern critical dynamics at magnetic fields below and above the multicritical point. These are described by model C at  $n = 1$  and model F at  $n = 2$ .

	$f_{\perp}^*$	$q^*$	$s^*$	$z_{\text{OP}}$	$z_m$
$\mathcal{B}$	1.232	$1.167 \cdot 10^{-86}$	0	2.048	1.131
$\mathcal{H}$	1.211	$3.324 \cdot 10^{-8}$	0	2.003	1.542
$\mathcal{B}$	1.232	$2.51 \cdot 10^{-782}$	0.705	2.048	1.131
$\mathcal{H}$	1.211	$3.16 \cdot 10^{-66}$	0.698	2.003	1.542
C [20]	–	–	–	2.18	2.18
F [21]	0.83	–	–	$\sim 1.5$	$\sim 1.5$

scales has to be the same. Therefore the approach to the multicritical dynamic FP is described by the flow in the limit  $w_{\perp} \rightarrow 0$ ,  $w_{\parallel} \rightarrow 0$  and  $v$  finite (asymptotic subspace). The flow in the complete dynamic parameter space and in this asymptotic subspace will be discussed afterwards.

The  $\zeta$ -function for the perpendicular OP relaxation might be complex,  $\zeta_{\Gamma_{\perp}} = \zeta'_{\Gamma_{\perp}} + i\zeta''_{\Gamma_{\perp}}$ . In order to obtain the usual asymptotic power laws for the relaxation coefficients  $\Gamma_{\parallel}$  and  $\Gamma_{\perp}$  the FP value of the imaginary part  $\zeta''^*$  has to be zero. In consequence the asymptotic flow of the real and imaginary part of  $v$  is governed by the same exponent  $\zeta'_{\Gamma_{\perp}} - \zeta'_{\Gamma_{\parallel}}$ .

If the FP value of the mode coupling  $f$  is different from zero and finite one has from eq. (9)  $\varepsilon + \zeta'_{\Gamma_{\perp}} + \zeta_{\lambda}^{(d)*} = 0$  and the relation [14] between dynamical and static critical exponents  $z_{\perp} + z_m = 2\frac{\phi}{\nu}$  (here the  $z$  exponents govern the corresponding scaling times and  $\phi$  and  $\nu$  are the crossover and correlation length exponents). The dynamical exponents are defined as  $z_o = 2 + \zeta_o^*$  with  $o = \perp, \parallel, m$ . Because  $v^*$  is finite and non-zero  $z_{\perp} = z_{\parallel} \equiv z_{\text{OP}}$ . This means that strong scaling with respect to the OPs, the components of the staggered magnetizations, but weak scaling with respect to the conserved density, the magnetization  $m$ , holds since  $z_m \neq z_{\text{OP}}$ .

The two-loop order values of the dynamic exponents together with the FP values of the time scales and the mode coupling are presented in table 1. For the model of the three-dimensional anisotropic antiferromagnet under consideration, the biconical FP  $\mathcal{B}$  ( $u_{\perp}^* \neq u_{\parallel}^* \neq u_{\times}^*$ ) has been shown to be stable. It governs the static critical behavior in the complete space of couplings (see, *e.g.*, [5]). Substituting their two-loop values obtained in [5] within generalized Padé-Borel resummation technique [22] into the flow equations (8), (9), we get two dynamical FPs, their coordinates are given in the first and third row of table 1. Although two different dynamical FPs are found (with zero and non-zero  $s^*$ ) this difference does not lead to a change in the corresponding FP values of the dynamical

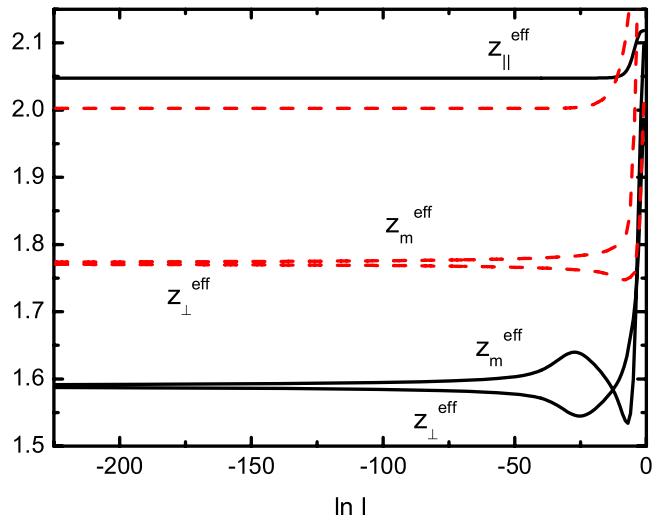


Fig. 1: (Colour on-line) Effective dynamic exponents in the background using the flow equations (8), (9) in the complete dynamical parameter space. The static FP values are taken for the Heisenberg FP (dashed curves) and for the biconical FP (solid curves).

exponents. This is because both FPs have extremely small but different  $q^*$ .

For comparison we have included, besides the biconical FP  $\mathcal{B}$  describing tetracritical behavior, the isotropic Heisenberg FP  $\mathcal{H}$  ( $u_{\perp}^* = u_{\parallel}^* = u_{\times}^*$ ) describing bicritical behavior. This FP is only reached in the subspace of the static couplings that lie in its attraction region (see fig. 3 in [5]). Again, as in the case of the biconical FP  $\mathcal{B}$ , we obtain two dynamical FPs with coordinates given in the second and fourth line of table 1. The numerical values of the dynamical exponents are practically equal in the different dynamical FPs  $\mathcal{H}$ . We further quote in table 1 the dynamical critical exponents on the two phase transition lines below and above the multicritical point, which are given by model C and model F, respectively.

The FP value of  $v$  is extremely small and therefore in the experimental accessible region one cannot prove strong scaling for the OP components. Indeed in the non-asymptotic region the dynamic parameters are described by the flow equations (8), (9) and from these dependencies the effective dynamic exponents can be calculated. The result is shown in fig. 1. The static parameters have been set already to their FP values and therefore the starting values of the effective exponents are different from  $z = 2$ . It turns out that the prefactor of the  $\ln v$ -terms in eqs. (11) and (12), which drive the flow of the dynamic parameters into the asymptotic subspace is reduced and the flow is almost like in one-loop order. Therefore weak scaling with  $z_{\parallel}^{eff} \sim 2.04$  and  $z_{\perp}^{eff} \sim z_m^{eff} \sim 1.6$  is observed<sup>1</sup>.

<sup>1</sup>Even for flow parameters  $\ln l \sim -10^6$  no visible changes in the values of the different  $z^{eff}$  occur although the dynamic parameters change. Thus, the asymptotic subspace is not reached for these extremely small values of  $l$ .

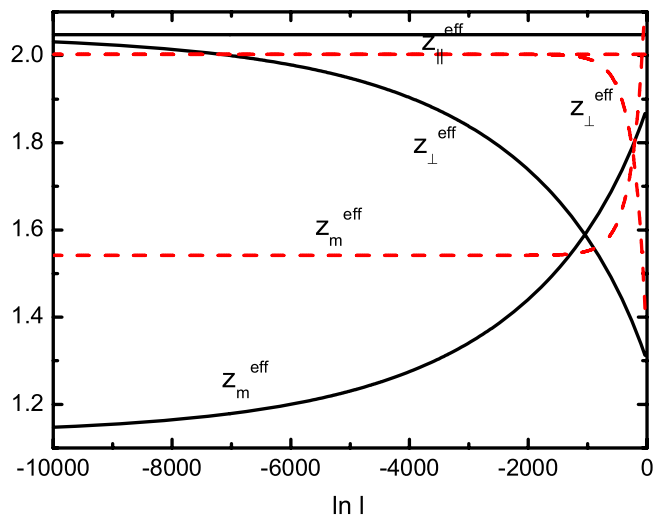


Fig. 2: (Colour on-line) Effective dynamic exponents in the asymptotic subspace  $w_{\parallel} = w_{\perp} = 0$  and  $v \equiv w_{\parallel}/w_{\perp} \neq 0$  and finite. Dashed and solid curves as in fig. 1.

The approach of the effective dynamical exponents in the asymptotic subspace  $w_{\perp} = w_{\parallel} = 0$  and  $v$  finite to their biconical FP values is shown in fig. 2. The background behavior is dominated by a behavior corresponding for the perpendicular components by model F and for the parallel components by model A with a finite value  $\text{Re}(v)$  whereas  $\text{Im}(v)$  is almost zero. Therefore for the biconical case even for flow parameter values  $\ln l \sim -10^4$  the two effective exponents do not reach their asymptotics:  $z_{\perp}^{\text{eff}} < z_{\text{OP}}$  and  $z_m^{\text{eff}} > z_m$ . This is different for the Heisenberg case where the FP values of the dynamical exponents are reached (see dashed curves in fig. 2).

Although from our calculation we conclude that the asymptotics (strong scaling) would be unobservable effective exponents as described in fig. 1 are observable. The complete two-loop calculation allowed us to calculate not only the values of the dynamic FP but also the effective exponents which are the quantities governing the behavior of the transport coefficients, *i.e.* the relaxation and diffusion coefficient of the staggered magnetization and magnetization, respectively. It is well known that near a dynamical stability boundary separating a strong scaling FP with a finite time scale ratio from a weak scaling FP with a vanishing time scale ratio also small dynamic transient exponents appear and effective critical behavior is observed. The case where the OPs have the component values  $n=2$  and  $n=1$  is located near the stability boundary between the biconical and decoupling FP as has been demonstrated in [5]. For the decoupling FP the time scales of the two OPs scale differently and weak scaling is expected. The effective values of the dynamical critical exponents (starting from different initial conditions) are driven to almost stationary values. These might be measured in neutron scattering experiments.

A natural question concerns the reliability of numerical predictions for the observables obtained in our study.

In particular, how will an increase of the order of the perturbation theory influence the numerical estimates. An estimate can be given by comparing results obtained in different perturbation theory (loop) orders. This can be done for the static part of the RG functions, which are currently known with a record five-loop accuracy [23]. As it was demonstrated in ref. [5], the two-loop approximation we consider here refined by the resummation is enough to catch the main features of the static phase transition (FP stability and respective universality classes) as well as to give reliable estimates for the observable quantities that govern the phase transition (leading exponents and corrections to scaling). It is well known that the dynamic RG calculations are technically much more complicated as static ones. In particular, no higher orders of the perturbation theory are known for the model we consider here. However, as known from the previous experience in the RG description of dynamical criticality [9] we expect also in this case that the two-loop calculation captures the essential dynamical properties, namely to be effectively in a weak scaling situation.

The comparison between experiment and theory for multicritical behavior is much less developed in dynamics than in statics (see, *e.g.*, the situation at the tricritical point [24]). But exploring even the critical dynamics along the two transition lines is of interest since the OP and the conserved density are experimentally accessible. Also computer simulation might be considered for a comparison [25] and explicit theoretical results are worthwhile for the interpretation of the numerical results. Not only exponents are necessary for a careful interpretation but also the calculation of the dynamic structure factors are of interest. This also concerns the neutron scattering experiments. The effective values of the time scale ratios, known from flow equations (like (8)), enter the shape functions and may change their shape from a Lorentzian considerably.

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